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ПЛИСКА

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HANKEL'S TRANSFORM AND SERIES IN LAGUERRE POLYNOMIALS

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It is proved that a complex function f, analytic in the region $\Delta(\lambda_0)=\{z\in\mathbf{C}:\operatorname{Re}(-z)^{1/2}<\lambda_0\}$, can be represented in this region by a series in Laguerre polynomials iff f is a Hankel's type transform of a suitable analytic function.

The problem of representation of analytic functions by series in Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ is solved by H. Pollard [1] in the case $\alpha=0$. It is interesting that the method used in [1] is closely related to this latter condition. As far as we know, the general case $\alpha>-1$ $(\alpha\neq0)$ is not considered and the aim of the paper is to give a result in this direction.

1. Laguerre polynomials and functions of second kind. The system of Laguerre polynomials $\{L_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ can be defined in the region $\mathbf{C}-(-\infty, 0]$ by means of the corresponding Rodrigues formula, namely [2, 10.12, (5)]

(1) $$L_n^{(\alpha)}(z)=(n!)^{-1}z^{-\alpha}\exp z\{z^{n+\alpha}\exp(-z)\}^{(n)}, \qquad n=0, 1, 2, \dots$$

Here $\alpha\neq-1, -2, \dots$ is an arbitrary complex number, but we shall consider only the case α real and greater than -1.

We shall deal with series of the kind

(2) $$\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z)$$

with arbitrary complex coefficients. Using the asymptotic formulas and inequalities for the Laguerre polynomials, it is not difficult to describe the region of convergence of the series (2). If

$$\lambda_0=-\limsup_{n\to+\infty}(2\sqrt{n})^{-1}\ln|a_n|>0,$$

the series (2) is absolutely convergent in the domain $\Delta(\lambda_0)=\{z\in\mathbf{C}:\operatorname{Re}(-z)^{1/2}<\lambda_0\}$ and diverges at every point $z\in\mathbf{C}-\overline{\Delta(\lambda_0)}$. Let us note that if $\lambda_0<+\infty$, $\Delta(\lambda_0)$ is the interior of the parabola $p(\lambda_0)=\{z\in\mathbf{C}:\operatorname{Re}(-z)^{1/2}=\lambda_0\}$ and $\Delta(+\infty)$ is the whole complex plane.

The system $\{M_n^{(\alpha)}(z)\}_{n=0}^{\infty}$ of Laguerre functions of second kind is defined in the region $\mathbf{C}-[0, +\infty)$ as follows

(3) $$M_n^{(\alpha)}(z)=-\int_0^{\infty}\frac{t^{\alpha}\exp(-t)L_n^{(\alpha)}(t)}{t-z}\,dt, \qquad n=0, 1, 2, \dots$$

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Using (1), from (3) after integration by parts we get that

$$M_n^{(\alpha)}(z)=-\int_0^\infty \frac{t^{n+\alpha}\exp(-t)}{(t-z)^{n+1}}\,dt,\ n=0,1,2,\dots$$

Let Re $z<0$ and $l(z)$ be the ray $\{\zeta\in\mathbf{C}:\zeta=(-z)t,\ 0\le t<+\infty\}$. Then the Cauchy inregral theorem gives that

$$M_n^{(\alpha)}(z)=-\int_{l(z)}\frac{\zeta^{n+\alpha}\exp(-\zeta)}{(\zeta-z)^{n+1}}\,d\zeta=-(-z)^\alpha\int_0^\infty\frac{t^{n+\alpha}\exp zt}{(1+t)^{n+1}}\,dt.$$

The last integral representation of the Laguerre functions of second kind leads in a natural way to a generating function for the system (3) namely (Re $z<0$, $w\in\mathbf{C}$)

(4) $$M^{(\alpha)}(z,w)=\sum_{n=0}^\infty\frac{M_n^{(\alpha)}(z)}{n!}w^n=-(-z)^\alpha\int_0^\infty\frac{t^\alpha}{1+t}\exp\left\{\frac{wt}{1+t}+zt\right\}dt.$$

2. The class $A(\sigma)$. If $0<\sigma\le+\infty$, with $A(\sigma)$ we denote the class of all entire functions Φ having the property

(5) $$\limsup_{|w|\to+\infty}(2\sqrt{|w|})^{-1}(\ln|\Phi(w)|-|w|)\le-\sigma.$$

Let us note that every entire function of exponential type less than one belongs to the class $A(+\infty)$. How wide is the class $A(\sigma)$ shows the following

