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A NOTE ON THE "CONSTRUCTING" OF NONSTATIONARY METHODS FOR SOLVING NONLINEAR EQUATIONS WITH RAISED SPEED OF CONVERGENCE.*

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Abstract. In this paper we give methodological survey of "contemporary methods" for solving the nonlinear equation $f(x) = 0$. The reason for this review is that many authors in present days rediscovered such classical methods. Here we develop one methodological schema for constructing nonstationary methods with a preliminary chosen speed of convergence.

1. Introduction. During the last several years, numerous papers [1], [3], [6]–[10], [12]–[28], [31]–[39], [41]–[44], [46]–[52], [54]–[79], [81], [86]–[88], [90]–[91], [94]–[98] devoted to iterative methods for solving nonlinear equations have appeared in various journals – *Appl. Math. Comput.*, *Nonlinear Analysis Forum*,

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This collection of papers is intended to give a survey on many hybrid combination methods, multi-point methods generated by composition, and properties of these methods (the cost of arithmetic operations, computational efficiency and order of convergence).

We will point out that the methodology of construction of these or similar iteration methods (with recursive generation) and the technique for receiving precise estimations of the order of convergence are well known and given in the literature – [93], [89], [80], [2], [45], [92], [85], [84], [83] and [49].

This issue was studied extensively by Prof. M. Petkovic and his coauthors.

In one paper L. Petkovic and M. Petkovic [82] made a serious analysis of the "scientific achievements" of one small part of the aforecited publications of this subject.

Obviously the rediscovering of classical and newer methods of solving nonlinear equations continues with a nondecreasing rate nowadays.

We will follow the idea of constructing iteration methods "with raised speed of convergence".

Newton's method for the calculation of a simple root ξ of nonlinear equation $f(x) = 0$ is probably the most widely used iterative method defined by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$n = 0, 1, 2, \ldots .$$

An interesting approach in constructing iteration methods is based on quadrature rules.

Several third-order methods based on quadratures are given in the literature.

A third-order variant of Newton's method appeared in Weerakon and Fernando [95] where rectangular trapezoidal approximations to the integral in Newton's theorem

$$f(x) = f(x_n) + \int_{x_n}^{x} f'(t)dt \tag{1}$$

were considered to rederive Newton's method and to obtain the cubical method

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(y_n)}, \qquad y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \tag{2}$$

$$n = 0, 1, 2, \dots .$$

The iteration function (IF)

$$\varphi = x - \frac{2f(x)}{f'(x) + f'(x - u(x))}, \qquad u(x) = \frac{f(x)}{f'(x)}$$

is Traub's method presented in [93].

Frontini and Sormany [33] considered the midpoint rule for the integral (1) to obtain the third-order method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'\left(\dfrac{1}{2}(x_n + y_n)\right)} \tag{3}$$

$$n = 0, 1, 2, \dots$$

It should be mentioned that the method (3) has be derived by Homeier [46] independently

$$x_{n+1} = x_n - \frac{f(x_n)}{f'\left(x_n - \dfrac{1}{2}u(x_n)\right)} \tag{4}$$

$$n = 0, 1, 2, \dots .$$

The method (4) follows from Traub [93].

The third-order iteration method

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left(\frac{1}{f'(x_n)} + \frac{1}{f'(y_n)}\right) \tag{5}$$

$$n = 0, 1, 2, \dots$$

was proposed by Osban [81] and Homeier [47].

The IF

$$\varphi = x - \frac{f(x)}{2}\left(\frac{1}{f'(x)} + \frac{1}{f'(x - u(x))}\right)$$

is Traub's method presented in [93].

We observe that the method (3) can be obtained using the midpoint value $f'(\frac{1}{2}(x_n + y_n))$ instead of the arithmetic mean of $f'(x_n)$ and $f'(y_n)$ in the method (2) (see, Kou, Li and Wang [62]).

Evidently, a modified method based on Simpson's rule will be

$$x_{n+1} = x_n - \frac{bf(x_n)}{f'(x_n) + (b-2)f'\left(\frac{1}{2}(x_n+y_n)\right) + f'(y_n)}$$

$$n = 0, 1, 2, \ldots,$$

where b is a free parameter.

Recently, Neta [75] (see, also Chun and Neta [26]) used the method of undetermined coefficients to obtain a new efficient modifications.

Kou and Li [57] considered the following modification of Jarratt's method

$$\begin{aligned} x_{n+1} &= z_n - \frac{f(z_n)}{\frac{3}{2}J_f(x_n)f'(v_n) + \left(1 - \frac{3}{2}J_f(x_n)\right)f'(x_n)}, \\ v_n &= x_n - \frac{2}{3}\frac{f(x_n)}{f'(x_n)}, \\ z_n &= x_n - J_f(x_n)\frac{f(x_n)}{f'(x_n)}, \\ J_f(x_n) &= \frac{3f'(v_n) + f'(x_n)}{6f'(v_n) - 2f'(x_n)}, \end{aligned} \tag{6}$$

$$n = 0, 1, 2, \ldots.$$

The method (6) is of order six.

