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ПЛИСКА
БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

PARAMETRIC APPROXIMATION OF PIECEWISE ANALYTIC FUNCTIONS

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The following estimate is obtained: if f is piecewise analytic in the interval $[-1, 1]$, then

$$\varepsilon_{n,n}(f)=O(\exp(-c(f)\sqrt{n}\ln n)),$$

where $\varepsilon_{n,n}(f)$ is the best uniform parametric approximation of order (n, n) and $c(f)>0$ is a constant depending only on f.

1. Parametric approximation of functions was introduced by Bl. Sendov [1]. Let H_n be the set of all algebraic polynomials of degree $\leqq n$ and let

$$\hat{H}_n=\{P \mid P\in H_n,\ P(-1)=-1,\ P(1)=1,\ P'(x)\geqq 0,\ x\in[-1, 1]\}.$$

For every function $f\in C[-1, 1]$ we set

$$\varepsilon_{m,n}(f)=\inf\{\|f(P(x))-Q(x)\|_{C[-1,1]} \mid P\in\hat{H}_m,\ Q\in H_n\},$$

where $\|f-g\|=\sup\{|f(x)-g(x)| \mid x\in[-1, 1]\}$.

It is easy to see that for every $f\in C[-1, 1]$ there exist two polynomials $P^*\in\hat{H}_m$, $Q^*\in H_n$, so that $\varepsilon_{m,n}(f)=\|f(P^*(x))-Q^*(x)\|$.

The polynomials in the couple (P^*, Q^*) are called best parametric approximation polynomials of order (m, n) of f. In the general case this couple is not unique.

In [2] Bl. Sendov proved the following

Theorem A. *Let the function $f\in C[-1, 1]$ be given by*

$$f(x)=\begin{cases} f_1(x) & \text{for } x\in[-1, 0],\\ f_2(x) & \text{for } x\in[0, 1],\ f_1(0)=f_2(0),\end{cases}$$

where f_i, $i=1, 2$, are analytic functions in the circle with radius $r>1$. Then we have

$$\varepsilon_{n,n}(f)=O(e^{-c(f)\sqrt{n}}),$$

where the constant $c(f)>0$ depends only on f.

In [3] J. Szabados generalized this result as follows:

Theorem B. *Let* $-1=\xi_0<\xi_1<\ldots<\xi_s=1$ *be a partition of the interval* $[-1, 1]$ *and let* $f\in C[-1, 1]$ *be such that in each interval* $[\xi_{i-1}, \xi_i]$, $i=1,\ldots,s$, f *is equal to the analytic function* f_i *in the circle* $c_i=\{z \mid (\xi_i-\xi_{i-1})r\geqq|2z-\xi_{i-1}-\xi_i|\}$, $r>1$, $i=1,\ldots,s$. *Then*

$$\varepsilon_{n,n}(f)=O(e^{-k(f)\sqrt{n}}),$$

where the constant $k(f)>0$ *depends only on* f.

More precisely J. Szabados proved that in this case we have

$$\varepsilon_{[\sqrt{n}],n}(f)=O(e^{-k(f)\sqrt{n}}).$$

Let us mention that there exists an analogue between the order of rational uniform approximations and the degree of parametric approximations of the class of piecewise analytic functions. P. Turan and P. Szüsz [4] have shown that for the best rationol uniform approximation of n-th degree of a function f piecewise analytic in $[-1, 1]$, one has

$$R_n(f)=O(e^{-d(f)\sqrt{n}}), \tag{1}$$

where $R_n(f)=\inf\{\|f-r\| \mid r\in R_n\}$ and R_n is the set of all rational functions of degree n, $d(f)>0$ is a constant, depending only on f. That the order in (1) is exact follows from Newman's result [5]: $e^{-c_1\sqrt{n}}\leqq R_n(|x|)\leqq e^{-c_1\sqrt{n}}$.

In connection with this the question arises whether the order $\exp(-c(f)\sqrt{n})$ in theorems A and B is exact. In this note we shall improve theorem B (and hence theorem A), showing that the order of the best parametric approximation of order (n, n) is better than the order of the rational uniform approximation of n-th degree for piecewise analytic functions: we shall show that for such functions we have

$$\varepsilon_{n,n}(f)=O(e^{-c(f)\sqrt{n\ln n}}). \tag{2}$$

The question whether the order in the estimate (2) is exact remains open.

