## PREFACE

Academician Obrechkoff (1896-1963) was a world-famous expert in the summation of divergent series, an area of studies started in the 19 -th and continued with a growing intensity in 20-th century engaging the efforts of a great number of outstanding mathematicians as N. Abel, S. Poisson, E. Borel, G. Mittag-Leffler, O. Hölder, E. Cesaro, L. Feijer, G. Hardy, J. Littlewood, F. Hausdorff, M. Riesz, N. Winner and many others. Obrechkoff himself also let a deep trace in the summation theory. One of his first remarkable results is the summation of the derivatives of a Fourier series which generalized the classical theorem of Riemann for these series. Remarkable are also his essential generalizations of Borel's and Mitag-Leffler's methods for summation as well as the summation on the boundaries of the regions of summability of Taylor's series by these methods.

In 1939 appeared in Bulgarian his paper Summation by Euler's transform of the series of Dirichlet, factorial series and the series of Newton, Annuaire Univ. Sofia, Phys.-Math. Fac., 1, 1-148. Probably, due to the language of this publication it remained almost unknown to the specialists in the field of divergent series.

The present issue, which is an English translated version of this paper, is published on the occasion of 120 years from the birth of Academician Obrechkoff, the 70 anniversary of the Institute of Mathematics and Informatics of Bulgarian Academy of Sciences (created October 1947) and 310 years from the birth of Leonard Euler (1717-1783).

In this work Obrechkoff deals with a contemporary version of Euler's summation due to Konrad Knopp. An arbitrary series

$$
\begin{equation*}
\sum_{n=0}^{\infty} a_{n}, a_{n} \in \mathbb{C}, \quad n=0,1,2, \ldots \tag{1}
\end{equation*}
$$

is called $E_{k}$-summable, if the series

$$
\begin{equation*}
\sum_{n=0}^{\infty} A_{n}^{k} \tag{2}
\end{equation*}
$$

with members defined by the formal expansions

$$
\sum_{n=0}^{\infty}\left(\frac{z}{q+1-q z}\right)^{n+1}=\sum_{n=0}^{\infty} A_{n}^{k} z^{n+1}
$$

i.e.,

$$
\begin{gathered}
A_{n}^{k}=\frac{1}{(q+1)^{n+1}} \sum_{\nu=0}^{n}\binom{n}{\nu} q^{n-\nu} a_{\nu} \\
q=2^{k}-1, \quad k>0, \quad n=0,1,2, \ldots
\end{gathered}
$$

is convergent. If the series (2) is absolutely convergent, then the series (1) is called $\left|E_{k}\right|$-summable. The classical Euler summation corresponds to $k=0$, i.e., when $q=1$.

In this Obrechkoff's paper the $E_{k}$-summation is applied to the series

$$
\begin{equation*}
\sum_{n=0}^{\infty} \frac{a_{n}}{(n+1)^{s}}, \quad \sum_{n=0}^{\infty} \frac{a_{n}}{s(s+1) \ldots(s+n)}, \quad \sum_{n=0}^{\infty} a_{n} s(s-1) \ldots(s-n) \tag{3}
\end{equation*}
$$

He has succeeded to "translate" the classical theory of these series into the language of the Euler-Knopp summation. Indeed, at first Abel's type theorems are established for the series (3). E.g., it is proved that if the Dirichlet series is $E_{k}$-summable for some $s_{0} \in \mathbb{C}$, then the same holds for each $s \in \mathbb{C}$ such that $\Re s>\Re s_{0}$. Moreover, for the holomorphic function $f(s)$, which is the $E_{k^{-}}$ transform of the Dirichlet series, it holds the representation

$$
f(s)=\frac{1}{\Gamma\left(s-s_{0}\right)} \int_{0}^{\infty} t^{s-s_{0}-1} \sum_{n=0}^{\infty} S_{n}^{k}\left(\frac{q+1}{q+e^{t}}\right)^{n+1} \frac{e^{t}-1}{e^{t}+q} d t, \Re s>\Re s_{0},
$$

where

$$
S_{n}^{k}=\sum_{\nu=0}^{n} A_{\nu}^{k}
$$

A consequence of the last statement is the existence of a real number $e_{k}$ such that the Dirichlet series is $E_{k}$-summable for each $s$ such that $\Re s>e_{k}$, and loses this property when $\Re s<e_{k}$. For this number, named abscissa of the $E_{k}$-summability of the Dirichlet series, Obrechkoff has proved that if $e_{k} \geq 0$, then

$$
e_{k}=\limsup _{n \rightarrow \infty} \frac{\log \left|A_{0}^{k}+A_{1}^{k}+\cdots+A_{n}^{k}\right|}{\log (n+1)}
$$

and

$$
e_{k}=\limsup _{n \rightarrow \infty} \frac{\log \left|A_{n}^{k}+A_{n+1}^{k} \ldots\right|}{\log (n+1)}
$$

when $e_{k}<0$, i.e., he has also got formulas of Cauchy-Hadamard's type for the $E_{k}$-summation of Dirichlet's series.

Further, it is proved that analogous statements hold also for the factorial and the Newton series as well as that they remain true for all the series (3) if the $E_{k}$-summability is replaced by the absolute one, i.e., $A_{n}^{k}, n=0,1,2, \ldots$ are replaced by $\left|A_{n}^{k}\right|, n=0,1,2, \ldots$.

At the end, Obrechkoff has applied the Borel integral method for summation of divergent series to the factorial series written in the form

$$
\begin{equation*}
\sum_{n=0}^{\infty} \frac{n!a_{n}}{s(s+1) \ldots(s+n)}, \quad s \neq 0,-1-2, \ldots \tag{4}
\end{equation*}
$$

and has got results similar to that for the Euler-Knopp summation of the same series. One of them is that if the series (4) is $B$-summable for some $s_{0} \in \mathbb{C}$, then the same holds for each $s \in \mathbb{C}$ such that $\Re s>\Re s_{0}$.

There is no doubt that the readers of this version of Obrechkoff's paper, already mentioned, will be deeply impressed not only by the beauty of his results but also by the brilliant analytic technique he has used.

Peter Rusev
Institute of Mathematics and Informatics Bulgarian Academy of Sciences