Another sixth-order improved Jarratt's method is given by Chun [22].

Kou, Li and Wang [62] obtained the following method

$$x_{n+1} = x_n - \frac{f(x_n)}{2}\left(\frac{1}{f'(x_n)} + \frac{1}{f'\left(\frac{1}{2}(x_n+y_n)\right) - f'(x_n)}\right). \tag{7}$$

$$n = 0, 1, 2, \ldots.$$

In [94], Ujevic obtained the following iteration

$$\begin{aligned} x_{n+1} &= x_n + 4(z_n - x_n)\frac{f(x_n)}{3f(x_n) - 2f(z_n)}, \\ z_n &= x_n - \alpha\frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{8}$$

$$n = 0, 1, 2, \ldots$$

where $0 < \alpha \leq 1$.

Following Traub's terminology (see, also L. Petkovic and M. Petkovic [82]), the combined methods of the type (8) are often called multi-point iteration methods generated by composition.

In [73] Nedzhibov, Hassanov and Petkov consider the following family

$$\begin{aligned} x_{n+1} &= x_n - u_n\left(1 + \frac{f(x_n - u_n)}{f(x_n) - 2\lambda f(x_n - u_n)}\right), \\ u_n &= \frac{f(x_n)}{f'(x_n)}, \end{aligned} \tag{9}$$

$$n = 0, 1, 2, \ldots,$$

where λ is an arbitrary real parameter.

In particular cases we get some well known formulas:

For $\lambda = 0$ we get the iteration

$$x_{n+1} = x_n - u_n - \frac{f(x_n - u_n)}{f'(x_n)} \tag{10}$$

$$n = 0, 1, 2, \ldots$$

studied by Traub.

For $\lambda = \frac{1}{2}$ we get the Newton-secant method

$$x_{n+1} = x_n - \frac{f^2(x_n)}{f'(x_n)(f(x_n) - f(x_n - u_n))} \tag{11}$$

$$n = 0, 1, 2, \ldots.$$

For $\lambda = 1$ we get

$$x_{n+1} = x_n - u_n\left(\frac{f(x_n) - f(x_n - u_n)}{f(x_n) - 2f(x_n - u_n)}\right). \tag{12}$$

$$n = 0, 1, 2, \ldots.$$

This formula is known as Ostrowski method.

In order to solve the nonlinear equation $f(x) = 0$ Gutierrez and Hernandez [40] considered the iterative formula

$$x_{n+1} = x_n - u_n\left(1 + \frac{1}{s(x_n) - \alpha}\right),$$

$$n = 0, 1, 2, \dots,$$

where α is a real parameter and

$$s(x_n) = \frac{2f'^2(x_n)}{f(x_n)f''(x_n)}.$$

The family converges cubically and includes, for example, Halley's method ($\alpha = 1$) and Chebyshev-Euler's method ($\alpha = 0$).

If $|\alpha|$ is very large, then the presented method behaves as Newton's method.

Kou [56] obtained the following fifth-order modifications of Newton's method:

$$\left| \begin{array}{lcl} z_{n+1} & = & x_n - \dfrac{f(x_n)}{f'\left(\dfrac{1}{2}(x_n + y_n)\right)}, \\ x_{n+1} & = & z_{n+1} - \dfrac{f(z_{n+1})}{f'(x_n)} \cdot \dfrac{f'\left(\dfrac{1}{2}(x_n + y_n)\right)}{3f'\left(\dfrac{1}{2}(x_n + y_n)\right) - 2f'(x_n)}, \end{array} \right.$$

$$n = 0, 1, 2, \dots,$$

$$\left| \begin{array}{lcl} z_{n+1} & = & x_n - \dfrac{f(x_n)}{2}.\left(\dfrac{1}{f'(x_n)} + \dfrac{1}{2f'\left(\dfrac{1}{2}(x_n + y_n)\right) - f'(x_n)}\right), \\ x_{n+1} & = & z_{n+1} - \dfrac{f(z_{n+1})}{f'(x_n)}.\dfrac{f'\left(\dfrac{1}{2}(x_n + y_n)\right)}{3f'\left(\dfrac{1}{2}(x_n + y_n)\right) - 2f'(x_n)}, \end{array} \right.$$

$$n = 0, 1, 2, \dots,$$

$$\left| \begin{array}{lcl} z_{n+1} & = & x_n - \dfrac{f(x_n)}{f'\left(\dfrac{1}{2}(x_n + y_n)\right)}, \\ x_{n+1} & = & z_{n+1} - f(z_{n+1}).\dfrac{2f'\left(\dfrac{1}{2}(x_n + y_n)\right) - f'(x_n)}{3f'^2\left(\dfrac{1}{2}(x_n + y_n)\right) - f'^2(x_n)}, \end{array} \right.$$

