PLISKA
STUDIA MATHEMATICA

ПЛИСКА
МАТЕМАТИЧЕСКИ
СТУДИИ

Pliska Stud. Math. **30** (2019), 21–28

PLISKA
STUDIA MATHEMATICA

GROMOV HYPERBOLICITY OF THE KOBAYASHI METRIC

Lyubomir Andreev, Nikolai Nikolov, Maria Trybuła

In this paper we discuss the global geometry of the Kobayashi metric on domains in $\mathbb{C}^n$. We gather all known results. We pay special attention to methods which distroy the hyperbolicity.

1. Part I

In this paper we study the Kobayashi distance on domains in $\mathbb{C}^n$. The Kobayashi distance k_Ω is a pseudodistance associated to every domain Ω, which has many important properties, for instance, the distance decreasing property. It is known that for bounded domains the Kobayashi pseudodistance is actually a distance. The Kobayashi distance is useful for studying holomorphic maps and the geometry of the Kobayashi distance has played an important role in tmany results in several complex variables.

An important feature of the Kobayashi metric for strongly pseudoconvex domains is that they are negatively curved. For general domains the Kobayashi

2010 *Mathematics Subject Classification:* 32F45, 53C23.

Key words: Kobayashi distance, Kobayashi metric, Gromov hyperbolicity, convexity, pseudoconvexity, quasi-geodesic.

The second named author was partially supported by the Bulgarian National Science Fund, Ministry of Education and Science of Bulgaria under contract DN 12/2. The third named author was supported by National Center of Science (Poland), grant no. 2013/10/A/ST1/00091.

metric is no longer Riemannian, and thus will no longer has a curvature. Instead one can consider a coarser notion of non-positive curvature from geometric group theory called Gromov hyperbolicity.

Gromov hyperbolic metric spaces have been intensively studied and have a number of important properties. For instance, it is known that in a Gromov hyperbolic space every quasi-geodesic is within a bounded distance of an actual geodesic (cf. [3, Chapter III.H, Theorem 1.7]). This property is known as the shadowing property. This gives us some information about the posiible geodesics since often segments to the boundary points are quasi-geodesics however it is difficult to find actual geodesics. Moreover, every quasi-isometry $f : \Omega_1 \to \Omega_2$ between Gromov hyperbolic metric spaces have a continuous extension to natural compactifications of Ω_1 and Ω_2 (cf. [3, Chapter III.H, Theorem, 3.9]).

Balogh and Bonk [2] proved that the Kobayashi metric is Gromov hyperbolic when the domain is strongly pseudoconvex.

Theorem 1.1. ([2]) *Suppose Ω is a strongly pseudoconvex domain. Then (Ω, k_Ω) is Gromov hyperbolic.*

For a long time it was one and only known example of hyperbolic spaces in complex analysis. It was Stephen M. Buckley who posed for the first time whether the Kobayashi metric on general smooth, bounded pseudoconvex domains with finite type is Gromov hyperbolic. Gaussier and Seshadri [4] proved this result for bounded convex domains (see also [10]). Extending this work, Zimmer gave a complete characterization of bounded convex domains for which the Kobayashi metric is Gromov hyperbolic.

Theorem 1.2. ([12]) *Suppose Ω is a bounded convex domain with $\mathcal{C}^\infty$ boundary. Then (Ω, k_Ω) is Gromov hyperbolic if and only if $\partial\Omega$ has finite type in the sense of D'Angelo.*

Part "if" of this result was later extended on a natural generalization of convex domains, i.e., $\mathbb{C}$-convex domains.

The study of the sensitivity of Gromov hyperbolicity to removing some part from a domain was initiated in [10] by Thomas and the last two named authors. They observed that under certain conditions on Ω_2, the space $(\Omega_1 \setminus \overline{\Omega}_2, k_{\Omega_1\setminus\overline{\Omega}_2})$ is hyperbolic if and only if (Ω_1, k_{Ω_1}) is hyperbolic.

Proposition 1.3. ([10, Proposition 6]) *Let Ω be a bounded domain in $\mathbb{C}^n$, $n \geq 2$. Assume that K is a linearly convex compact subset of Ω. Then $\Omega \setminus K$ is a domain such that $k_{\Omega\setminus K}$ is quasi-isometrically equivalent to $k_\Omega|_{(\Omega\setminus K)\times(\Omega\setminus K)}$.*

In particular, if (Ω, k_Ω) is Gromov hyperbolic, then so is $(\Omega \setminus K, k_{\Omega\setminus K})$.

