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CONVOLUTIONS, MULTIPLIERS AND COMMUTANTS RELATED TO DOUBLE COMPLEX DIRICHLET EXPANSIONS

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An algebraic approach to the problem of expanding of functions of several complex variables in multiple Dirichlet series in polydomains is proposed. An explicit representation of the coefficient convolutions and multipliers of Gromov-Leontiev's expansion is found. By the way, the commutant of the operators for partial differentiation in certain invariant subspaces is determined.

Introduction. Let $L(\lambda)$ and $M(\mu)$ are entire functions of exponential type (i. e. of order 1 and of normal type) with infinite sequences $\{\lambda_n\}_{n=1}^{\infty}$, and $\{\mu_n\}_{n=1}^{\infty}$ of simple zeros. Without loss of generality we may assume $L(0)=1$ an $M(0)=1$. Let $\gamma(u)$ and $\delta(v)$ are the Borel transforms of $L(\lambda)$ and $M(\mu)$, respectively. Let U and V are bounded convex domains in $\mathbf{C}^1$, such that $\bar{U}$ and $\bar{V}$ contain all singularities of $\gamma(u)$ und $\delta(v)$, correspondingly. We denote $D=U\times V$ and assume $(0, 0)\in D$. By $A(\bar{D})$ we denote the space of the functions $f(u, v)$, analytic on $\bar{D}=\bar{U}\times\bar{V}$.

The problem of expanding of function $f(u, v)\in A(\bar{D})$ in double Dirichlet series of the type

$$f(u, v)=\sum_{l,m=1}^{\infty} a_{l,m}e^{\lambda_l u+\mu_m v} \tag{1}$$

had been studied by V. P. Gromov [1] and by A. F. Leontiev [2]. Here an attempt for developing of an algebraic approach to this and to related problems is made. By means of a suitable convolution, an explicit representation of the coefficient multipliers and coefficient convolutions of the Gromov-Leontiev's expansion is found. By the way, another way of derivation of V. P. Gromov's formulas for the coefficients is given. A convolutional representation of all continuous linear operators in $A(\bar{D})$, commuting with $\partial/\partial u$ and $\partial/\partial v$ in certain invariant subspaces is found. For sake of simplicity, the considerations are made for functions of two complex variables only, but there are no difficulties to transfer them to functions of n complex variables in polydomains.

Before proceeding to our main aim, we should formulate some well-known facts about the topology of the space $A(\bar{D})$, not in their full generality, but in a form, needed for our considerations.

Let E be convex compact set in $\mathbf{C}^1$ or $\mathbf{C}^2$. Let $\mathcal{O}^E$ be a family of open convex domains, containing E. For $O\in\mathcal{O}^E$, by $A(O)$ we denote the space of the analytical functions in O, with the usual compact topology. Two functions

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$f, g \in \bigcup_{O \in \mathcal{O}^E} A(O)$ are said to be equivalent iff there exists $O \in \mathcal{O}^E$ such that $f|_O = g|_O$. The set of these equvalence classes is denoted by $A(E)$. In the linear space $A(E)$ we introduce the inductive topology, determined by the family $\{A(O)\}_{O \in \mathcal{O}^E}$ and the canonical maps $\varphi_O : A(O) \to A(E)$ which map each function $f \in A(O)$ in the equivalence class $\tilde{f} \in A(E)$, such that $f \in \tilde{f}$. Thus, we introduce the strongest local convex topology in $A(E)$, such that all maps φ_O are continuous. It is well-known that this topology is separable, and it coinsides with the inductive topology of the countable family $\{A(O_n)\}_{n=1}^{\infty}$, where $\{O_n\}_{n=1}^{\infty}$ is a countable base of open, relative compact, convex sets, containing E. An extensive study of such kind of spaces can be found in [3, pp. 378–381].

In our case of $E = \bar{D} = \bar{U} \times \bar{V}$, $U \subset \mathbf{C}^1$, $V \subset \mathbf{C}^1$ as a countable base $\{O_{n,m}\}_{n,m=1}^{\infty}$ of E we can choose $O_{n,m} = G_n \times \Omega_m$, where $\{G_n\}_{n=1}^{\infty}$ and $\{\Omega_m\}_{m=1}^{\infty}$ are such bases for $\bar{U}$ and $\bar{V}$.

Lemma 1 (Sebastião e Silva [4]). *Let E be a compact convex set in $\mathbf{C}^1$. If F is a continuous linear functional on $A(E)$, then there exists an unique function $\chi \in A(\mathbf{C}^1 \setminus E)$ with $\chi(\infty) = 0$, such that for each $f \in A(E)$ the representation*

$$F(\tilde{f}) = \frac{1}{2\pi i} \int_{\Gamma} f(\xi)\, \chi(\xi)\, d\xi \tag{2}$$

holds. Here Γ is a contour lying in the domain of analyticity of the representative $f \in \tilde{f}$, such that Γ contains E inside. Conversely, if $\chi \in A(\mathbf{C}^1 \setminus E)$, $\chi(\infty) = 0$ is arbitrary, (2) *defines a continuous linear functional on $A(E)$.*

For a proof see [3, p. 380]. Let us note that the functional F defined by (2) does not depend neither on the special choice $f \in \tilde{f}$ nor on the choice on the contour Γ.