Lemma. *The function*

(6) $$\Phi(w)=\sum_{n=0}^\infty\frac{a_n}{n!}w^n$$

is in the class $A(\sigma)$ *iff*

(7) $$\limsup_{n\to+\infty}(2\sqrt{n})^{-1}\ln|a_n|\le-\sigma.$$

Proof. First we shall consider the case $0<\sigma<+\infty$. If (5) holds, then for every $\delta>0$ there exist $B(\delta)>0$ and $N(\delta)>0$ such that for $n>N(\delta)$

$$|a_n|\le n!\,n^{-n}\max_{|w|=n}|\Phi(w)|\le B(\delta)n!\,n^{-n}\exp[n-2(\sigma-\delta)\sqrt{n}]$$

and Stirling's formula gives that $\limsup_{n\to+\infty}(2\sqrt{n})^{-1}\ln|a_n|\le-\sigma+\delta$.

To prove that (7) is sufficient for the function (6) to be in the class $A(\sigma)$, we shall use the asymptotic formula for the system of the Laguerre functions of second kind namely [3, p. 272, (11)]

(8) $$M_n^{(\alpha)}(z)=-\sqrt{\pi}\exp(z/2)(-z)^{\alpha/2-1/4}n^{\alpha/2-1/4}\exp\{-2\sqrt{n}(-z)^{1/2}\}\{1+\mu_n^{(\alpha)}(z)\},$$

where $\{\mu_n^{(\alpha)}(z)\}_{n=0}^\infty$ are complex functions analytic in the region $\mathbf{C}-[0,+\infty)$ and $\lim_{n\to+\infty}\mu_n^{(\alpha)}(z)=0$ uniformly on every compact subset of this region.

If the sequence $\{a_n\}_{n=0}^\infty$ satisfies (7) with $\sigma<+\infty$, it follows that for every $0<\delta<\sigma$ there exists $C(\delta)$ such that $|a_n|\le C(\delta)\{-M_n^{(0)}[-(\sigma-\delta)^2]\}$, $n=0,1,2,.$ Having (4) in view, we get that

$$|\Phi(w)|\le C(\delta)M^{(0)}[-(\sigma-\delta)^2,\ |w|]$$

and therefore,

$$|\Phi(w)| = O\{\int_0^\infty \exp[|w|t(1+t)^{-1}-(\sigma-\delta)^2 t]\,dt\}$$

$$= O\{\exp|w| \int_1^\infty \exp[-(\sigma-\delta)^2 t - |w|t^{-1}]\,dt\}$$

$$= O\{\sqrt{|w|}\exp|w| \int_{(\sigma-\delta)/\sqrt{|w|}}^\infty \exp[-(\sigma-\delta)\sqrt{|w|}(t+t^{-1})]\,dt\}$$

$$= O\{\sqrt{|w|}\exp|w| \int_{(\sigma-\delta)\sqrt{|w|}}^1 \exp[-(\sigma-\delta)\sqrt{|w|}(t+t^{-1})\,dt$$

$$+\sqrt{|w|}\exp|w| \int_1^\infty \exp[-(\sigma-\delta)\sqrt{|w|}(t+t^{-1})]\,dt\}$$

$$= O\{\sqrt{|w|} \int_1^{\sqrt{|w|}/(\sigma-\delta)} t^{-2}\exp[-(\sigma-\delta)\sqrt{|w|}(t+t^{-1})\,dt$$

$$+\sqrt{|w|}\exp|w| \int_1^\infty \exp[-(\sigma-\delta)\sqrt{|w|}(t+t^{-1})]\,dt\}$$

$$= O\{\sqrt{|w|}\exp|w| \int_1^\infty \exp[-(\sigma-\delta)\sqrt{|w|}(t+t^{-1})\,dt\}$$

$$= O\{\sqrt{|w|}\exp|w| \int_2^\infty [1+t(t^2-4)^{-1}]\exp[-(\sigma-\delta)\sqrt{|w|}\,t]dt\}$$

$$= O\{\sqrt{|w|}\exp[|w|-2(\sigma-\delta)\sqrt{|w|}]\}.$$

The proof of the Lemma in the case $\sigma=+\infty$ is analogous to the given above and we shall omit it.