Motivated by this result, Haggui and Chrih in [6] considered a situation when we remove a non-compact subset from the domain.

Theorem 1.4. ([6, Theorem 3.1]) *Let Ω be a bounded convex open set in $\mathbb{C}^n$ and S be a complex affine hyperplane such that $\Omega \cap S$ is not empty. Then $(\Omega \setminus S, k_{\Omega\setminus S})$ is not Gromov hyperbolic.*

If A is relatively closed in Ω, and A and very small, i.e., it is negligible with respect to the $(2n-2)$-dimensional Hausdorff measure, then

$$k_{\Omega\setminus A} = k_\Omega|_{(\Omega\setminus A)\times(\Omega\setminus A)}$$

(cf. [8, Theorem 3.4.2]). The case considered by Haggui and Chrih is interesting since removing any hyperplane from a convex domain does not destroy the pseudoconvexity . Nikolov and Trybuła refined the construction used in [6], and showed that what is really needed in Theorem 1.4 is $\mathbb{C}$-convexity.

Theorem 1.5. ([11]) *Let $\Omega \subset \mathbb{C}^n$ be a bounded $\mathbb{C}$-convex domain and S a complex affine hyperplane such that $\Omega \cap S$ is not empty. Then $(\Omega \setminus S, k_{\Omega\setminus S})$ is not Gromov hyperbolic.*

Althought $\Omega \setminus S$ is generally not $\mathbb{C}$-convex Nikolov with Trybuła showed that a slight modification of the proof of Theorem 1.5 allows to carry on the cutting procedure, and the resulting domain is not Gromov hyperbolic as well.

Corollary 1.6. ([11]) *Let $\Omega \subset \mathbb{C}^n$ be a bounded $\mathbb{C}$-convex domain in $\mathbb{C}^n$ and $\mathcal{S} \neq \varnothing$ be a finite family of complex affine hyperplanes intersecting Ω. Then $(\Omega \setminus \bigcup \mathcal{S}, k_{\Omega\setminus\bigcup\mathcal{S}})$ is not Gromov hyperbolic.*

Since convex domains are $\mathbb{C}$-convex, Corollary 1.6 can be applied to convex domains also.

Corollary 1.7. ([11]) *Let $\Omega \subset \mathbb{C}^n$ be a bounded convex domain and $\mathcal{S} \neq \varnothing$ a finite family of complex affine hyperplanes intersecting Ω. Then $(\Omega \setminus \bigcup \mathcal{S}, k_{\Omega\setminus\bigcup\mathcal{S}})$ is not Gromov hyperbolic.*

It was also noticed in ([11]) that Theorem 1.5 can not be extended on weakly linearly convex domains.

Theorem 1.5 and its consequences hold due to a simple observation that the convex hull of any union of n orthogonal discs in Ω is entirely contained in Ω. However, this is not the only tool which was used there. Also important are the estimates of invariant metrics for $\mathbb{C}$-convex domains that are false fo general complex convex domains.

Since $\mathbb{C}$-convexity of smooth bounded domains can be expressed locally, one might ask if the last results have localization versions. It turned out that the answer to this question is positive. However, from one point of view such a situation is irrelevant because of the lack of pseudoconvexity and k-compactness.

Theorem 1.8. ([11]) *Let $\Omega \subset \mathbb{C}^n$ be a convex domain containing no complex line. Suppose there is a closed subset S' of Ω such that there are a complex affine line S, a point $\zeta \in \partial\Omega \cap S$ and $R > 0$ so that*

$$\Omega \cap \mathbb{B}(\zeta, R) \cap S' = \Omega \cap \mathbb{B}(\zeta, R) \cap S$$

and $\partial\Omega$ is strongly convex near ζ, i.e., $\partial\Omega$ contains no segment in some neighbourhood of ζ. Then $(\Omega \setminus S', k_{\Omega\setminus S'})$ is not Gromov hyperbolic.

2. Part II

2.1. Basic notation

- For $X \subset \mathbb{C}$ let $X_* = X \setminus \{0\}$.