1. A convolution in $A(\bar{D})$. Multipliers. Let Φ and Ψ are two arbitrary linear continuous functionals in $A(\bar{U})$ and $A(\bar{V})$, respectively with $\Phi(1) = \Psi(1) = 1$. According to lemma 1

$$\Phi(\tilde{f}) = \frac{1}{2\pi i} \int_{\Gamma} f(u)\, \gamma(u)\, du, \qquad \tilde{f} \in A(\bar{U}) \tag{3}$$

and

$$\Psi(\tilde{g}) = \frac{1}{2\pi i} \int_{\Delta} g(v)\, \delta(v)\, dv, \qquad \tilde{g} \in A(\bar{V}) \tag{4}$$

with some $\gamma \in A(\mathbf{C}^1 \setminus \bar{U})$, $\gamma(\infty) = 0$ and $\delta \in A(\mathbf{C}^1 \setminus \bar{V})$, $\delta(\infty) = 0$. The contours Γ and Δ are chosen to lye in analiticity domains of $f \in \tilde{f}$ and $g \in \tilde{g}$, respectively, containing $\bar{U}$ and $\bar{V}$ inside.

By means of the functionals we define the following two linear operators:

$$l_1 f(u) = \int_0^u f(\sigma)\, d\sigma - \Phi_\xi\{\int_0^\xi f(\sigma)\, d\sigma\}, \qquad f(u) \in A(\bar{U}) \tag{5}$$

and

$$l_2 f(v) \stackrel{\text{def}}{=} \int_0^v f(\tau)\, d\tau - \Psi_\eta\{\int_0^\eta f(\tau)\, d\tau\}, \qquad f(v) \in A(\bar{V}). \tag{6}$$

They are right inverse operators of d/du and d/dv in $A(\bar{U})$ and $A(\bar{V})$, respectively. Let us introduce the following two operations:

$$f \overset{(u)}{*} g = \Phi_\xi\{\int_\xi^u f(u+\xi-\sigma)g(\sigma)d\sigma\}; \qquad f, g \in A(\bar{U}) \tag{7}$$

and

$$f \overset{(v)}{*} g = \Psi_\eta\{\int_\eta^v f(v+\eta-\tau)g(\tau)d\tau\}; \qquad f, g \in A(\bar{V}) \tag{8}$$

in $A(\bar{U})$ and $A(\bar{V})$, respectively. As it is shown in [5], (7) and (8) are continuous, bilinear, commutative and associative operations in $A(\bar{U})$ and $A(\bar{V})$, respectively. l_1 and l_2 can be represented in the form $l_1 f=\{1\}\overset{(u)}{*}f$ and $l_2 f=\{1\}\overset{(v)}{*}f$. From the associativity, it follows that $l_1(f\overset{(u)}{*}g)=(l_1 f)\overset{(u)}{*}g = f\overset{(u)}{*}(l_1 g)$ for $f, g \in A(\bar{U})$ and $l_2(f\overset{(v)}{*}g)=(l_2 f)\overset{(v)}{*}g = f\overset{(v)}{*}(l_2 g)$ for $f, g \in A(\bar{V})$.

The operators l_1 and l_2 make sense in $A(\bar{D})$ too

$$l_1\{f(u, v)\} \overset{\text{def}}{=} \int_0^u f(\sigma, v)d\sigma - \Phi_\xi\{\int_0^\xi f(\sigma, v)d\sigma\} \tag{7'}$$

and

$$l_2\{f(u, v)\} \overset{\text{def}}{=} \int_0^v f(u, \tau)d\tau - \Psi_\eta\{\int_0^\eta f(u, \tau)d\tau\}. \tag{8'}$$

In this case, l_1 and l_2 are right inverse operators of $\partial/\partial u$ and in $\partial/\partial v$, respectively, and $\Phi_u\{l_1 f(u, v)\}=0$ for $v \in \bar{V}$ and $\Psi_v\{l_2 f(u, v)\}=0$ for $u \in \bar{U}$.

The basic means in our approach is a convolution in $A(\bar{D})$ for the operators l_1 and l_2. First we remind a general definition of convolution of linear operator, which maps a linear space into itself.

Definition 1 (see [6]). *A bilinear, commutative and associative operation* $*: X \times X \to X$ *in a linear space* X *is said to be a convolution of a linear operator* $T: X \to X$ *iff the relation*

$$T(f*g)=(Tf)*g=f*(Tg) \tag{9}$$

holds for all $f, g \in X$.

