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A FRACTIONAL INEQUALITY ON COMPACT MANIFOLDS AND LIE GROUPS

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We show that, on any compact Riemaniann manifold or any unimodular Lie group, the space $H^\sigma \cap L^\infty$ is a fractional algebra, by proving the following inequality

$$\|f|f|^\mu\|_{H^\sigma} \leq C \|f\|_{H^\sigma} \|f\|^\mu_{L^\infty},$$

for any $f \in H^\sigma \cap L^\infty$, with $\mu > 0$, $C > 0$ and provided that $0 < \sigma < 1 + \mu$.

1. Introduction

Let us consider first a Riemaniann manifold $\mathcal{M}^k$ (at least complete) with $k \geq 1$, endowed with a standard smooth metric g. Introduce then Δ_y, the Laplace-Beltrami operator associated to the manifold $\mathcal{M}^k$, defined in local coordinates by

$$\frac{1}{\sqrt{|g(y)|}}\partial_{y_i}\sqrt{|g(y)|}g^{hi}(y)\partial_{y_i},$$

where $g_{hi}(y)$ is the metric tensor, $|g(y)| = \det(g_{hi}(y))$ and $g^{hi} = (g_{hi}(y))^{-1}$. For any $\sigma \in \mathbb{R}$, we denote by $H^\sigma_y = W^{\sigma,2}_y$, where

$$W^{\sigma,q}_y = W^{\sigma,q}(\mathcal{M}^k) = (1-\Delta_y)^{-\frac{\sigma}{2}} L^q_y. \tag{1}$$

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Here $L^q_y = L^q(\mathcal{M}^k)$, indicating by $f \in L^q(\mathcal{M}^k)$, for $1 \leq q < \infty$, if

$$\|f\|^q_{L^q(\mathcal{M}^k)} = \int_{\mathcal{M}^k} |f(y)|^q \, d\mathrm{v}_g < +\infty,$$

(one can has obvious modification when $q = \infty$) with $d\mathrm{v}_g$ the volume element of $\mathcal{M}^k$ which reads in local coordinates as $\sqrt{|g(y)|}dy$. Furthermore the h-th component of the gradient operator ∇_y is given in local coordinates by $g^{hi}(y)\partial_{y_i}$. We introduce also the quantity $|\nabla_y|^\sigma$ defined as $(-\Delta_y)^{\frac{\sigma}{2}}$.

We need the following assumptions:

- Doubling property. One says that $\mathcal{M}^k$ satisfies the doubling property (D) if there exists a constant $C > 0$, such that for all $y \in M^k$, $r > 0$ we have

(2) $$\mathrm{v}_g(B(y, 2r)) \leq C\mathrm{v}_g(B(y, r)),$$

  where $B(y, r)$ is the open ball of center $y \in \mathcal{M}^k$ and radius r.

- Lie groups. We assume that there exists a constant $c > 0$ such that for all $y \in \mathcal{M}^k$

(3) $$\mathrm{v}_g(B(y, r)) \geq c.$$

The above assumptions imply to the following facts.

Let $\mathcal{M}^k$ be a Riemannian manifold satisfying (D). Denote by M the uncentered Hardy-Littlewood maximal function over open balls $\mathcal{Q}$ of $\mathcal{M}^k$ defined by

(4) $$\mathrm{M}(f)(y) = \sup_{\substack{\mathcal{Q} \\ y \in \mathcal{Q}}} \frac{1}{\mathrm{v}_g(\mathcal{Q})} \int_{\mathcal{Q}} |f| \, d\mathrm{v}_g.$$

Then for every $p \in (1, \infty]$, M is L^p bounded and moreover of weak type $(1, 1)$. Observe that if $\mathcal{M}^k$ satisfies (D) then $\mathrm{diam}(\mathcal{M}^k) < \infty$ if and only if $\mathrm{v}_g(\mathcal{M}^k) < \infty$. Therefore if $\mathcal{M}^k$ is a non-compact Riemannian manifold satisfying (2) then $\mathrm{v}_g(\mathcal{M}^k) = \infty$. In the framework of Lie groups, due to the homogeneous type of the manifold $\mathcal{M}^k$, this is equivalent to a control from below of the volume of balls

(5) $$\mathrm{v}_g(B(y, r)) \geq r^k,$$

for $0 < r \leq 1$.