$$n = 0, 1, 2, \ldots,$$

$$\left|\begin{array}{lcl} z_{n+1} & = & x_n - \dfrac{f(x_n)}{2}.\left(\dfrac{1}{f'(x_n)} + \dfrac{1}{2f'\left(\dfrac{1}{2}(x_n+y_n)\right) - f'(x_n)}\right), \\ x_{n+1} & = & z_{n+1} - f(z_{n+1}).\dfrac{2f'\left(\dfrac{1}{2}(x_n+y_n)\right) - f'(x_n)}{3f'^2\left(\dfrac{1}{2}(x_n+y_n)\right) - f'^2(x_n)}, \end{array}\right.$$

$$n = 0, 1, 2, \ldots$$

and new methods with order of convergence 6:

$$\left|\begin{array}{lcl} z_{n+1} & = & x_n - \dfrac{2f(x_n)}{f'(y_n) + f'(x_n)}, \\ x_{n+1} & = & z_{n+1} - \dfrac{f(z_{n+1})}{f'(x_n)} \cdot \dfrac{f'(y_n) + f'(x_n)}{3f'(y_n) - f'(x_n)}, \end{array}\right.$$

$$n = 0, 1, 2, \ldots,$$

$$\left|\begin{array}{lcl} z_{n+1} & = & x_n - \dfrac{f(x_n)}{2}\left(\dfrac{1}{f'(x_n)} + \dfrac{1}{f'(y_n)}\right), \\ x_{n+1} & = & z_{n+1} - \dfrac{f(z_{n+1})}{f'(x_n)} \cdot \dfrac{f'(y_n) + f'(x_n)}{3f'(y_n) - f'(x_n)}, \end{array}\right.$$

$$n = 0, 1, 2, \ldots,$$

$$\left|\begin{array}{lcl} z_{n+1} & = & x_n - \dfrac{2f(x_n)}{f'(y_n) + f'(x_n)}, \\ x_{n+1} & = & z_{n+1} - f(z_{n+1}) \cdot \dfrac{2f'(y_n)}{f'^2(y_n) + 2f'(y_n)f'(x_n) - f'^2(x_n)}, \end{array}\right.$$

$$n = 0, 1, 2, \ldots,$$

$$\left|\begin{array}{lcl} z_{n+1} & = & x_n - \dfrac{f(x_n)}{2}\left(\dfrac{1}{f'(x_n)} + \dfrac{1}{f'(y_n)}\right), \\ x_{n+1} & = & z_{n+1} - f(z_{n+1}) \cdot \dfrac{2f'(y_n)}{f'^2(y_n) + 2f'(y_n)f'(x_n) - f'^2(x_n)}, \end{array}\right.$$

$$n = 0, 1, 2, \dots, .$$

Rafig, Ahmad and Hussain [86] obtained some"new" sixth-order variants of Newton's method:

$$\left| \begin{array}{lcl} z_n & = & x_n - \dfrac{2f(x_n)}{f'(x_n) + f'(y_n)}, \\[2ex] x_{n+1} & = & z_n - \dfrac{f(x_n)}{f'(x_n)} \cdot \dfrac{f(z_n)}{f(x_n) - (f'(y_n) - f'(x_n))(z_n - x_n)}, \end{array} \right.$$

$$n = 0, 1, 2, \dots,$$

$$\left| \begin{array}{lcl} z_n & = & x_n - \dfrac{f(x_n)}{f'\left(\dfrac{1}{2}(x_n + y_n)\right)}, \\[2ex] x_{n+1} & = & z_n - \dfrac{f(x_n)}{f'(x_n)} \cdot \dfrac{f(z_n)}{f(x_n) - (f'(y_n) - f'(x_n))(z_n - x_n)}, \end{array} \right.$$

$$n = 0, 1, 2, \dots,$$

$$\left| \begin{array}{lcl} z_n & = & x_n - \dfrac{f(x_n)}{2}\left(\dfrac{1}{f'(x_n)} + \dfrac{1}{f'(y_n)}\right), \\[2ex] x_{n+1} & = & z_n - \dfrac{f(x_n)}{f'(x_n)} \cdot \dfrac{f(z_n)}{f(x_n) - (f'(y_n) - f'(x_n))(z_n - x_n)}, \end{array} \right.$$

$$n = 0, 1, 2, \dots,$$

$$\left| \begin{array}{lcl} z_n & = & x_n - \dfrac{f(x_n)}{2}\left(\dfrac{1}{f'(x_n)} + \dfrac{1}{f'\left(\dfrac{1}{2}(x_n + y_n)\right) - f'(x_n)}\right), \\[2ex] x_{n+1} & = & z_n - \dfrac{f(x_n)}{f'(x_n)} \cdot \dfrac{f(z_n)}{f(x_n) - (f'(y_n) - f'(x_n))\,(z_n - x_n)}, \end{array} \right.$$

$$n = 0, 1, 2, \dots.$$

The third-order method

$$x_{n+1} = x_n + \frac{-f(x_n) \pm \sqrt{f'^2(x_n) - 2f(x_n)f''(x_n)}}{f''(x_n)}$$

$$n = 0, 1, 2, \ldots$$

derived by Fang et al. [31] is a very old method described by Euler [29] and Cauchy [11].