Theorem 1. *The expression*

$$f*g \overset{\text{def}}{=} \Phi_\xi\Psi_\eta\{\int_\xi^u\int_\eta^v f(u+\xi-\sigma, v+\eta-\tau)g(\sigma, \tau)d\sigma d\tau\} \tag{10}$$

defines a continuous convolution for l_1 *and* l_2 *in* $A(\bar{D})$, *such that*

$$l_1 l_2 f=\{1\}*f. \tag{11}$$

Proof. First, we shall show that (10) defines a function of $A(\bar{D})$. Indeed if $\tilde{f}, \tilde{g} \in A(\bar{D})$ we choose representatives $f \in A(O_1)$ and $g \in A(O_2)$, where $O_i \in \mathcal{O}^{\bar{D}}$, $O_i=G_i\times\Omega_i$, $G_i \in \mathcal{O}^{\bar{U}}$, $\Omega_i \in \mathcal{O}^{\bar{V}}$. Then the statement of the theorem should be understood as

(12) $$\widetilde{f}*\widetilde{g}\overset{\text{def}}{=}\overline{f*g}.$$

Here $f*g\in A(O_1\cap O_2)$ is defined by (10) for $f\in A(O_1)$, $g\in A(O_2)$. It is easy to see that (12) defines an operation which does not depend on the representatives $f\in\widetilde{f}$ and $g\in\widetilde{g}$.

Let B denotes the bilinear operator $A(\overline{D})\times A(\overline{D})\to A(\overline{D})$ induced by (12). We shall prove that B is continuous. Let O_1 and O_2 are from $\mathcal{O}^{\overline{D}}$ and are of the form $O_i=G_i\times\Omega_i$, $G_i\in\mathcal{O}^{\overline{U}}$, $\Omega_i\in\mathcal{O}^{\overline{V}}$. Let B_{O_1,O_2} be the operation $f*g$ considered as a bilinear map $A(O_1)\times A(O_2)\to A(O_1\cap O_2)$. It is clear that B_{O_1,O_2} is continuous operator with respect to the compact topology in $A(O_i)$, $A(O_1\cap O_2)$. Let $\varphi_{O_1}\times\varphi_{O_2}$ is the correspondence $(f, g)\to(\varphi_{O_1}f, \varphi_{O_2}g)=(\widetilde{f}, \widetilde{g})$, where φ_{O_1}, φ_{O_2} are the canonical maps. Since another form of (8) is $B\circ\varphi_{O_1}\times\varphi_{O_2}=\varphi_{O_1\cap O_2}\circ B_{O_1,O_2}$, then $B\circ\varphi_{O_1}\times\varphi_{O_2}$ is a continuous bilinear map $A(O_1)\times A(O_2)\to A(\overline{D})$ and from a well-known theorem (see [7, p. 670]) it follows that B is a continuous operator.

The bilinearity and the commutativity of $f*g$ are evident. We give an elaborate proof of the associativity only. First, we shall prove that the relation $(f*g)*h=f*(g*h)$ holds in $A(O)$, where O is an arbitrary convex domain of the form $O=G\times\Omega$ with $G\in\mathcal{O}^{\overline{U}}$ and $\Omega\in\mathcal{O}^{\overline{V}}$. It is easy to see that for functions of the form

(13) $$f(u, v)=f_1(u)f_2(v);\ f_1\in A(G),\ f_2\in A(\Omega)$$

we have

(14) $$\{f(u, v)\}*\{g(u, v)\}=[f_1(u)\overset{(u)}{*}g_1(u)]\,[f_2(v)\overset{(v)}{*}g_2(v)],$$

where $\overset{(u)}{*}$ and $\overset{(v)}{*}$ are one-dimensional operations (7) and (8). Due to the associativity of these operations (see [5]), we obtain

$$(f*g)*h=[(f_1(u)\overset{(u)}{*}g_1(u)).(f_2(v)\overset{(v)}{*}g_2(v))]*[h_1(u)h_2(v)]$$
$$=[(f_1\overset{(u)}{*}g_1)\overset{(u)}{*}h_1][(f_2\overset{(v)}{*}g_2)\overset{(v)}{*}h_2]=[f_1\overset{(u)}{*}(g_1\overset{(u)}{*}h_1)]\,[f_2\overset{(v)}{*}(g_2\overset{(v)}{*}h_2)]=f*(g*h),$$

i. e. the associativity relation holds for functions of the form (13). Therefore, from the bilinearity of $f*g$, the associativity holds for polynomials of u and v. But, according to a well-known variant of Runge's approximation theorem [8, p. 53], these polynomials are dense in $A(O)$. Hence, the associativity holds in $A(O)$.

Let now $\widetilde{f}$, $\widetilde{g}$, $\widetilde{h}\in A(\overline{D})$ and f, g, h are their representatives in a space $A(O)$ with some $O=G\times\Omega$. Then, by (8), we have

$$(\widetilde{f}*\widetilde{g})*\widetilde{h}=(\overline{f*g})*\widetilde{h}=\overline{(f*g)*h}=\overline{f*(g*h)}=\widetilde{f}*(\overline{g*h})=\widetilde{f}*(\widetilde{g}*\widetilde{h}).$$

Hence, the associativity of the operation $\widetilde{f}*\widetilde{g}$ in $A(\overline{D})$ is proved.