3. The main result. Our "starting point" is the following integral representation of Laguerre polynomials namely [2, 10.12, (21)]

$$L_n^{(\alpha)}(z) = (n!)^{-1} z^{-\alpha/2} \exp z \int_0^\infty \exp(-t)\, t^{n+\alpha/2} J_\alpha(2\sqrt{zt})\,dt, \tag{9}$$

where J_α is the Bessel function of the first kind with parameter α. The above equality shows that the complex function $n!\,z^{\alpha/2}\exp(-z)L_n^{(\alpha)}(z)$, which is analytic in the region $\mathbf{C}-(-\infty, 0]$, is the image of the function $\exp(-t)t^{n+\alpha/2}$ under a transformation of Hankel's type. This observation gives rise to propose that if an analytic function $f(z)$ is represented by a series of the kind (2), the function $z^{\alpha/2}\exp(-z)f(z)$ must be also Hankel's transform of a suitable complex function. We shall see that this is realy the fact.

Theorem. *Let* $0<\lambda_0\leqq+\infty$ *and* $\alpha>-1$. *The complex function* f, *which is analytic in the region* $\Delta(\lambda_0)$, *can be expanded in this region in series of Laguerre polynomials with parameter* α *iff in the region* $\tilde{\Delta}(\lambda_0)=\Delta(\lambda_0)-(-\lambda_0^2, 0]$ *holds a representation*

$$f(z) = z^{-\alpha/2}\exp z \int_0^\infty t^{\alpha/2}\exp(-t)\,\Phi(t)\,J_\alpha(2\sqrt{zt})\,dt, \tag{10}$$

where $\Phi\in A(\lambda_0)$.

Proof. First of all we note that if the function $\Phi \in A(\lambda_0)$, the integral in (10) is absolutely convergent at every point $z \in \tilde{\Delta}(\lambda_0)$. We shall prove this in the case $\lambda_0 < +\infty$. If $\operatorname{Re}(-z)^{1/2} = \lambda$, from the asymptotic formula [4, 7.13, (3)] follows that $|J_\alpha(2\sqrt{zt})| = O\{\exp(2\lambda\sqrt{t})\}$. If $\delta = (\lambda_0 - \lambda)/2$ and $0 \leq t < +\infty$, the inequality (5) gives that

$$|\Phi(t)| = O\{\exp[t - 2(\lambda_0 - \delta)\sqrt{t}]\} = O\{\exp[t - (\lambda_0 + \lambda)\sqrt{t}]\},$$

therefore, if $t \to +\infty$

$$t^{\alpha/2}\exp(-t)\,|\Phi(t)J_\alpha(2\sqrt{zt})| = O\{t^{\alpha/2}\exp(-2\delta\sqrt{t})\}. \tag{11}$$

Let us suppose that the condition of the theorem is fullfilled. Then, if the function Φ is represented by the power series (6), according to the Lemma holds the inequality (7) with $\sigma = \lambda_0$. Therefore, the series (2) is convergent in the region $\Delta(\lambda_0)$.

Further, if we define for $\nu = 0, 1, 2, \ldots$ and $z \in \tilde{\Delta}(\lambda_0)$

$$R_\nu(z) = f(z) - \sum_{n=0}^{\nu} a_n L_n^{(\alpha)}(z)$$

from (9) and (10) follows that

$$R_\nu(z) = z^{-\alpha/2}\exp z \int_0^\infty t^{\alpha/2}\exp(-t)\left\{\sum_{n=\nu+1}^{\infty}\frac{a_n t^n}{n!}\right\} J_\alpha(2\sqrt{zt})\,dt. \tag{12}$$

The function $\Phi^*(t) = \sum_{n=0}^{\infty}(n!)^{-1}|a_n|\,|w|^n$ is also in the class $A(\lambda_0)$. That is why, if we replace Φ by Φ^* in the inequality (11), we can assert that for every $\varepsilon > 0$ there exists $T = T(\varepsilon) > 0$ such that

$$\int_T^\infty t^{\alpha/2}\exp(-t)\,\Phi^*(t)\,|J_\alpha(2\sqrt{zt}\,|)\,dt < \varepsilon.$$

Then, for every $\nu = 0, 1, 2, \ldots$

$$\left|\int_T^\infty t^{\alpha/2}\exp(-t)\left\{\sum_{n=\nu+1}^{\infty}\frac{a_n t^n}{n!}\right\} J_\alpha(2\sqrt{zt})\,dt\right| \tag{13}$$

$$\leq \int_T^\infty t^{\alpha/2}\exp(-t)\,\Phi^*(t)\,|J_\alpha(2\sqrt{zt})|\,dt < \varepsilon.$$