Popowski's family of methods to obtain a simple root of the nonlinear equation $f(x) = 0$ is given by

$$x_{n+1} = x_n - (1-e)\frac{f'_n}{f''_n}\left(\left(1 - \frac{e}{e-1}u_n\frac{f''_n}{f'_n}\right)^{\frac{1}{e}} - 1\right),$$

$$n = 0, 1, 2, \ldots$$

where

$$f_n^{(i)} = f^{(i)}(x_n),\ i = 0, 1, 2.$$

For $e = -1$, the method is due to Halley. For $e = 2$ the method is due to Cauchy. For $e = \frac{1}{2}$, the method is due to Chebyshev.

Kou, Li and Wang [65] have modified Halley's method as follows:

$$x_{n+1} = x_n - u_n\frac{\theta^2 f_n}{(\theta^2 - \theta + 1)f_n - f(y_n)},$$

$$n = 0, 1, 2, \ldots,$$

where θ is a nonzero real number, and $y_n = x_n - \theta u_n$.

Kou and Li [58] developed an extension of Chebyshev's method

$$x_{n+1} = x_n - u_n\left(\frac{\theta^2 + \theta - 1}{\theta^2} + \frac{f(y_n)}{\theta^2 f_n}\right),$$

$$n = 0, 1, 2, \ldots.$$

In [75], Neta derived the following method

$$x_{n+1} = x_n - (1-e)\frac{\theta^2 f_n^2}{2f'_n(f(y_n) - (1-\theta)f_n)}\left(\left(1 - \frac{2e}{e-1}\frac{f(y_n) - (1-\theta)f_n}{\theta^2 f_n}\right)^{\frac{1}{e}} - 1\right),$$

$$n = 0, 1, 2, \ldots .$$

For other results, see Kou and Li [59], [60], Kou, Li and Wang [66], [67], Grau and Diaz-Barrero [37], [38], Grau and Noguera [39], Sharma and Guha [91], Chun and Ham [25], Costabile, Guattieri and Luceri [28], Gutierrez and Hernandez [41], Amat, Busquier and Gutierrez [3], Ezquerro, Gutierrez, Hernandez and Salanova [30], Grau [36], Bathi Kasturiarachi [8], Noor and Ahmad [77].

We note that the methodology of construction of the family of iterative methods without employing derivatives for solving nonlinear equations can be found in the book [83]

$$\begin{aligned} x_{n+1} &= \alpha x_{n-1} + (1-\alpha)x_n \\ &\quad - (x_n - x_{n-1}) \frac{\alpha |f(x_{n-1})|^{\beta\gamma} sgn f(x_{n-1}) + (1-\alpha)|f(x_n)|^{\gamma\delta} sgn f(x_n)}{|B|^{\delta} sgn B}, \end{aligned}$$

where

$$B = |f(x_n)|^{\beta} sgn f(x_n) - |f(x_{n-1})|^{\gamma} sgn f(x_{n-1})$$

and α, β, γ and δ are real parameters.

In the paper by Xu Liangzang and Mi Xiangjiang [68] refined conditions of convergence for the difference analogue of Halley's method

$$x_{n+1} = x_n - \frac{f(x_n)}{f(x_n, x_{n-1}) + f(x_n, x_{n-1}, x_{n-2})(x_n - x_{n-1})}, \tag{13}$$

$$n = 0, 1, 2, \ldots$$

for solving nonlinear equation in R^1 are given.

In [49] the following nonstationary iteration algorithm without derivatives is obtained:

$$\left| \begin{array}{lcl} x_{2n+1} & = & x_{2n} - (x_{2n-1} - x_{2n}) \dfrac{f(x_{2n})}{f(x_{2n-1}) - f(x_{2n})}, \\ \\ x_{2n+2} & = & x_{2n+1} - \dfrac{f(x_{2n+1})}{f(x_{2n+1}, x_{2n}) + f(x_{2n+1}, x_{2n}, x_{2n-1})(x_{2n+1} - x_{2n})}, \end{array} \right. \tag{14}$$

$$n = 0, 1, 2, \ldots$$

with order of convergence $\lambda = 3$.