Since $l_i\widetilde{f}=\overline{l_if}$, then it is enough to prove the convolution relations

(15) $$l_i(\widetilde{f}*\widetilde{g})=(l_i\widetilde{f})*\widetilde{g}$$

in a space $A(O)$ with $O=G\times\Omega$ only. This could be proved in the same way as the associativity by approximation.

At last, (11) can easily be verified directly.

Henceforth we shall not make any difference between the elements of $A(\bar{D})$ and their representatives from the family $\{A(O)\}_{O\in\mathcal{O}\bar{D}}$. Thus, we instead of (12) shall use directly (10) as a convolution in $A(\bar{D})$. In doing so, no confusion would arise.

Now, we shall find all the multipliers of convolution (10).

Definition 2. *An operator* $T: A(\bar{D})\to A(\bar{D})$ *is said to be a multiplier of the convolution* (10) *iff*

(16) $$(Tf)*g=f*(Tg)$$

holds for all $f, g\in A(\bar{D})$.

Lemma 2. *The polynomials of* u *and* v *are dense in* $A(\bar{D})$.

Proof. If $\tilde{f}\in A(\bar{D})$, then there exists $f\in A(O)$ with $O=G\times\Omega$, where $G\supset\bar{U}$, and $\Omega\supset\bar{V}$ are convex open sets. There exists a polynomial sequence $\{P_n\}_{n=1}^{\infty}$, such that $P_n\xrightarrow{A(O)}f$ and then $\tilde{P}_n=\varphi_O(P_n)\xrightarrow{A(\bar{D})}\varphi_O(f)=\tilde{f}$ since $\varphi_O: A(O)\to A(\bar{D})$ is continuous.

Theorem 2. *The following four propositions are equivalent*:

a) $T: A(\bar{D})\to A(\bar{D})$ *is a multiplier of the convolution* (10).

b) $T: A(\bar{D})\to A(\bar{D})$ *is an operator of the form*

(17) $$Tf=(\partial^2/\partial u\partial v)(r*f) \text{ with } r=T\{1\}\in A(\bar{D}).$$

c) $T: A(\bar{D})\to A(\bar{D})$ *is an operator of the form*

(17′) $$Tf=\varrho_0 f(u,v)+\{\varrho_1(u)\}\overset{(u)}{*}\{f(u,v)\}+\{\varrho_2(v)\}\overset{(v)}{*}\{f(u,v)\}+\{\varrho(u,v)\}*\{f(u,v)\},$$

where $\varrho_0=const$, $\varrho_1(u)\in A(\bar{U})$, $\varrho_2(v)\in A(\bar{V})$ *and* $\varrho(u,v)\in A(\bar{D})$.

d) $T: A(\bar{D})\to A(\bar{D})$ *is a linear continuous operator, commuting with* l_1 *and* l_2 in $A(\bar{D})$.

Proof. a) $\Rightarrow$ b). Let $T: A(\bar{D})\to A(\bar{D})$ be a multiplier of (10). Then $T(1)*f=1*Tf=l_1l_2Tf$, and (17) follows immediately, with $r=T\{1\}$.

b) $\Rightarrow$ c). With direct differentiation under the integral sign (using representations (3) and (4)) we get after some elementary algebra, the formula

(18) $$\frac{\partial^2}{\partial u\partial v}(f*g)=\frac{\partial^2 f}{\partial u\partial v}*g+\{\frac{\partial}{\partial u}\Psi_\eta[f(u,\eta)]\}\overset{(u)}{*}\{g(u,v)\}$$
$$+\{\frac{\partial}{\partial v}\Phi_\xi[f(\xi,v)]\}\overset{(v)}{*}\{g(u,v)\}+\Phi_\xi\Psi_\eta\{f(\xi,\eta)\}\,.\,g(u,v).$$

By means of (18), we get at once (17′) with

$$\varrho_0=\Phi_\xi\Psi_\eta\{f(\xi,\eta)\},\ \varrho_1(u)=\Psi_\eta[\frac{\partial}{\partial u}r(u,\eta)],\ \varrho_2(v)=\Phi_\xi[\frac{\partial}{\partial v}r(\xi,v)],\ \varrho(u,v)=\frac{\partial^2 r}{\partial u\partial v}\cdot$$

c) $\Rightarrow$ d). It is evident that each operator of the form (17′) is a continuous linear operator in $A(\bar{D})$. The commutation relations $Tl_1=l_1T$ and $Tl_2=l_2T$

can be verified easily, using the fact that $\overset{(u)}{f*g}$ is a convolution of l_1, $\overset{(v)}{f*g}$ is a convolution of l_2, and $f*g$ is a convolution of both in $A(\bar{D})$.

d) $\Longrightarrow$ a). From the obvious identity $T(1)*\{1\}=\{1\}*T(1)$ and the commutating of T and l_i, $i=1, 2$, it follows

$$T[l_1^p l_2^q\{1\}]*l_1^r l_2^s\{1\}=l_1^p l_2^q\{1\}*T[l_1^r l_2^s\{1\}]$$

for $p, q, r, s=0, 1, 2, \ldots$. Therefore, the multiplier relation (16) holds for functions of the form $l_1^p l_2^q\{1\}=u^p v^q/p!\,q!$, $p, q=0, 1, 2, \ldots$. From the bilinearity of $f*g$ it follows that (16) holds for polynomials. By lemma 2, (16) holds in $A(\bar{D})$. The theorem is proved.