At this point we can illustrate our main target. We want to see that for $\mathcal{M}^k$ a compact Riemaniann manifold (or for a Lie Group), the space $H^\sigma_y \cap L^\infty_y$ is a fractional algebra. Namely, our main contribution is the following (see also [6]).

Theorem 1. *Assume $\mathcal{M}^k$ a compact manifold (resp. Lie group) having dimension (resp. local dimension) $k \geq 1$. Then for any $f \in H^\sigma_y \cap L^\infty_y$ let $G(f) = f|f|^\mu$ be a function with $\mu > 0$. Then one has*

$$\|G(f)\|_{H^\sigma_y} \leq C \|f\|_{H^\sigma_y} \|f\|^\mu_{L^\infty_y}, \tag{6}$$

with $C > 0$, provided that $0 < \sigma < 1 + \mu$.

Remark 1. Let $\mathcal{M}^k = \mathcal{G}$ be a unimodular connected Lie group endowed with its Haar measure $d\nu = dx$ and assume that it has a polynomial volume growth. Recall that "unimodular" means that dx is both left-invariant and right-invariant. Let $\mathcal{L}$ be the Lie algebra of $\mathcal{G}$. Consider a family $Y = Y_1, \dots, Y_M$ of left-invariant vector fields on $\mathcal{G}$ satisfying the Hörmander condition. By "left-invariant" one means that, for any $g \in \mathcal{G}$ and $f \in C^\infty_0(\mathcal{G})$,$Y(\tau_g f) = \tau_g(Yf)$, where τ_g is the left-translation operator. Then we can build the Carnot-Carathéodory metric on $\mathcal{G}$. This allows us to equip the group $\mathcal{G}$ with a sub Riemannian structure. The left-invariance of the Y_i implies the left-invariance of the distance. So that for every r, the volume of the ball $B(x, r)$ does not depend on $x \in \mathcal{G}$. It is well known that (G, d) is then a space of homogeneous type. Particular cases are Carnot groups, where the vector fields are given by a Jacobian basis of its Lie algebra and satisfy Hörmander condition. In this situation, two cases may occur: either the manifold enjoy (2) and (5) or the volume of the balls admit an exponential growth. For example, nilpotents Lie groups satisfy the doubling property as well as the Heisenberg group $\mathcal{H}^k$ that is a Carnot group of dimension $2k + 2$. Then Theorem 1 works exactly in the same fashion as in the compact geometry setting.

We briefly explain why we are seeking for estimates like the (6). Consider the following family of Cauchy problems:

$$\begin{cases} i\partial_t u + \Delta_y u + \lambda|u|^\mu u = 0, & (t, y) \in \mathbb{R} \times \mathcal{M}^k \\ u(0, y) = f(y) \in H^\sigma_y, \end{cases} \tag{7}$$

where the nonlinearity parameter μ satisfies $4/k < \mu < \mu^*(k)$, for some $\mu^*(k) > 0$. Then it is possible to investigate some relevant questions, for instance as the local and global existence as well as the persistence of regularity for the map data-solution $f \to u(t, \cdot)$ assuming the initial data in the space H^σ_y. In order to do that, we need to close a the fixed point argument. This means that we have

Let Ω be an open connected subset of $\mathbb{R}^d$ and Y family of real-valued, infinitely differentiable vector fields. Y is said to satisfy Hörmander condition in Ω if the family of commutators of vector fields in Y $(Y_i, [Y_i, Y_j], \dots)$ span $\mathbb{R}^d$ at every point of Ω.

to show

$$\|u|u|^{\mu}(t,x,\cdot)\|_{H^{\sigma}_y} \leq \|u(t,x,\cdot)\|^{\mu+1}_{H^{\sigma}_y},$$

in general achieved by using kind of a Sobolev fractional inequality like the (6) in combination with the embedding $H^{\sigma}_y \subset L^{\infty}_y$, valid for $\frac{k}{2} < \sigma$. We underline that the above inequality (1.) is clear when μ is an odd integer because of the fact that H^{σ}_y is an algebra. But totally unclear if μ is fractional. The literature, in this direction, is not so extended. I want to cite the following result of [5], with $\mathcal{M}^k = \mathbb{R}^k$ and [7], valid when one choses $\mathcal{M}^k = \mathbb{T}$.