Two algorithms are almost universally known. For the former,

$$
\begin{aligned}
y_{n+1} &= y_n - \frac{2f(y_n)}{\delta + [\delta^2 - 4f(y_n)f(y_n, y_{n-1}, y_{n-2})]^{\frac{1}{2}}}, \\
\delta &= f(y_n, y_{n-1}) + (y_n - y_{n-1})f(y_n, y_{n-1}, y_{n-2}),
\end{aligned}
\tag{15}
$$

$$n = 0, 1, 2, \ldots,$$

while for the latter,

$$
z_{n+1} = z_n - f(z_n)\left[\frac{1}{f(z_n, z_{n-1})} + \frac{1}{f(z_n, z_{n-2})} - \frac{1}{f(z_{n-1}, z_{n-2})}\right],
\tag{16}
$$

$$n = 0, 1, 2, \ldots.$$

The iteration (15) differs only in form from Muller's iteration over which it enjoys a number of advantages (see [93]).

The following iterative nonstationary schemes can be obtained using the approach given in [49]:

$$
\left|
\begin{aligned}
y_{2n+1} &= y_{2n-1} + (y_{2n} - y_{2n-1})\frac{f(y_{2n-1})}{f(y_{2n-1}) - f(y_{2n})}, \\
y_{2n+2} &= y_{2n+1} - \frac{2f(y_{2n+1})}{\delta + [\delta^2 - 4f(y_{2n+1})f(y_{2n+1}, y_{2n}, y_{2n-1})]^{\frac{1}{2}}}, \\
\delta &= f(y_{2n+1}, y_{2n}) + (y_{2n+1} - y_{2n})f(y_{2n+1}, y_{2n}, y_{2n-1}),
\end{aligned}
\right.
\tag{17}
$$

$$n = 0, 1, 2, \ldots$$

(18)

$$
\left|
\begin{aligned}
z_{2n+1} &= z_{2n-1} + (z_{2n} - z_{2n-1})\frac{f(z_{2n-1})}{f(z_{2n-1}) - f(z_{2n})}, \\
z_{2n+2} &= z_{2n+1} - f(z_{2n+1})\left[\frac{1}{f(z_{2n+1}, z_{2n})} + \frac{1}{f(z_{2n+1}, z_{2n-1})} - \frac{1}{f(z_{2n}, z_{2n-1})}\right],
\end{aligned}
\right.
$$

$$n = 0, 1, 2, \ldots.$$

The methodological survey and constructing of multi-point nonstationary algorithms with memory which are generated by estimations for f'' using $f(x_{2n})$, $f(x_{2n-1})$, $f'(x_{2n})$, and $f'(x_{2n-1})$ are given in [49].

The order of convergence of the iteration

$$(19)\quad \left|\begin{aligned} x_{2n+1} &= x_{2n} - \frac{f(x_{2n})}{f'(x_{2n})} - \frac{f^2(x_{2n})}{2f'^3(x_{2n})}\left(\frac{f'(x_{2n}) - f'(x_{2n-1})}{x_{2n} - x_{2n-1}}\right), \\ x_{2n+2} &= x_{2n+1} - \frac{f(x_{2n+1})}{f(x_{2n+1}, x_{2n}) + f(x_{2n+1}, x_{2n}, x_{2n-1})(x_{2n+1} - x_{2n})}, \end{aligned}\right.$$

$$n = 0, 1, 2, \ldots$$

is $\lambda = 2 + \sqrt{5}$.

Kou, Li and Wang [67] presented a family of variants of Ostrowski's method with order of convergence 7, given by:

$$(20)\quad \begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ H_2(x_n, y_n) &= (f(x_n) - 2f(y_n))^{-1} f(y_n), \\ z_n &= y_n - H_2(x_n, y_n)(x_n - y_n), \\ H_\alpha(y_n, z_n) &= (f(y_n) - \alpha f(z_n))^{-1} f(z_n), \\ x_{n+1} &= z_n - \left((1 + H_2(x_n, y_n))^2 + H_\alpha(y_n, z_n)\right)\frac{f(z_n)}{f'(x_n)}, \end{aligned}$$

$$n = 0, 1, 2, \ldots,$$

where $\alpha \in R$ is a constant.

Bi, Ren and Wu [10] presented a new family of methods with order of convergence 7 as follows:

$$(21)\quad \left|\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ z_{n+1} &= y_n - \frac{f(y_n)}{f'(x_n)} \cdot \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2)f(y_n)}, \\ x_{n+1} &= z_n - \frac{f(z_n)}{f(z_n, y_n) + f(z_n, x_n, x_n)(z_n - y_n)} \end{aligned}\right.$$

$$n = 0, 1, 2, \dots,$$

where β is a constant.

Remark. The method (21) is based on King's fourth-order method [53]:

$$\left|\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)}, \\ x_{n+1} &= y_n - \frac{f(y_n)}{f'(x_n)} \cdot \frac{f(x_n) + \beta f(y_n)}{f(x_n) + (\beta - 2) f(y_n)}, \end{aligned}\right. \tag{22}$$

$$n = 0, 1, 2, \dots .$$

Observe that in some cases the multi-point nonstationary algorithms are two-sided algorithms.