R e m a r k. The multiplier operators $T: A(\bar{D})\to A(\bar{D})$ satisfy not only the relation (16), but the convolution relation (9) too. Indeed, if $f, g\in A(\bar{D})$, then from the chain of identities

$$l_1 l_2 T(f*g)=1*T(f*g)=T(1)*(f*g)$$
$$=(T(1)*f)*g=(1*Tf)*g=1*(Tf*g)=l_1 l_2(Tf*g)$$

it follows $l_1 l_2 T(f*g)=l_1 l_2(Tf*g)$. By applying $\partial^2/\partial u\partial v$ we get $T(f*g)=(Tf)*g$.

2. A comutant of $\partial/\partial u$ and $\partial/\partial v$. Our next aim is to find all continuous linear operators $T: A(\bar{D})\to A(\bar{D})$ which commute with $\partial/\partial u$ in A_Φ and with $\partial/\partial v$ in A_Ψ, where A_Φ and A_Ψ are invariant subspaces of $A(\bar{D})$ defined by

$$A_\Phi \overset{\text{def}}{=} \{f\in A(\bar{D}): \Phi_u\{f(u, v)\}=0 \quad \text{for} \quad v\in \bar{V}\} \tag{19}$$

and

$$A_\Psi \overset{\text{def}}{=} \{f\in A(\bar{D}): \Psi_v\{f(u, v)\}=0 \quad \text{for} \quad u\in \bar{U}\}. \tag{20}$$

The solution is given by the following theorem.

T h e o r e m 3. *A continuous linear operator $T: A(\bar{D})\to A(\bar{D})$ with invariant subspaces A_Φ and A_Ψ commutes with $\partial/\partial u$ in A_Φ and with $\partial/\partial v$ in A_Ψ iff T is a multiplier of convolution* (10), (*and hence it can be represented by* (17) *or* (17')).

P r o o f. First, we shall show that the two conditions $T[A_\Phi]\subset A_\Phi$ and $(\partial/\partial u)Tf=T(\partial/\partial u)f$ for $f\in A_\Phi$ are equivalent to the condition $Tl_1=l_1T$ in $A(\bar{D})$. Let $g\in A(\bar{D})$, then $l_1 g\in A_\Phi$ and hence $(\partial/\partial u)Tl_1 g=T(\partial/\partial u)l_1 g=Tg$. Since $Tl_1 g\in A_\Phi$, then $l_1 Tg=l_1(\partial/\partial u)Tl_1 g=Tl_1 g-\Phi_u\{Tl_1 g\}=Tl_1 g$. Conversely, let $l_1T=Tl_1$ in $A(\bar{D})$. If $f\in A_\Phi$, then there exists a $g\in A(\bar{D})$, such that $f=l_1 g$. Then $Tf=l_1 Tg\in A_\Phi$. Hence $T[A_\Phi]\subset A_\Phi$. With the same f we get $(\partial/\partial u)Tf=(\partial/\partial u)Tl_1 g=(\partial/\partial u)l_1 Tg=Tg=T(\partial/\partial u)f$. Analogously, the conditions $T[A_\Psi]\subset A_\Psi$ and $(\partial/\partial v)Tf=T(\partial/\partial v)f$ for $f\in A_\Psi$ are equivalent to $l_2T=Tl_2$ in $A(\bar{D})$.

Hence, according to theorem 2, T is a multiplier of the convolution (10).

3. Coefficient multipliers of Gromov-Leontiev's expansion. In this section we suppose that the functionals Φ and Ψ are defined by (3) and (4) in the special case, when $\gamma(u)$ and $\delta(v)$ are the Borel transforms of the given entire functions $L(\lambda)$ and $M(\mu)$ (see the Introduction). It is well known that [9, p. 24]:

$$L(\lambda)=\Phi_u\{e^{\lambda u}\}=\frac{1}{2\pi i}\int_\Gamma \gamma(u)e^{\lambda u}du,$$

(21)

$$M(\mu)=\Psi_v\{e^{\mu v}\}=\frac{1}{2\pi i}\int_\Delta \delta(v)e^{\mu v}dv.$$

In [5] is shown that this choice of the functionals Φ and Ψ is closely connected with the exponent systems $\{e^{\lambda_l u}\}_{l=1}^\infty$ and $\{e^{\mu_m v}\}_{m=1}^\infty$ by the formulas

(22) $$e^{\lambda_l u}\overset{(u)}{*}e^{\lambda_i u}=\begin{cases} L'(\lambda_l)e^{\lambda_l u}, & l=i;\\ 0, & l\neq i:\end{cases} \quad\text{and}\quad e^{\mu_m v}\overset{(u)}{*}e^{\mu_j v}=\begin{cases} M'(\mu_m)e^{\mu_m v}, & m=j;\\ 0, & m\neq j.\end{cases}$$

It happens that such "convolutional orthogonality" holds for the two-dimensional exponent systems too.