2. Proof of the main theorem

We present here the proof in the compact geometry setting only, because the one concerning the Lie Groups is similar. Before to start, we recall the following basics.

Given any compact manifold $\mathcal{M}^k$, we have:

- the doubling property (D) is satisfied;
- the curvature tensor (with its derivatives) is bounded;
- the Ricci curvature tensor is bounded from below;
- the injectivity radius is positive.

At this point we can proceed by proving the Theorem 1.

Proof. By an use of the above facts then we are allowed to represent the fractional derivative as $|\nabla_y|^{\sigma} = (-\Delta_y)^{\frac{\sigma}{2}}$, when $0 < \sigma < 1$ as

$$|\nabla_y|^{\sigma} f(y) = \left(\int_0^{\infty} \left(\frac{1}{t^{\sigma} \mathrm{v}_g(B(y,t))} \int_{B(y,t)} |f(x) - f(y)| \, d\mathrm{v}_g(x) \right)^2 \frac{dt}{t} \right)^{\frac{1}{2}}, \tag{8}$$

with $B(y,t)$ we indicate the open ball of center $y \in \mathcal{M}^k$ and radius $t > 0$ (we remand to [2] and [4]). We can go over to the next step, that is the proof of the following lemma, which is a generalization of a result appeared in the paper [8] and valid in the euclidean setting.

Lemma 1. *Assume $\mathcal{M}^k$ is a compact manifold with dimension $k \geq 1$ and let ϕ be an Hölder continuous function of order $0 < \mu < 1$. Thus, for any $0 < s < \mu$, $1 < q < \infty$ and $\frac{s}{\mu} < \sigma < 1$ we get*

$$\||\nabla_y|^s \phi(f)\|_{L^q_y} \leq C \left\| |f|^{\mu - \frac{s}{\sigma}} \right\|_{L^{q_1}_y} \||\nabla_y|^\sigma f\|^{\frac{s}{\sigma}}_{L^{q_2 \frac{s}{\sigma}}_y}, \tag{9}$$

with $C > 0$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ and $\left(1 - \frac{s}{\mu\sigma}\right) q_1 > 1$.

P r o o f o f t h e l e m m a. It comes out from the pointwise inequality

$$|\nabla_y|^s \phi(f)(y) \leq C (M(|f|^\mu)(y))^{1 - \frac{s}{\mu\sigma}} (|\nabla_y|^\sigma f(y))^{\frac{s}{\sigma}}, \tag{10}$$

with $|\nabla_y|^s$ defined as in (8), where

$$M(f)(y) = \sup_{t>0} \frac{1}{\mathrm{v}_g(B(y,t))} \int_{B(y,t)} |f(x)| \, d\mathrm{v}_g(x),$$

is the ceneterd Hardy-Littlewood maximal operator (equivalent to (4)) defined on $\mathcal{M}^k$ (for additional details we refer to [2] and [3]). Because of ϕ is μ-Hölder continuous,

$$|\phi(f(x)) - \phi((y))| \leq C |f(x) - f(y)|^\mu \leq C |f(x)|^\mu + |f(y)|^\mu, \tag{11}$$

for some $C > 0$. We use both estimates; the first one for small values of t, and the second for large values of t. The meaning of both "small" and "large" is y-dependent. For t small, we can use the Hölder inequality, getting the following

$$\int_0^{A(y)} \left(\frac{1}{t^s \mathrm{v}_g(B(y,t))} \int_{B(y,t)} |\phi(f(x)) - \phi((y))| \, d\mathrm{v}_g(x) \right)^2 \frac{dt}{t} \tag{12}$$

$$\leq C \int_0^{A(y)} \left(\frac{1}{\mathrm{v}_g(B(y,t))} \int_{B(y,t)} |f(x) - f(y)|^\mu \, d\mathrm{v}_g(x) \right)^2 \frac{dt}{t^{1+2s}}$$

$$\leq C \int_0^{A(y)} \left(\frac{1}{\mathrm{v}_g(B(y,t))} \int_{B(y,t)} |f(x) - f(y)| \, d\mathrm{v}_g(x) \right)^{2\mu} \frac{dt}{t^{1+2s}}.$$