2. We need some lemmas. Let $-1=\xi_0<\xi_1<\ldots<\xi_s=1$ be a partition of the interval $[-1, 1]$, $s>1$. Throughout the paper this partition remains fixed. Let us set $v=\min\{\xi_i-\xi_{i-1} \mid 1\leqq i\leqq s\}$. The following lemma is proved in [3]:

Lemma 1. *Let* $k>0$ *be an arbitrary natural number. There exists an algebraic polynomial* $P\in\hat{H}_r$, $r\leqq c_1(v, s)k$, *such that* $P(\xi_i)=\xi_i$, $i=0,\ldots,s$; $P^{(j)}(\xi_i)=0$, $j=1,\ldots,2k$, $i=0,\ldots,s$.

Lemma 2. *Let* $[a, b]\subset[c, d]$, $P\in H_n$. *Then*

$$\max_{x\in[c,d]}|P(x)|\leqq\left[\frac{2(d-c)}{b-a}\right]^n\max_{x\in[a,b]}|P(x)|.$$

Proof. It is known that for every polynomial $q\in H_n$ and every $x\in[-1, 1]$ we have $|q(x)|\leqq|T_n(x)|\,\|q\|$, where T_n is the Chebyshev polynomial of degree n: $T_n(x)=\cos(n\arccos x)$. For $x\notin[-1, 1]$ we have

(3) $$T_n(x) = (x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n\ 2 \leqq (2|x|)^n .$$

Applying the linear transformation $T:[a,\ b] \to [-1,\ 1]$ $(Tx=(2x-(a+b))/(b-a))$, we see that $T([c,d]) \subset [-(d-c)/(b-a),\ (d-c)/(b-a)]$, which together with (3) implies the lemma.

Lemma 3. *Let P be the polynomial from lemma 1. Then for $x \in [\xi_i-\tau,\ \xi_i+\tau] \cap [-1,\ 1]$, $\tau < v/4$ we have $P'(x) \leqq (c_2(v,\ s))^k \tau^{2k}$.*

Proof. We may assume that $x \in [\xi_i - \tau]$, $1 \leqq i \leqq s$. Using Markov's inequality we obtain for $x \in [-1,\ 1]$:

(4) $$P'(x) \leqq k^2 (c_1(v,\ s))^2\ P = (c_1(v,\ s)\,k)^2 .$$

By lemma 1 we have

(5) $$P'(x) = (x-\xi_i)^{2k} P^*(x),\quad P^* \in H_{r_1},\quad r_1 = c_1(v,\ s)\,k - 2k - 1.$$

From (4) we obtain $(x-\xi_i)^{2k}\ P^*(x) \leqq (c_1(v,\ s)\,k)^2$ and therofore

(6) $$\max\{P^*(x)\ \ \xi_{i-1} \leqq x \leqq (\xi_{i-1}+\xi_i)/2\}$$
$$\leqq 4^{2k}(c_1(v,\ s)\,k)^2/(\xi_i-\xi_{i-1})^{2k} \leqq 4^{2k}(c_1(v,\ s))^2 k^2/v^{2k}.$$

Using lemma 2 we obtain from (6)

(7) $$\max\{P^*(x)\ \ \xi_{i-1} \leqq x \leqq \xi_i\} \leqq 4^{c_1(v,\ s)\,k}(c_1(v,\ s)\,k)^2 (4/v)^{2k} .$$

Finally (5) and (7) give us for $x \in [\xi_i-\tau,\ \xi_i]$ $P'(x) \leqq \tau^{2k}(c_2(v,\ s))^k$ that proves the lemma.

Lemma 4. *For every natural number $n > 1$ and every $\delta \in [8 \ln n/n,\ 1)$ there exists an algebraic polynomial $\sigma_{n,\delta} \in H_{2n+1}$ such that*

$$|\sigma_{n,\delta}(x)| \leqq \frac{1}{2} e^{-n\delta 4} \ \textit{for}\ x \in [-1,\ -\delta],$$

$$1 - \sigma_{n,\delta}(x) \leqq \frac{1}{2} e^{-n\delta 4} \ \textit{for}\ x \in [\delta,\ 1],$$