Theorem 4. *The exponent system* $\{e^{\lambda_l u+\mu_m v}\}_{l,m=1}^\infty$ *is convolutionally orthogonal with respect to convolution* (10), *i. e.*

(23) $$\{e^{\lambda_l u+\mu_m v}\}*\{e^{\lambda_i u+\mu_j v}\}=\begin{cases} L'(\lambda_l)M'(\mu_m)e^{\lambda_l u+\mu_m v}; & l=i,\ m=j;\\ 0; & (l,m)\neq(i,j).\end{cases}$$

Proof. From (14) it follows $\{e^{\lambda_l u+\mu_m v}\}*\{e^{\lambda_i u+\mu_j v}\}=[e^{\lambda_l u}\overset{(u)}{*}e^{\lambda_i u}][e^{\mu_m v}\overset{(v)}{*}e^{\mu_j v}]$. Then, using (22), we get (23).

Lemma 3. *If* $f(u,v)\in A(\bar D)$ *and* λ, μ *are complex numbers, then*

(24) $$f*\{e^{\lambda u+\mu v}\}=e^{\lambda u+\mu v}\{L(\lambda)M(\mu)\int_0^u\int_0^v e^{-\lambda\sigma-\mu\tau}f(\sigma,\tau)d\sigma d\tau$$

$$-L(\lambda)\int_0^u\Psi_\eta[e^{\mu\eta}\int_0^\eta e^{-\lambda\sigma-\mu\tau}f(\sigma,\tau)d\tau]d\sigma-M(\mu)\int_0^v\Phi_\xi[e^{\lambda\xi}\int_0^\xi e^{-\lambda\sigma-\mu\tau}f(\sigma,\tau)d\sigma]d\tau$$

$$+\Phi_\xi\Psi_\eta[\int_0^\xi\int_0^\eta e^{\lambda\sigma+\mu\tau}f(\xi-\sigma,\eta-\tau)d\sigma d\tau]\}.$$

Proof. It is easy to verify the identities

(25) $$f\overset{(u)}{*}e^{\lambda u}=e^{\lambda u}\{L(\lambda)\int_0^u e^{-\lambda\sigma}f(\sigma,v)d\sigma-\Phi_\xi[\int_0^\xi e^{\lambda\sigma}f(\xi-\sigma,v)d\sigma]\}$$

and

(26) $$f\overset{(v)}{*}e^{\mu v}=e^{\mu v}\{M(\mu)\int_0^v e^{-\mu\tau}f(u,\tau)d\tau-\Psi_\eta[\int_0^\eta e^{\mu\tau}f(u,\eta-\tau)d\tau]\}.$$

From the evident identity

(27) $$f*\{e^{\lambda u+\mu v}\}=[f(u,v)\overset{(u)}{*}e^{\lambda u}]\overset{(v)}{*}e^{\mu v}$$

and from (25) and (26) we get at once (24).

Theorem 5. *If* λ_l *and* μ_m *are zeros of* $L(\lambda)$ *and* $M(\mu)$ *respectively, then*

(28) $$f*\{e^{\lambda_l u+\mu_m v}\}=\omega_{l,m}(f)\,e^{\lambda_l u+\mu_m v}$$

where

$$\omega_{l,m}(f)=\Phi_{\xi}\Psi_{\eta}\{\int_0^{\xi}\int_0^{\eta} e^{\lambda_l\sigma+\mu_m\tau}f(\xi-\sigma,\ \eta-\tau)d\sigma d\tau\} \tag{29}$$

are continuous linear functionals in $A(\bar{D})$.

Proof. Since $L(\lambda_l)=M(\mu_m)=0$, then (28) follows immediately from (24). As in theorem 1 we can prove that for an arbitrary $O\in\mathcal{O}^{\bar{D}}$ of the form $O=G\times\Omega$, $G\in\mathcal{O}^{\bar{U}}$, $\Omega\in\mathcal{O}^{\bar{V}}$ the linear functionals $\omega_{l,m}\circ\varphi_O$ (given also by (29)) are continuous with respect to the compact topology in $A(O)$. Hence $\omega_{l,m}(f)$ is continuous in $A(\bar{D})$.