The last term of the above chain of inequalities is bounded by

$$(A(y))^{2(\sigma\mu - s)} \left(\int_0^{A(y)} \left(\frac{1}{\mathrm{v}_g(B(y,t))} \int_{B(y,t)} |f(x) - f(y)| \, d\mathrm{v}_g(x) \right)^2 \frac{dt}{t^{1+2\sigma}} \right)^\mu$$

$$\leq C(A(y))^{2(\sigma\mu-s)}(|\nabla_y|^s f(y))^{2\mu}.$$

The penultimate step requires $\sigma\mu - s > 0$. For large t, we first note that

$$\frac{1}{\mathrm{v}_g(B(y,t))}\int_{B(y,t)}|u(x)|^{\mu}\,d\mathrm{v}_g(x) \leq CM(|u|^{\mu})(y).$$

Consequently, we get that

$$\int_{A(y)}^{\infty}\left(\frac{1}{t^s\mathrm{v}_g(B(y,t))}\int_{B(y,t)}|\phi(f(x))-\phi((y))|\,d\mathrm{v}_g(x)\right)^2\frac{dt}{t},$$

can be bounded by the following

$$\int_{A(y)}^{\infty}\left(\frac{1}{\mathrm{v}_g(B(y,t))}\int_{B(y,t)}|f(x)|^{\mu}+|f(y)|^{\mu}d\mathrm{v}_g(x)\right)^2\frac{dt}{t^{1+2s}}$$

$$\leq C\int_{A(y)}^{\infty}\frac{dt}{t^{1+2s}}(M(|u|^{\mu})(y))^2 \leq C(A(y))^{-2s}(M(|u|^{\mu})(y))^2.$$

Picking up $A(y) = (M(|f|^{\mu})(y))^{\frac{1}{\mu\sigma}}(|\nabla_y|^{\sigma}f(y))^{-\frac{1}{\sigma}}$ one gets immediately (10). The lemma is a straightforward consequence of Hölder inequality and boundedness of the maximal operator; the latter requires $\left(1-\dfrac{s}{\mu\sigma}\right)q_1 > 1$. $\square$

We turn back to the proof of the main theorem.

Case $0 < \sigma < 1$. In this regime one has

$$\|f\|_{H_y^{\sigma}} \sim \|f\|_{L_y^2} + \||\nabla_y|^{\sigma}f\|_{L_y^2}, \tag{13}$$

with $|\nabla_y|^{\sigma}$ as in (8). The result is given by an application of the elementary inequality

$$|f(x)|f(x)|^{\mu} - f(y)|f(y)|^{\mu}| \leq C\,\||f|^{\mu}\|_{L_y^{\infty}}|f(x)-f(y)| \leq C\,\|f\|_{L_y^{\infty}}^{\mu}|f(x)-f(y)|.$$

Case $\sigma = 1$. This is given by the fact that the $L^{\infty}\cap H_y^{\sigma}$ is an algebra when μ is an even integer. Otherwise we can include it in the next case because the embedding $H_y^{\sigma}\subset H_y^1$.

Case $\sigma > 1$. We will only give the details for $\mu < 1$. The argument works also in the case $\mu > 1$ with some further restriction on σ. Because of $\sigma > 1$ we can write $\sigma = 1 + s$ with $0 < s < 1$, then by the definition of the H_y^{σ}-norm we arrive at

$$\|f|f|^{\mu}\|_{H_y^{\sigma}} \leq C\,\|f|f|^{\mu}\|_{H_y^s} + \|\nabla_y(f|f|^{\mu})\|_{H_y^s} \lesssim \|f|f|^{\mu}\|_{H_y^s} + \||f|^{\mu}\nabla_y f\|_{H_y^s}. \tag{14}$$