$$0 \leqq \sigma_{n,\delta}(x) \leqq 1 \ \textit{for}\ |x| \leqq \delta .$$

Proof. The lemma follows from the results in (6), but for completeness we shall give the full proof. Let $T_n(x) = \cos(n \arccos x)$ be the Chebyshev polynomial of degree n. Then $P_n(x) = T_n((2x^2-(1+\delta^2))/(1-\delta^2))$ have the following properties: $|P_n(x)| \leqq 1$ for $\delta \leqq x \leqq 1$,

$$P_n(x) = (-1)^n\{(\sqrt{1-x^2}+\sqrt{\delta^2-x^2})^{2n} + (\sqrt{1-x^2}-\sqrt{\delta^2-x^2})^{2n}\}/(2(1-\delta^2)^n)$$

for $|x| \leqq \delta$.

Consequently for even n the polynomial $P_n(x)$ is even, $P_n(x) \geqq 1$ for $|x| \leqq \delta$ and $P_n(x)$ is monotone decreasing in $[0,\ \delta]$; $P^n(x)$ is even for odd n, $P_n(x) \leqq -1$ for $|x| \leqq \delta$ and $P_n(x)$ is monotone increasing in $[0,\ \delta]$.

Since $\sqrt{1-x^2} \geqq 1-x^2$ for $|x| \leqq 1$, we have for $|x| \leqq \delta/2$

(8) $$|P_n(x)| \geqq (1-x^2+\delta\sqrt{3}/2)^{2n}/2 \geqq (1+\delta/2)^{2n}/2 .$$

Let us denote

$$\sigma_{n,\delta}(x)=\{\int_{-1}^{x}P_n(t)\,dt-\int_{-1}^{-\delta}P_n(t)\,dt\}/(\int_{-\delta}^{\delta}P_n(t)\,dt).$$

We have from (8)

$$\int_{-\delta}^{\delta}P_n(t)\,dt\geq\delta(1+\delta/2)^{2n}/2\geq\delta e^{n\delta/2}/2. \tag{9}$$

Then, since $\delta\geq 8\ln n/n$ and $n\geq 2$, we obtain from (9):
For $-1\leqq x\leqq-\delta$:

$$\sigma_{n,\delta}(x)\leqq 2\,(\int_{-\delta}^{\delta}P_n(t)\,dt)^{-1}\leqq 4\,e^{-n\delta/2}/\delta$$

$$\leqq\frac{n}{2\ln n}e^{-n\delta/4}e^{-2\ln n}\leqq\frac{1}{2n\ln n}e^{-n\delta/4}\leqq\frac{1}{2}e^{-n\delta/4},$$

for $|x|\leqq\delta$:

$$0\leqq\sigma_{n,\delta}(x)=(\int_{-\delta}^{x}P_n(t)\,dt)/(\int_{-\delta}^{\delta}P_n(t)\,dt)\leqq 1$$

and for $\delta\leqq x\leqq 1$:

$$1-\sigma_{n,\delta}(x)=\int_{\delta}^{x}P_n(t)\,dt\,/(\int_{-\delta}^{\delta}P_n(t)\,dt)\leqq(x-\delta)\,e^{-n\delta/4}/2\leqq e^{-n\delta/4}/2.$$

Since $\sigma_{n,\delta}\in H_{2n+1}$, the lemma is proved.

3. Theorem 1. *Let $f\in C[-1,1]$ be such that there exists a partition $-1=\xi_0<\xi_1<\ldots<\xi_s=1$ with $f(x)=f_i(x)$ for $x\in[\xi_{i-1},\xi_i]$, $i=1,\ldots,s$ ($f_i(\xi_i)=f_{i+1}(\xi_i)$, $i=1,\ldots,s-1$), where f_i is analytic in $C_i=\{z:|z-\eta_i|\leqq(\xi_i-\xi_{i-1})r,\ \eta_i=(\xi_{i-1}+\xi_i)/2,\ r>1,\ i=1,\ldots,s\}$. Then*

$$\varepsilon_{m,n}(f)=O(\exp(-c_1(f)\sqrt{n\ln n}))\ \textit{for}\ m\geq c_2(f)\sqrt{n/\ln n}.$$

In particular $\varepsilon_{n,n}(f)=O(\exp(-c_3(f)\sqrt{n\ln n}))$.