Theorem 6. *The continuous linear functionals* $\omega_{l,m}(f)$; $l, m=1, 2,\ldots$ *ale multiplicative in* $A(\bar{D})$ *with respect to the convolution* $f*g$, i. e.

$$\omega_{l,m}(f*g)=\omega_{l,m}(f)\omega_{l,m}(g),\qquad l,m=1, 2,\ldots \tag{30}$$

Proof. Using (23) and (28) we get

$$L'(\lambda_l)M'(\mu_m)\omega_{l,m}(f)\omega_{l,m}(g)e^{\lambda_l u+\mu_m v}=\omega_{l,m}(f)\omega_{l,m}(g)[e^{\lambda_l u+\ \ m^v}*e^{\lambda\ u+\mu_m v}]$$
$$=[f*\{e^{\lambda_l u+\mu_m v}\}]*[g*\{e^{\lambda_l u+\mu_m v}\}]=(f*g)*[e^{\lambda_l u+\mu_m v}*e^{\lambda_l u+\mu_m v}]$$
$$=L'(\lambda_l)M'(\mu_m)(f*g)*\{e^{\lambda_l u+\mu_m v}\}=L'(\lambda_l)M'(\mu_m)\omega_{l,m}(f*g)\,e^{\lambda_l u+\mu_m v}$$

and (30) holds.

Definition 3. *If* $f(u, v)\in A(\bar{D})$, *then a formal Gromov-Leontiev's expansion of* f *on the exponent system* $\{e^{\lambda_l u+\mu_m v}\}_{l,m=1}^{\infty}$ *is said to be the corre - pondence*

$$f(u,\ v)\sim\sum_{l,m=1}^{\infty} a_{l,m}(f)^{\lambda_l u+\mu_m v} \tag{31}$$

with the coefficients

$$a_{l,m}(f)=\omega_{l,m}(f)/L'(\lambda_l)M'(\mu_m). \tag{32}$$

From (30) *now it is clear that*

$$a_{l,m}(f*g)=L'(\lambda_l)M'(\mu_m)a_{l,m}(f)a_{l,m}(g). \tag{33}$$

Remark. The functionals $a_{l,m}(f)$ are exactly the coefficients found by V. P. Gromov [1], since the functional in (28):

$$\Phi_{\xi}\Psi_{\eta}\{f(\xi,\ \eta)\}=-\frac{1}{4\pi^2}\int_{\Gamma}\int_{\Delta}f(\xi,\ \eta)\gamma(\xi)\delta(\eta)d\xi d\eta.$$

Now we shall prove a uniqueness theorem for Gromov-Leontiev's expansion.

Theorem 7. *If* $f\in A(\bar{D})$ *and* $a_{l,m}(f)=0$ *for* $l, m=1, 2,\ldots$, *then* $f(u, v)\equiv 0$.

Proof. It is clear that $a_{l,m}(f)=0$ iff $\omega_{l,m}(f)=0$ or equivalently, $f*\{e^{\lambda_l u+\mu_m v}\}=0$.

We shall use the following two systems of one-dimensional continuous linear functionals $\overset{(1)}{\omega}_l(f)$, $\overset{(2)}{\omega}_m(f)$, defined in $A(\bar{U})$ and $A(\bar{V})$, respectively (see [5]) by

(34) $\overset{(1)}{\omega_l}(f)e^{\lambda_l u} = f \overset{(u)}{*} \{ e^{\lambda_l u}\},\ f \in A(\bar{U})$; $\overset{(2)}{\omega_m}(f)\, e^{\mu_m v} = f \overset{(v)}{*} \{ e^{\mu_m v}\},\quad g \in A(\bar{V})$.

Let $\overset{(1)}{a_l}(f)$ and $\overset{(2)}{a_m}(f)$ are the corresponding coefficients in one-dimensional Dirichlet expansions on the systems $\{e^{\lambda_l u}\}_{l=1}^{\infty}$ and $\{e^{\mu_m v}\}_{m=1}^{\infty}$ in $A(\bar{U})$ and $A(\bar{V})$, respectively. They are

(35) $$\overset{(1)}{a_l}(f) \overset{\text{def}}{=} \overset{(1)}{\omega_l}(f)/L'(\lambda_l),\quad \overset{(2)}{a_m}(f) \overset{\text{def}}{=} \overset{(2)}{\omega_m}(f)/M'(\mu_m),\quad l,\ m = 1,\ 2, \ldots$$

Let $F_l(u, v) \overset{\text{def}}{=} f(u, v) \overset{(u)}{*} e^{\lambda_l u}$, $l = 1, 2, \ldots$. Then, by (27), it follows that $0 = f * \{e^{\lambda_l u + \mu_m v}\} = \{F_l(u, v)\} \overset{(v)}{*} e^{\mu_m v} = [\overset{(2)}{\omega_m}]_v\ [F_l(u, v)]e^{\mu_m v}$, i. e. $[\overset{(2)}{\omega_m}]_v[F_l(u, v)] = 0$, $m = 1, 2, \ldots$ for $u \in \bar{U}$ and for fixed l. Then, according to A. F. Leontiev's uniqueness theorem (see [9, p. 255]) for the one-variable case, we obtain $F_l(u, v) = 0$, $l = 1, 2, \ldots, u \in \bar{U}$ for each $v \in \bar{V}$, i. e. $[\overset{(1)}{\omega_l}]_u[f(u, v)] = 0$ for each $v \in \bar{V}$, $l = 1, 2, \ldots$. Using again A. F. Leontiev's uniqueness theorem, we conclude that $f(u, v) \equiv 0$.