D. Jiang and D. Han [50] proposed one construction of rational iterative algorithms for solving nonlinear equations.

Chebyshev's classical method for solving nonlinear equation is written as:

$$x_{i+1} = x_i - u_i \sum_{j=0}^{m} u_i^j Y_j,$$

$$i = 0, 1, 2, \dots; \quad m = 0, 1, \dots,$$

where

$$Y_j = \frac{(-1)^j}{(j+1)!} (f_i')^{j+1} g_i^{(j+1)}, \quad u_i = \frac{f_i}{f_i'},$$

$$f_i = f(x_i), \quad Y_0 = 1; \quad g_i = g(y_i)$$

and g is the inverse function of f.

In [4] Andreev and Kyurkchiev obtained the following two-sided analog of Chebyshev's method for a given integer positive number m:

$$T_1(x_i) = x_i - u_i \sum_{j=0}^{m} u_i^j Y_j,$$

$$T_2(x_i) = x_i - u_i \left(\sum_{j=0}^{m} u_i^j Y_j + 2 u_i^{m+1} Y_{m+1} \right),$$

$$i = 0, 1, 2, \ldots; \quad m = 0, 1, \ldots$$

with order of convergence $\lambda = m + 2$, i.e,

$$(T_1(x_i) - \xi)\,(T_2(x_i) - \xi) \leq 0,$$

$$|T_1(x_i) - \xi| = O\left(q^{(m+2)^i}\right),$$

$$|T_2(x_i) - \xi| = O\left(q^{(m+2)^i}\right),$$

$$0 < q < 1;\ \ i = 0, 1, 2, \ldots.$$

Nowadays we witness the rediscovery of some two-sides and interval iteration algorithms.

We will point out that the methodology of construction of these or similar two-sided, interval and rational iterative methods with high order of convergence for solving nonlinear equations is known (see [83], [5]).

2. Main results. Only one passing glance over the list at the beginning of this article shows in what direction scientific investigations are made:

— "one-parameter family of third-order of convergence methods for solving nonlinear equations";
— "two-parameter family of third-order of convergence methods for solving nonlinear equations";
— "three-step iterative methods";
— "multi-point methods";
— "construction of rational iterative algorithms";
— "predictor-corrector methods";
— "leap-frogging methods";
— "modifications by the method of undetermined coefficients";
— "iterative methods without derivative";
— "iterative methods free from second derivative";
— "iterative methods with fourth-order of convergence";
— "iterative methods with fifth-order of convergence";
— "iterative methods with sixth-order of convergence";
— "iterative methods with seventh-order of convergence";
— "iterative methods with eight-order of convergence".

In this fact there is nothing unnatural if for every one of the offered methods with convergence rate $\lambda = 3, 4, 5, 6, 7, 8$ the index of effectiveness and complexity in Traub – Wojnyakowski – Wassilkowski sense was investigated.

Unfortunately in most of the cited papers profound investigations are not presented in this direction and thus it has not become clear what advantages and disadvantages of proposed algorithms are.

Here we give a methodological construction of nonstationary algorithms with a raised speed of convergence.

We will pose the following problem:

Let us construct an iteration procedure (with memory) with order of convergence $\lambda = 10$ using:

a) a system of two initial approximations x_{-1} and x_0;

b) information about f and f'.

For solving this task it is appropriate to use the following basic fourth-order IF

$$\varphi(x) = x - a_1\omega_1(x) - a_2\omega_2(x) - a_3\omega_3(x),$$

where

$$\omega_1(x) = \frac{f(x)}{f'(x)}, \quad \omega_2(x) = \frac{f(x)}{f'(x + \beta\omega_1(x))}, \quad \omega_3(x) = \frac{f(x)}{f'(x + \gamma\omega_1(x) + \delta\omega_2(x))}$$

was proposed by Traub [93].

Let

$$a_1 = a_2 = \frac{1}{6}; \quad a_3 = \frac{4}{6}; \quad \beta = -1; \quad \gamma = \delta = -\frac{1}{4}$$

then

$$\varphi(x) = x - \frac{f(x)}{6f'(x)} - \frac{f(x)}{6f'\left(x - \dfrac{f(x)}{f'(x)}\right)} - \frac{4f(x)}{6f'\left(x - \dfrac{f(x)}{4f'(x)} - \dfrac{f(x)}{4f'\left(x - \dfrac{f(x)}{f'(x)}\right)}\right)}.$$

It is also proper to use well-known multi-point algorithm (see, [93])

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n)\,(2f'(x_n) + f'(x_{n-1}) - 3f(x_n, x_{n-1}))}{f'^3(x_n)(x_n - x_{n-1})}$$

$$n = 0, 1, 2, \ldots.$$

Consider the following iterative nonstationary algorithm for solving the nonlinear equation $f(x) = 0$:

(23)

$$x_{2n+1} = x_{2n} - \frac{f(x_{2n})}{f'(x_{2n})} - \frac{f^2(x_{2n})\,(2f'(x_{2n}) + f'(x_{2n-1}) - 3f(x_{2n}, x_{2n-1}))}{f'^3(x_{2n})(x_{2n} - x_{2n-1})},$$

$$x_{2n+2} = x_{2n+1} - \frac{f(x_{2n+1})}{6f'(x_{2n+1})} - \frac{f(x_{2n+1})}{6f'\left(x_{2n+1} - \frac{f(x_{2n+1})}{f'(x_{2n+1})}\right)}$$

$$- \frac{4f(x_{2n+1})}{6f'\left(x_{2n+1} - \frac{f(x_{2n+1})}{4f'(x_{2n+1})} - \frac{f(x_{2n+1})}{4f'\left(x_{2n+1} - \frac{f(x_{2n+1})}{f'(x_{2n+1})}\right)}\right)},$$

$$n = 0, 1, 2, \ldots .$$

Here $f(x, y)$ denote the finite difference. Let

$$\epsilon_i = x_i - \xi, \quad i = -1, 0, 1, \ldots; \quad A_k(\xi) = \frac{f^{(k)}(\xi)}{k!f'(\xi)}.$$

It is well known that for the error ϵ_i [93] is valid

$$\epsilon_{2n+1} \sim A_4(\xi)\epsilon_{2n}^2\epsilon_{2n-1}^2,$$

(24)

$$\epsilon_{2n+2} \sim \tfrac{1}{3}A_2^3(\xi)\epsilon_{2n+1}^4,$$

where $\sim$ denotes the asymptotical equation when $n \to \infty$.

Let

$$K = \max\left\{|A_4(\xi)|, |\tfrac{1}{3}A_2^3(\xi)|\right\},$$

$$d_{2n-1} = K^{\frac{1}{3}}|\epsilon_{2n-1}|,$$

$$d_{2n} = K^{\frac{1}{3}}|\epsilon_{2n}|$$

and let $d > 0$, and x_{-1} and x_0 be chosen so that the following inequalities

$$d_{-1} = K^{\frac{1}{3}}|x_{-1} - \xi| \le d < 1,$$

$$d_0 = K^{\frac{1}{3}}|x_0 - \xi| \le d < 1$$

hold true.

From (24), we have

$$
\begin{aligned}
d_{2n+1} &= K^{\frac{1}{3}}|\epsilon_{2n+1}| \le K^{\frac{1}{3}}K\epsilon_{2n}^2\epsilon_{2n-1}^2 = K^{\frac{2}{3}}\epsilon_{2n}^2K^{\frac{2}{3}}\epsilon_{2n-1}^2 = d_{2n}^2d_{2n-1}^2, \\
d_{2n+2} &= K^{\frac{1}{3}}|\epsilon_{2n+2}| \le K^{\frac{1}{3}}K\epsilon_{2n+1}^4 = d_{2n+1}^4.
\end{aligned} \tag{25}
$$

Our results concerning the order of convergence generated by (23) are summarized in the following theorem.

Theorem. *Assume that the initial approximations x_0, x_{-1} are chosen so that $d_{-1} \le d < 1$ and $d_0 \le d < 1$. Then for the error of the sequences $\{x_{2n}\}_{n=0}^{\infty}$ and $\{x_{2n-1}\}_{n=0}^{\infty}$ determined by* (23)*, we have*

$$
\begin{aligned}
d_{2n-1} &\le d^{4.10^{n-1}}, \\
d_{2n} &\le d^{16.10^{n-1}}, \quad n = 1, 2, \ldots
\end{aligned} \tag{26}
$$

and the order of convergence of the iteration (20) *is $\lambda = 10$.*

P r o o f. The proof is by induction with respect to the iteration number n. For $n = 0$, from (25), we find

$$
\begin{aligned}
d_1 &\le d^2.d^2 = d^4, \\
d_2 &\le d^{16}
\end{aligned}
$$

and (26) is fulfilled.

Let (26) be fulfilled for $n \le m$. For $n = m+1$, from (25) and (26), we have

$$
\begin{aligned}
d_{2(m+1)-1} &= d_{2m+1} \le d_{2m}^2d_{2m-1}^2 \le d^{2.16.10^{m-1}+2.4.10^{m-1}} = d^{40.10^{m-1}} = d^{4.10^m}, \\
d_{2(m+1)} &= d_{2m+2} \le d_{2m+1}^4 < d^{16.10^m}
\end{aligned}
$$

which completes the induction.