Then we can see that it is enough to deal with the last term in the previous estimate, that is

$$\| |f|^{\mu} \nabla_y f \|_{H^s_y} \tag{15}$$

$$\leq C \, \|\nabla_y f\|_{H^s_y} \, \|f\|^{\mu}_{L^\infty_y} + \|\nabla_y f\|_{L^{2\sigma}_y} \, \| |f|^{\mu} \|_{W^{s,2\frac{\sigma}{s}}_y} \, ,$$

where we used the first estimate of Theorem 27 in [4]. Since the interpolation bound

$$\|\nabla_y f\|_{L^{2\sigma}_y} \lesssim \|f\|^{\frac{1}{\sigma}}_{H^\sigma_y} \, \|f\|^{1-\frac{1}{\sigma}}_{L^\infty_y} \, , \tag{16}$$

we need only to prove the following one

$$\| |\nabla_y|^s |f|^{\mu} \|_{L^{2\frac{\sigma}{s}}_y} \lesssim \left\| |\nabla_y|^{\overline{\sigma}} f \right\|^{\frac{s}{\overline{\sigma}}}_{L^{2\frac{\sigma}{\overline{\sigma}}}_y} \, \|f\|^{\mu-\frac{s}{\overline{\sigma}}}_{L^\infty_y} \, , \tag{17}$$

for some $\dfrac{s}{\mu} < \overline{\sigma} < 1$. Assume the above (17) true, then by the interpolation estimate (we refer to the Propositions 31, 32 in [4])

$$\left\| |\nabla_y|^{\overline{\sigma}} f \right\|_{L^{2\frac{\sigma}{\overline{\sigma}}}_y} \leq C \, \| |\nabla_y|^{\sigma} f \|^{\frac{\overline{\sigma}}{\sigma}}_{L^2_y} \, \|f\|^{1-\frac{\overline{\sigma}}{\sigma}}_{L^\infty_y} \, , \tag{18}$$

one achieve

$$\| |\nabla_y|^s |f|^{\mu} \|_{L^{2\frac{\sigma}{s}}_y} \leq C \left(\| |\nabla_y|^{\sigma} f \|^{\frac{\overline{\sigma}}{\sigma}}_{L^2_y} \, \|f\|^{1-\frac{\overline{\sigma}}{\sigma}}_{L^\infty_y} \right)^{\frac{s}{\overline{\sigma}}} \|f\|^{\mu-\frac{s}{\overline{\sigma}}}_{L^\infty_y} \tag{19}$$

$$\leq C \, \| |\nabla_y|^{\sigma} f \|^{\frac{s}{\sigma}}_{L^2_y} \, \|f\|^{\frac{s}{\overline{\sigma}}-\frac{s}{\sigma}}_{L^\infty_y} \, \|f\|^{\mu-\frac{s}{\overline{\sigma}}}_{L^\infty_y} \leq C \, \|f\|^{\frac{s}{\sigma}}_{H^\sigma_y} \, \|f\|^{\frac{s}{\overline{\sigma}}-\frac{s}{\sigma}}_{L^\infty_y} \, \|f\|^{\mu-\frac{s}{\overline{\sigma}}}_{L^\infty_y} \, .$$

Therefore (16) in connection with (19), by recalling again the definition of the H^σ_y-norm, brings to

$$\|\nabla_y f\|_{L^{2\sigma}_y} \, \| |f|^{\mu} \|_{W^{s,2\frac{\sigma}{s}}_y} \leq C \, \|f\|_{H^\sigma_y} \, \|f\|^{\mu}_{L^\infty_y} \tag{20}$$

Finally combining (14), (15) and (20) one arrive at (6). It remains to consider the estimate (17). Since the function $|f|^{\mu}$ is Hölder continuous of order $0 < \mu < 1$, one is in position to apply Lemma 1 with $q = q_2 = 2\dfrac{\sigma}{s}$, $q_1 = \infty$ and $\sigma = \overline{\sigma}$ getting

$$\| |\nabla_y|^s |f|^{\mu} \|_{L^{2\frac{\sigma}{s}}_y} \leq C \left\| |\nabla_y|^{\overline{\sigma}} f \right\|^{\frac{s}{\overline{\sigma}}}_{L^{2\frac{\sigma}{s}\frac{s}{\overline{\sigma}}}_y} \left\| |f|^{\mu-\frac{s}{\overline{\sigma}}} \right\|_{L^\infty_y} \, ,$$

that is the desired (17). This completes the proof of the theorem. □

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