There exists $N = N(\varepsilon) > 0$ with the property that if $\nu > N$ and $0 \leq t \leq T$, $|\sum_{n=\nu+1}^{\infty}(n!)^{-1}a_n t^n| < \varepsilon$, therefore

$$\left|\int_0^T t^{\alpha/2}\exp(-t)\left\{\sum_{n=\nu+1}^{\infty}(n!)^{-1}a_n t^n\right\} J_\alpha(2\sqrt{zt})\,dt\right| \tag{14}$$

$$= O\left\{\varepsilon\int_0^\infty t^{\alpha/2}\exp(-t)\,|J_\alpha(2\sqrt{zt})|\,dt\right\} = O(\varepsilon).$$

From (12), (13) and (14) we get that $R_\nu(z) = O(\varepsilon)$, $\nu > N$, i. e. the series (2) represents the function f in the region $\tilde{\Delta}(\lambda_0)$ and, therefore, also in the region $\Delta(\lambda_0)$.

Let us suppose now, that the function f is analytic in the region $\Delta(\lambda_0)$ and can be represented in this region by the series (2). Then, from the Lemma follows that the function (6) is in the class $A(\lambda_0)$. This function by means of the integral transformation (10) defines a complex function $\tilde{f}$ analytic in the region $\tilde{\Delta}(\lambda_0)$. But we have just seen that the function $\tilde{f}$ is represented in this region by the series (2) and, therefore $f=\tilde{f}$.

4. An example. Let f be an entire function of exponential type $\tau<1$ and $h(f;\,0)<1/2$ ($h(f;\,\theta)$ is the indicator function of f). If $F(f;\,\zeta)$ is the Borel transform of f, we define ($w\in\mathbf{C}$)

$$\Phi_\alpha(f;\,w)=\frac{1}{2\pi i}\int_\Gamma (1-\zeta)^{-1-\alpha}\exp\left(-\frac{w\zeta}{1-\zeta}\right)F(f;\,\zeta)d\zeta,$$

where Γ is a Jordan rectifiable curve lying in the domain $D=\{|\zeta|<1\}\cap\{\operatorname{Re}\zeta<1/2\}$ and containing the conjugate diagram of f in its interior. Then it follows immediately that the entire function $\Phi_\alpha(f;\,w)\in A(+\infty)$. Now we shall see that for $z\in\mathbf{C}-(-\infty,0]$,

$$f(z)=z^{-\alpha/2}\exp(z)\int_0^\infty t^{\alpha/2}\exp(-t)\,\Phi_\alpha(f;\,t)J_\alpha(2\sqrt{zt})dt.$$

Indeed, having in view that if $z\in\mathbf{C}-(-\infty,0]$ and $\zeta\in D$

$$\int_0^\infty t^{\alpha/2}\exp[-t(1-\zeta)^{-1}]\,J_\alpha(2\sqrt{zt})\,dt=z^{\alpha/2}(1-\zeta)^{1+\alpha}\exp[-z(1-\zeta)],$$

we get easily that

$$z^{-\alpha/2}\exp z\int_0^\infty t^{\alpha/2}\exp(-t)\,\Phi_\alpha(f;\,t)J_\alpha(2\sqrt{tz})dt$$

$$=\frac{1}{2\pi i}\int_\Gamma \exp(z\zeta)F(f;\,\zeta)d\zeta=f(z).$$

Therefore, f can be represented in the whole complex plane by a series in Laguerre polynomials with parameter α [4, p. 40, (X)].

Let us note that the last statement false if $h(f;\,0)=1/2$. Indeed, the entire function $\exp(z/2)$ cannot be represented by a convergent series in the polynomials $\{L_n^{(1/2)}(z)\}_{n=0}^\infty$ [5, 9.5, (7)].

REFERENCES

1. H. Pollard. Representation of an analytic function by a Laguerre series. *Ann. Math.*, **43**, 1947, 358-365.
2. H. Bateman, A. Erdelyi. Higher transcendental functions, v. 2. New York, 1953.
3. P. Rusev. Laguerre's functions of the second kind. *Ann. Univ. Sofia, Fac. Math. et Mech.*, **67**, 1972/73, 269-283.
4. R. P. Boas, R. C. Buck. Polynomial expansions of analytic functions. *Ergebnisse der Mathematik und ihrer Grenzgebiete*, **19**, Berlin, 1958.
5. G. Szegö. Orthogonal polynomials. New York, 1959.

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