On the other hand,

$$
\begin{aligned}
d_{2n-1} &= K^{\frac{1}{3}}|\epsilon_{2n-1}|, \\
d_{2n} &= K^{\frac{1}{3}}|\epsilon_{2n}|
\end{aligned}
$$

and equation (26) can be written as

$$\begin{aligned} |\epsilon_{2n-1}| &\leq K^{-\frac{1}{3}} d^{4.10^{n-1}}, \\ |\epsilon_{2n}| &\leq K^{-\frac{1}{3}} d^{16.10^{n-1}}, \ n = 1, 2, \ldots, \end{aligned}$$

and the order of convergence of iteration (23) is equal to 10.

Thus, the theorem is proved. □

3. Numerical examples. We used the test functions

$$\begin{aligned} f(x) &= 10xe^{-x^2} - 1 \\ f_1(x) &= x^3 + 4x^2 - 15 \\ f_2(x) &= \sin x - \tfrac{1}{2}x \end{aligned}$$

as Bi, Ren and Wu [10] with initial approximation $x_0 = 1.8$ (for $f(x)$); $x_0 = 2.$ (for $f_1(x)$) and $x_0 = 2.$ (for $f_2(x)$) for comparison of various algorithms.

The following Table 1 is given in [10]:

Table 1

	Newton's method	method (22)	method (20)	method (21)
$f(x)$				
$\lvert x_n - x^*\rvert$	4.42e-58	4.20e-237	4.84e-282	1.73e-337
$\lvert f(x_n)\rvert$	1.22e-57	1.16e-236	1.34e-281	4.77e-337
$f_1(x)$				
$\lvert x_n - x^*\rvert$	3.91e-55	4.87e-230	5.03e-276	4.18e-320
$\lvert f_1(x_n)\rvert$	8.23e-54	1.03e-228	1.06e-274	8.79e-320
$f_2(x)$				
$\lvert x_n - x^*\rvert$	1.89e-80	6.25e-313	0.00e+00	0.00e+00
$\lvert f_2(x_n)\rvert$	1.54e-80	5.12e-313	3.00e-350	3.00e-350

Here x^* is the exact root computed with 350 significant digits.

It may be remarked that for the comparison of various iterative methods, the following principle should be applied [82]:

"Method A is superior to method B if A attains the same accuracy of approximations as B but has less computational cost (expressed, say, by the total CPU (central processor unit) time, or by the total number of function evaluations).

It is assumed that "pathological" examples (meaning the choice of "awkward" functions or very inconvenient initial approximations) should be neglected."

Using initial approximations $x_{-1} = 1.5$ and $x_0 = 1.6$ (for the test function $f(x)$) and the computational scheme (23), we receive the results shown in Table 2.

Table 2

n	$\lvert x_{2n+2} - x_{2n+1} \rvert$	$\lvert f(x_{2n+2}) \rvert$
0	0.0396	2.49e-7
1	3.27e-14	9.22e-55
2	4.41e-109	3.04e-434
3	4.80e-868	4.29e-3470

Using formula (23), we receive the root x^*

$x^* =$ 1.679630610428449940674920338837970397829008946378045524066483282
89497355427088761068810276830643502683679719165399983047220534523
96700757895643172911738713037666235788899591848426460257248419353
21561866784894372916713351798007472354475884295762479761489869577
01606999353050339025980077584126054444507048659462786597626187972
76104631691080255918147021224221142011456693559307546018430592201
23788922039887808526415271812414684245346400940297492834277578732
36969836193754437561134906857639417651729050332319925983174578145
55116823041740168839795781919418092440266447702216122498252029724
05533821478773277420684538180705516309305903317463947383916968348
58596822416779155823419229227428705255511037757952383989425365246
31030091974318263892244234515363234055318238743632572201702618794
50817211235617623858800370011531895554280591734055011872437151579
1150263789281982994066

with 866 significant digits after 4 iterations (see, Table 2).

For the test function $f_1(x)$ at the initial approximations $x_{-1} = 1.4$ and $x_0 = 1.6$, using the computational scheme (23), we receive the results shown in Table 3.

Table 3

n	$\|x_{2n+2} - x_{2n+1}\|$	$\|f_1(x_{2n+2})\|$
3	5.52e-839	4.95e-3354

For the test function $f_2(x)$ at the initial approximations $x_{-1} = 1.8$ and $x_0 = 2.$, using computational scheme (23), we receive the results shown in Table 4.

Table 4

n	$\|x_{2n+2} - x_{2n+1}\|$	$\|f_2(x_{2n+2})\|$
3	1.43e-1026	5.e-4000

Obviously, the iteration (23) with order of convergence $\lambda = 10$ has good computational effectiveness.

Is the method (23) really superior than the Ujevic's method (8)?

Calculating the computational efficiency by Ostrowski's formula, we obtain

$$E(23) = 10^{\frac{1}{7}} \approx 1.389 > E(8) = 2^{\frac{1}{3}} \approx 1.26.$$

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