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APPROXIMATION OF PLANE COMPACTA BY MEANS OF POLYNOMIAL CURVES

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The class G_α of all compact and connected sets in the plane with a given metric dimension α, $1<\alpha<2$, is considered. It is shown that the estimate $o(n^{(\delta-1)/\alpha})$ for the best approximation by means of polynomial curves (δ is an arbitrary, strictly positive number) cannot be improved in the whole class G_α.

We shall consider a problem of approximating in the metric space G of all plane compacta with the Hausdorff distance

$$r(F_1, F_2)=\max\{\max_{a\in F_1}\min_{b\in F_2}\rho(a, b),\ \max_{b\in F_2}\min_{a\in F_1}\rho(a, b)\},$$

$F_1\in G$, $F_2\in G$, where $\rho(a, b)$ is the distance between the points a and b.

An element γ_n of G is said to be a polynomial curve of order n if there exist two algebraic polynomials $P(t)$ and $Q(t)$ of degree at most n, such that $\gamma_n=\{(x, y): x=P(t),\ y=Q(t),\ -1\leqq t\leqq 1\}$. Let us denote by Γ_n the set of all polynomial curves of order n. Γ_n is a subset of G.

The best approximation $E_{n,\Gamma}(F)$ of $F\in G$ by means of elements of Γ_n is defined as

$$E_{n,\Gamma}(F)=\inf_{\gamma_n\in\Gamma_n} r(F, \gamma_n).$$

If F is connected then $E_{n,\Gamma}(F)$ tends to zero and may be estimated using the metric dimension of F.

The definition of the metric dimension [1] needs some additional notions:

A system of sets $\{U_\nu\}$ is said to be an ε-covering of F if $F\subseteq\bigcup_\nu U_\nu$ and the diameter of each U_ν is at most 2ε; a subset F^* of F is said to be ε-distinguishable if $\rho(a, b)\geqq\varepsilon$ for each $a\in F^*$, $b\in F^*$, $a\neq b$.

Let us denote by $N_\varepsilon(F)$ the minimal number of sets in the ε-coverings of F and by $M_\varepsilon(F)$ — the maximal number of points in the ε-distinguishable subsets of F.

Then the upper and lower metric dimension of F are defined respectively as

$$\overline{\mathrm{dm}}(F)=\limsup_{\varepsilon\to 0}\{\mathrm{ld}\, N_\varepsilon(F)/\mathrm{ld}\, 1/\varepsilon\},\quad \underline{\mathrm{dm}}(F)=\liminf_{\varepsilon\to 0}\{\mathrm{ld}\, N_\varepsilon(F)/\mathrm{ld}\, 1/\varepsilon\}$$

($\log_2(\cdot)=\mathrm{ld}(\cdot)$) and the following equalities hold:

$$\overline{\mathrm{dm}}(F)=\limsup_{\varepsilon\to 0}\{\mathrm{ld}\, M_\varepsilon(F)/\mathrm{ld}\, 1/\varepsilon\},\quad \underline{\mathrm{dm}}(F)=\liminf_{\varepsilon\to 0}\{\mathrm{ld}\, M_\varepsilon(F)/\mathrm{ld}\, 1/\varepsilon\}.$$

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If $\overline{\mathrm{dm}}(F)=\mathrm{dm}(F)=\underline{\mathrm{dm}}(F)$ then $\mathrm{dm}(F)$ is said to be the metric dimension of F.

Let G_α be the class of all compact and connected sets in the plane with a given metric dimension α, $1<\alpha<2$. We shall consider the best approximation in G_α by means of polynomial curves. It is shown in [2] that if $F \in G_\alpha$ then for each $\delta>0$ holds $E_{n,\Gamma}(F)=O(n^{-1/(\alpha+\delta)})$. An equivalent formulation of this estimate is easy to be obtained:

$$E_{n,\Gamma}(F)=o((n^{\delta-1})^{1/\alpha})$$

for each $\delta>0$.

A slight modification of the proof of theorem 2 from [2] gives the following

Theorem. *Let α be a number, $1<\alpha<2$, and $\psi(x)$ be a function such that $\psi(x)=o(x^\delta)$ when $x\to\infty$ for each $\delta>0$. Then there exists a set $F^*_{\alpha,\psi}\in G_\alpha$ with the property*

$$\limsup_{n\to\infty}\{E_{n,\Gamma}(F^*_{\alpha,\psi})/(\psi(n)/n)^{1/\alpha}\}=\infty.$$

Proof. The set $F^*_{\alpha,\psi}$ will be obtained as the intersection of a family $\{F_k\}_{k=k_0}^{\infty}$, where F_k is a connected compact set and $F_{k+1}\subset F_k$.

Let us define the function $\varphi(x)$ by $\varphi(x)=\mathrm{ld}\,\psi(2^x)$.

We shall prove that $\varphi(x)=o(x)$ when $x\to\infty$. Indeed let us suppose, on the contrary, that there exist a sequence $\{x_i\}_{i=1}^{\infty}$ and a number $\eta>0$, such that $x_i\to\infty$ and $\varphi(x_i)>\eta\,.\,x_i$. Then we have $\mathrm{ld}\,\psi(2^{x_i})>\eta\,.\,x_i$. Let $x_i=\mathrm{ld}\,y_i$, so $\mathrm{ld}\,\psi(y_i)>\eta\,.\,\mathrm{ld}\,y_i=\mathrm{ld}\,(y_i)^\eta$. We get $\psi(y_i)>y_i^\eta$, where $y_i\to\infty$, $\eta>0$, which contradicts with $\psi(x)=o(x^\eta)$.