Proof. We may assume that $-1=\xi_0<\xi_1<\ldots<\xi_s=1$ is the partition of 2. From the condition of the theorem it follows that there exists a number τ_0, $\tau_0=\tau_0(v,r)$, $v=\min\{(\xi_i-\xi_{i-1}): 1\leqq i\leqq s\}$, such that f_i is analytic in the closed interval $\Delta_i=[\xi_{i-1}-\tau,\ \xi_i+\tau]$, $i=1,\ldots,s$, $\tau\leqq\tau_0$. We may suppose also that $\tau<v/4$. Therefore there exist algebraic polynomials Q_i, $Q_i\in H_m$, $i=1,\ldots,s$, and q, $0<q<1$, such that

$$\max\{|f_i(x)-Q_i(x)|: x\in\Delta_i\}\leqq cq^m,\quad c=\mathrm{const}=c(f). \tag{10}$$

Using the notations of 2, let us consider the algebraic polynomial Q of a degree at most $2[n/4]+c_1(v,s)\,km$

$$Q(x)=Q_1(P(x))+\sum_{i=1}^{s-1}\sigma_{[n/4],\tau}(x-\xi_i)\{Q_{i+1}(P(x))-Q_i(P(x))\},$$

where P is the polynomial of lemma 1. Let us estimate $|f(P(x))-Q(x)|$

a) If $x \in [\xi_{i_0}-\tau, \xi_{i_0}+\tau]$ for some i_0, $0 \leqq i_0 \leqq s$, then

$$(11)\qquad |f(P(x))-Q(x)|$$

$$\leqq |Q_1(P(x))+\sum_{i=1}^{i_0-1}\sigma_{[n/4],\tau}(x-\xi_i)\{Q_{i+1}(P(x))-Q_i(P(x))\}-Q_{i_0}(P(x))|$$

$$+|f(P(x))-(1-\alpha)Q_{i_0}(P(x))-\alpha Q_{i_0+1}(P(x))|$$

$$+|\sum_{i=i_0+1}^{s-1}\sigma_{[n/4],\tau}(x-\xi_i)\{Q_{i+1}(P(x))-Q_i(P(x))\}|,$$

where $\alpha=\sigma_{[n/4],\tau}(x-\xi_{i_0})$ and therefore $0\leqq\alpha\leqq 1$ (see lemma 4). Using lemma 2 we obtain for every i, $i=1,\dots,s$:

$$(12)\qquad \max_{x\in[-1,1]}|Q_i(x)|\leqq(4/v)^m\max_{x\in[\xi_{i-1},\xi_i]}|Q_i(x)|\leqq 2\|f\|(4/v)^m.$$

Using (11), (12) and lemma 4 we obtain

$$(13)\qquad |f(P(x))-Q(x)|\leqq 2s\|f\|(4/v)^m e^{-[n/4]\tau/4}$$

$$+(1-\alpha)|f(P(x))-Q_{i_0}(P(x))|+\alpha|f(P(x))-Q_{i_0+1}(P(x))|,\quad 0\leqq\alpha\leqq 1.$$

We have $x\in[\xi_{i_0}-\tau, \xi_{i_0}]$ or $x\in[\xi_{i_0}, \xi_{i_0}+\tau]$. Consider the first case, the second one may be treated in the same way. In this case we have $P(x) \in [\xi_{i_0-1}, \xi_{i_0}]$, therefore $f(P(x))=f_{i_0}(P(x))$ and consequently obtain from (10)

$$(14)\qquad |f(P(x))-Q_{i_0}(P(x))|\leqq cq^m.$$

In order to estimate $|f(P(x))-Q_{i_0+1}(P(x))|$ we mention first that since f_i are analytic in Δ_i, $i=1,\dots,s$, then f_i are Lipschitz functions, e. g. $|f_i(x)-f_i(y)|\leqq K_f|x-y|$ for $x,y\in\Delta_i$ and for some K_f, independent of i, $i=1,\dots,s$. Since $f_{i_0}(\xi_{i_0})=f_{i_0+1}(\xi_{i_0})$, $\xi_{i_0}=P(\xi_{i_0})$, we have

$$|f(P(x))-Q_{i_0+1}(P(x))|=|f_{i_0}(P(x))-Q_{i_0+1}(P(x))|$$

$$\leqq|f_{i_0+1}(P(x))-Q_{i_0+1}(P(x))|+|f_{i_0+1}(P(x))-f_{i_0+1}(\xi_{i_0})|+|f_{i_0+1}(\xi_{i_0})-f_{i_0}(P(x))|$$

$$\leqq cq^m+K_f|P(x)-P(\xi_{i_0})|+|f_{i_0}(\xi_{i_0})-f_{i_0}(P(x))|\leqq cq^m+2K_f\tau\max_{x\in[\xi_{i_0}-\tau,\xi_{i_0}]}|P'(x)|.$$