- For $r > 0$ let $\mathbb{D}(r) = \{z \in \mathbb{C} : |z| < r\}$, $\mathbb{D}(1) = \mathbb{D}$.

- For $z \in \mathbb{C}^n$ let $\|z\|$ denote the standard Euclidean norm of z.

- For $\zeta \in \mathbb{C}^n$, $r > 0$ let $\mathbb{B}(\zeta, r) = \{z \in \mathbb{C}^n : \|z - \zeta\| < r\}$, $\mathbb{B}(0, 1) = \mathbb{B}_n$.

- Given an open set $\Omega \subset \mathbb{C}^n$, $z \in \Omega$, $X \in \mathbb{C}^n$ let

$$\text{dist}_\Omega(z) = \inf\{\|z - w\| : w \in \partial\Omega\},$$

$$\text{dist}_\Omega(z; X) = \inf\{\|z - w\| : w \in (z + \mathbb{C}X) \cap \partial\Omega\}.$$

2.2. Complex convexity

A domain is said to be:

- $\mathbb{C}$-*convex* if any non-empty intersection with a complex line is a simply connected domain.
- *linearly convex* (respectively: *weakly linearly convex*) if each point in its complement (respectively: boundary) belongs to a complex hyperplane disjoint from the domain.

The following implications hold:

$$\mathbb{C}\text{-convexity} \Rightarrow \text{linearly convexity} \Rightarrow \text{weak linearly convexity.}$$

Let us note that all three notions of complex convexity are different, and do weaker than the ordinary convexity. However, for bounded domains with $\mathcal{C}^1$-smooth boundaries they coincide. In the general case their place is between convexity and pseudoconvexity.

Althought, complex convexity is in general a weaker property than convexity, we observe that the following phenomenon remains true for them:

Lemma 2.1. ([11, 13]) *Suppose that a weakly linearly convex domain $G \subset \mathbb{C}^n$ contains the origin and balanced sets $G_1 \ldots, G_d$, $G_j \subset \{0\}^{n_1+\ldots+n_{j-1}} \times \mathbb{C}^{n_j} \times \{0\}^{n-(n_j+\ldots+n_d)}$, $n_j \in \mathbb{N}_*$, $n_1 + \ldots + n_d = n$. Then G contains the convex hull of $G_1, \ldots, G_d$.*

More properties of complex convex domains can be found in [1], [7].

2.3. The Kobayashi metric and distance

Given a domain $\Omega \subset \mathbb{C}^n$ the *Kobayashi pseudometric* is defined as follows

$$\kappa_\Omega(z; X) = \inf\big\{|\lambda| : \text{there exists } f \in H(\mathbb{D}, \Omega) \text{ so that } f(0) = z,\ \lambda f'(0) = X\big\}.$$

If $\alpha : [a, b] \to \Omega$ is a $\mathcal{C}^1$ piecewise curve we can define the *length of* α to be

$$l_{\kappa_\Omega}(\alpha) = \int_a^b \kappa_\Omega\big(\alpha(t); \alpha'(t)\big)\, dt.$$

One can then define the *Kobayashi pseudodistance* to be

$$k_\Omega(p,q) = \inf \big\{ l_{\kappa_\Omega}(\alpha) : \alpha : [0,1] \to \Omega \text{ is a piecewise } \mathcal{C}^1 \text{ smooth with } \alpha(0)=p,\ \alpha(1)=q \big\}.$$

An important property of the Kobayashi pseudodistance is the *holomorphic contractibility:* if $f : \Omega_1 \to \Omega_2$ is holomorphic, then

$$k_{\Omega_2}(f(z), f(w)) \leq k_{\Omega_1}(z,w).$$

We say (Ω, k_Ω) is *k-finitely compact* if every ball with finite radius is relatively compact in Ω

For further information on the Kobayashi metric/distance we refer the reader to [8], [9].

2.4. Gromov hyperbolicity

Let (X,d) be a metric space.

A curve $\alpha : [a,b] \to X$ is a *geodesic* if $d(\alpha(s),\alpha(t)) = |s-t|$ for all s, $t \in [a,b]$. A *geodesic triangle* in the metric space is a choice of three points in X and geodesic segments connecting these points. A geodesic triangle is said to be M*-thin* if any point on any side of the triangle is within the distance of M from the other two sides. A geodesic metric space is said to be (*Gromov*) *hyperbolic* if there exists $M > 0$ so that every geodesic triangle is M-thin.