Now determine an integer $s>1$ for which $s+3\leqq\alpha s$ and denote $\gamma_k=s(\alpha+\varphi(2s(k+1))-\varphi(2sk))$. Let k_0 be such that for $k>k_0$ holds

$$\alpha+\varphi(2sk)/k<2,\ \alpha+\varphi(2s(k+1))-\varphi(2sk)<2,\ \gamma_k\leqq 2s-1.$$

The integer k_0 exists because of $1<\alpha<2$, $\varphi(2sx)=o(x)$ and we may suppose, without loss of generality, that $\varphi(x)$ is convex upwards and an increasing function.

We shall construct the sets F_k $(k=k_0, k_0+1,\ldots)$ by an iterative procedure. Let S_k be a square with side ε_k (Fig. 1) and let us have a partition of S_k consisting of 2^{2s} squares with side $\varepsilon_k\,.\,2^{-s}$. If $k>k_0$ we take $[2^{\gamma_k}]$ of them in such a way that their union is a connected subset of the set E (hatched on Fig. 1) and contains the contour of E. This may be done because of the inequalities $8\,.\,2^s\leqq 2^{\gamma_s}\leqq 2^{2s-1}$.

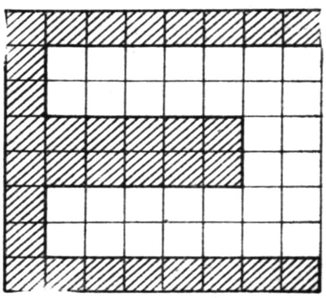

Fig. 1

Now let us denote $\varepsilon_k=2^{-sk}$, $\delta_k=\varphi(2sk)/k$, $N_k=2^{sk(\alpha+\delta_k)}$ and let F_{k_0} be a connected set consisting of N_{k_0} squares with side ε_{k_0}. If F_k is determined as the union of N_k squares with side ε_k, then we obtain F_{k+1} taking in each of them, after the manner above mentioned, $[2^{\gamma_k}]$ squares with side $\varepsilon_k\,.\,2^{-s}$. Thus F_{k+1} is connected and consists of $N_k\,.\,2^{\gamma_k}=N_{k+1}$ squares with side ε_{k+1}.

The set $F^*_{\alpha,\psi}=\bigcap_{k=k_0}^{\infty} F_k$ is a connected compactum and has an ε_k-distinguishable subset consisting of N_k points (the apexes of the squares of F_k). We shall prove that $F^*_{\alpha,\psi}\in G_\alpha$.

Given a sufficiently small $\varepsilon>0$, let $k>k_0$ be such that $\varepsilon_{k+1}\leqq\varepsilon\leqq\varepsilon_k$. Then

$$\mathrm{ld}\, N_\varepsilon(F^*_{\alpha,\psi})/\mathrm{ld}\, 1/\varepsilon\leqq \mathrm{ld}\, N_{k+1}/\mathrm{ld}\, 1/\varepsilon=s(k+1)\,(\alpha+\delta_{k+1})/(sk),$$

$$\mathrm{ld}\, M_\varepsilon(F^*_{\alpha,\psi})/\mathrm{ld}\, 1/\varepsilon\geqq \mathrm{ld}\, N_k/\mathrm{ld}\, 1/\varepsilon=sk(\alpha+\delta_k)/(s(k+1)).$$

If $\varepsilon\to 0$ then $k\to\infty$ and $\delta_k\to 0$, thus $\overline{\mathrm{dm}}(F^*_{\alpha,\psi})\leqq\alpha$, $\underline{\mathrm{dm}}(F^*_{\alpha,\psi})\geqq\alpha$ and we get $\mathrm{dm}(F^*_{\alpha,\psi})=\alpha$.

It is easy to see (taking account of Fig. 1) that if γ_m is a polynomial curve of order m,

$$\gamma_m=\{(x,y): x=P(t),\ y=Q(t),\ -1\leqq t\leqq 1\}$$

and $r(F^*_{\alpha,\psi},\gamma_m)<\varepsilon_k/8$, then $P'(t)$ must have at least N_k zeros, one for each square of F_k. Thus $m>N_k$.

Now let $n=N_k$. We have $E_{n,\Gamma}(F^*_{\alpha,\psi})\geqq\varepsilon_k/8$, so

$$E_{n,\Gamma}(F^*_{\alpha,\psi})/(\psi(n)/n)^{1/\alpha}\geqq\frac{1}{8}\cdot 2^{-sk}/\{\psi(N_k)/N_k\}^{1/\alpha}$$

$$=\frac{1}{8}\cdot 2^{-sk}.\{2^{sk(\alpha+\delta_k)}/\psi(N_k)\}^{1/\alpha}=\frac{1}{8}\cdot 2^{sk\delta_k/\alpha}/(\psi(N_k))^{1/\alpha}$$

$$=\frac{1}{8}\,(2^{s\cdot\varphi(2sk)}/2^{\varphi(\log_2 N_k)})^{1/\alpha}=\frac{1}{8}(2^{s\cdot\varphi(2sk)}/2^{\varphi(sk(\alpha+\delta_k))})^{1/\alpha}\geqq\frac{1}{8}\cdot 2^{(s-1)\varphi(2sk)/\alpha}.$$

If $k\to\infty$ then $2^{(s-1)\varphi(2sk)/\alpha}\to\infty$. The theorem is proved.

A similar theorem holds in the case of the approximation by means of trigonometrical polynomial curves.

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