Using lemma 3 we obtain

$$(15)\qquad |f(P(x))-Q_{i_0+1}(P(x))|\leqq cq^m+2K_f\tau^{2k+1}(c_2(v,s))^k.$$

From (13)—(15) it follows for $x\in[\xi_{i_0}-\tau, \xi_{i_0}+\tau]$

$$(16)\qquad |f(P(x))-Q(x)|\leqq 2s\|f\|(4/v)^m e^{-[n/4]\tau/4}+cq^m+2K_f\tau^{2k+1}(c_2(v,s))^k.$$

The case when $x\in[\xi_{i_0-1}+\tau, \xi_{i_0}-\tau]$ is not so difficult:

(17) $$f(P(x))-Q(x)$$

$$\leqq Q_1(P(x))+\sum_{i=1}^{i_0-1}\sigma_{[n/4],\tau}(x-\xi_i)\{Q_{i+1}(P(x))-Q_i(P(x))\}-Q_{i_0}(P(x))$$

$$+f_{i_0}(P(x))-Q_{i_0}(P(x))+\sum_{i=i_0}^{s-1}\sigma_{[n/4],\tau}(x-\xi_i)\{Q_{i+1}(P(x))-Q_i(P(x))\}$$

$$\leqq 2s\ f\ (4/v)^m e^{-[n/4]\tau/4}+cq^m.$$

Finally from (16) and (17) we obtain

(18) $$f(P(x))-Q(x)\leqq 2s\ f\ (4/v)^m e^{-[n/4]\tau/4}+cq^m+2K_f\tau^{2k+1}(c_2(v,\ s))^k.$$

Moreover Q is an algebraic polynomial of degree at most $2[n/4]+c_1(v,\ s)km$ and $P\in\hat{H}_t$, $t=c_1(v,\ s)k$.

Let us set $m=[\frac{1}{2}\sqrt{n\ln n}]$, $k=[c_1^{-1}(v,\ s)\sqrt{n/\ln n}]$, $\tau=32(1+\ln\frac{4}{v})\times\sqrt{\ln n/n}$.

Obviously $\tau<v/4$ and $\tau\leqq\tau_0$ for sufficiently large n. Moreover, $Q\in H_n$ and $P\in\hat{H}_t$, $t\leqq\sqrt{n}\ln n$. We have for this choice of m, k and τ

(19) $$2s\ f\ (4/v)^m e^{-[n/4]\tau/4}$$

$$=O((\frac{4}{v})^{\frac{1}{2}\sqrt{n\ln n}}(\frac{v}{4})^{\sqrt{n\ln n}}e^{-\sqrt{n\ln n}})=O(e^{-\sqrt{n\ln n}}),$$

$$cq^m=O(e^{-c_4(f)\sqrt{n\ln n}}),$$

$$2K_f\tau^{2k+1}(c_2(v,\ s))^k=O((c_5(f)\sqrt{\ln n/n})^{[\sqrt{n\ln n}/c_1(v,\ s)]})=O(e^{-c_5(f)\sqrt{n\ln n}}),$$

since if $(c_5(f)\sqrt{\ln n/n})^{[\sqrt{n/\ln n}/c_1(v,\ s)]}=e^{-\alpha}$, then

$$\alpha=[\sqrt{n/\ln n}/c_1(v,\ s)]\ (\frac{1}{2}\ln n-\ln(c_5(f)\sqrt{\ln n})$$

$$\geqq c_6(f)\sqrt{n\ln n}-c_7(f),\ c_6(f)>0.$$

Therefore (18) and (19) give $f(P(x))-Q(x)=O(e^{-c_8(f)\sqrt{n\ln n}})$, where $P\in\hat{H}_t$, $t\leqq\sqrt{n/\ln n}$, $Q\in H_n$. This proves theorem 1.

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