By definition, an (A,B)*-quasi-geodesic* $\beta : [c,d] \to X$ satisfies the condition

$$A^{-1}|s-t| - B \leq d(\beta(s),\beta(t)) \leq A|s-t| + B$$

for all s, $t \in [c,d]$. An (A,B)*-quasi-geodesic triangle* consists of three (A,B)-quasi-geodesics (its sides). Such a triangle is said to be M*-thin* if each side lies in the M-neighbourhood of the union of the other two sides.

Proposition 2.2. ([3]) *If (X,d) is hyperbolic, then every (A,B)-quasi-geodesic triangle is M-thin for some $M > 0$ depending on A, B.*

The book [3] by Bridson and Haefliger is one of the standard references for Gromov hyperbolic metric spaces.

REFERENCES

[1] M. Andersson, M. Passare, R. Sigurdsson. Complex convexity and analytic functionals. Progress in Mathematics vol. **225**. Basel, Birkhäuser Verlag, 2004.

[2] Z. M. Balogh, M. Bonk. Gromov hyperbolicity and the Kobayashi metric on strictly pseudoconvex domains. *Comment. Math. Helv.* **75**, *3* (2000), 504–533.

[3] M. R. Bridson, A. Haefliger. Metric spaces of non-positive curvature. Grundlehren der Mathematischen Wissenschaften vol. **319**. Berlin, Springer-Verlag, 1999.

[4] H. Gaussier, H. Seshadri. On the Gromov hyperbolicity of convex domains in $\mathbb{C}^n$. *Comput. Methods Funct. Theory.* **18**, *4* (2018), 617–641.

[5] M. Gromov. Hyperbolic groups. Essays in group theory, 75–263, Math. Sci. Res. Inst. Publ., vol. **8**. New York, Springer, 1987.

[6] F. Haggui, A. Chrih. On the Gromov hyperbolicity of certains domains in $\mathbb{C}^n$. *C. R. Math. Acad. Sci. Paris* **355**, *5* (2017), 493–498.

[7] L. Hörmander. Notions of convexity. Progress in Mathematics vol. **127**. Boston, MA, Birkhauser Boston, Inc., 1994.

[8] M. Jarnicki, P. Pflug. Invariant distances and metrics in complex analysis. 2nd extended ed. De Gruyter Expositions in Mathematics vol. **9**. Berlin, Walter de Gruyter GmbH& Co. KG, 2013.

[9] S. Kobayashi. Hyperbolic complex spaces. Grundlehren der Mathematischen Wissenschaften vol. **318**. Berlin, Springer-Verlag, 1998.

[10] N. Nikolov, P. J. Thomas, M. Trybuła. Gromov (non)hyperbolicity of certain domains in $\mathbb{C}^2$. *Forum Math.* **28**, *4* (2016), 783–794.

[11] N. Nikolov, M. Trybuła. Gromov hyperbolicity of the Kobayashi metric on $\mathbb{C}$-convex domains. *J. Math. Anal. Appl.* **468**, *2* (2018), 1164–1178.

[12] A. Zimmer. Gromov hyperbolicity and the Kobayashi metric on convex domains. *Math Ann.* **365**, *3–4* (2016), 1425–1498.

[13] S. V. Znamenskiĭ, L. N. Znamenskaya. Spiral connectedness of the sections and projections of ℂ-convex sets. *Math. Notes* **59**, *3–4* (1996), 253–260.

Lyubomir Andreev
Faculty of Mathematics and Informatics
"St. Kl. Ohridski" University of Sofia
5, J. Bourchier Blvd
1164 Sofia, Bulgaria
e-mail: lvandreev@fmi.uni-sofia.bg

Nikolai Nikolov
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev Str., Bl. 8
1113 Sofia, Bulgaria
and
Faculty of Information Sciences
State University of Library Studies and Information Technologies
69A, Shipchenski prohod
1574 Sofia, Bulgaria
e-mail: nik@math.bas.bg

Maria Trybuła
Faculty of Mathematics and Informatics
Adam Mickiewicz University
87, Umultowska
61-614 Poznań, Poland
e-mail: maria.h.trybula@gmail.com