Definition 4. *An operator $T: A(\bar{D}) \to A(\bar{D})$ is said to be a coefficient multiplier of the formal Gromov-Leontiev's expansion* (31), *iff there exists a double numerical sequence $\{\tau_{l,m}\}_{l,m=1}^{\infty}$ such that*

(36) $$a_{l,m}(Tf) = \tau_{l,m} a_{l,m}(f) \text{ for each } f \in A(\bar{D}),\ l,\ m = 1,\ 2, \ldots$$

Of course, the sequence $\{\tau_{l,m}\}$ should not depend on the function f.

Theorem 8. *An operator $T: A(\bar{D}) \to A(\bar{D})$ is a coefficient multiplier af Gromov-Leontiev's expansion iff it is a multiplier of the corresponding convolution $f*g$, i. e. T is a continuous linear operator, commuting with l_1 and l_2, and having representations of the form* (17) *or* (17′).

Proof. The functionals $\overset{(1)}{\omega_l}(f)$, $\overset{(2)}{\omega_m}(f)$, defined by (34) have the following property

(37) $$\omega_{l,m}[f_1(u) f_2(v)] = \overset{(1)}{\omega_l}(f_1)\overset{(2)}{\omega_m}(f_2) \text{ for } f_1 \in A(\bar{U}) \text{ and } f_2 \in A(\bar{V}).$$

As it is shown in [5], the functionals $\overset{(1)}{\omega_l}(f)$ and $\overset{(2)}{\omega_m}(f)$ are multiplicative in $A(\bar{U})$ and $A(\bar{V})$ with respect to the convolutions $f \overset{(u)}{*} g$ and $f \overset{(v)}{*} g$, respectively

Using (17′), (33), (35) and (37), it is not difficult to prove that if $T: A(\bar{D}) \to A(\bar{D})$ is a multiplier of the convolution $f*g$, then the identity

(38) $$a_{l,m}(Tf) = [\varrho_0 + L'(\lambda_l)\overset{(1)}{a_l}(\varrho_1) + M'(\mu_m)\overset{(2)}{a_m}(\varrho_2) + L'(\lambda_l)M'(\mu_m)a_{l,m}(\varrho)]a_{l,m}(f)$$

holds for $f(u, v) = f_1(u)f_2(v)$, $f_1 \in A(\bar{U})$, $f_2 \in A(\bar{V})$. Then, by approximation, it follows that (38) holds for an arbitrary $f \in A(\bar{D})$. Hence, each multiplier of $f*g$ is a coefficient multiplier of Gromov-Leontiev's expansion too.

Conversely, let $T: A(\bar{D}) \to A(\bar{D})$ be a coefficient multiplier, and let us form the function $h = Tf*g - f*Tg$ with $f,\ g\ A(\bar{D})$. Then

$$a_{l,m}(h) = a_{l,m}[Tf*g] - a_{l,m}[f*Tg] = L'(\lambda_l)M'(\mu_m)\{a_{l,m}(Tf)a_{l,m}(g) - a_{l,m}(f)a_{l,m}(Tg)\}$$
$$= L'(\lambda_l)M'(\mu_m)\{\tau_{l,m}a_{l,m}(f)a_{l,m}(g) - \tau_{l,m}a_{l,m}(f)a_{l,m}(g)\} = 0$$

and by our uniqueness theorem 7, it follows that $h \equiv 0$, q. e. d.

4. Coefficient convolutions of Gromov-Leontiev's expansion.

Definition 5. *A binary operation* $\tilde{*}: A(\bar{D}) \to A(\bar{D})$ *is said to be a coefficient convolution of Gromov-Leontiev's expansion* (31), *iff there exists a sequence* $\{\sigma_{l,m}\}_{l,m=1}^{\infty}$, *such that*

(39) $$a_{l,m}(f \tilde{*} g) = \sigma_{l,m} a_{l,m}(f) a_{l,m}(g) \text{ for all } f, g \in A(\bar{D})$$

$l, m = 1, 2, \ldots$ *The sequence* $\{\sigma_{l,m}\}_{l,m=1}^{\infty}$ *should not depend on the functions* f *and* g.

Theorem 9. *If an operation* $\tilde{*}: A(\bar{D}) \times A(\bar{D}) \to A(\bar{D})$ *is a coefficient convolution of Gromov-Leontiev's expansion, then it is continuous, bilinear, commutative and associative operation in* $A(\bar{D})$ *with a representation of the form*

(40) $$f \tilde{*} g = \left(\frac{\partial^2}{\partial u \partial v}\right)^2 (r * f * g) \text{ with } r = \{1\} \tilde{*} \{1\} \in A(\bar{D}).$$

Conversely, for each $r \in A(\bar{D})$, *the operation* $f \tilde{*} g$, *defined by* (40), *is a coefficient convolution of Gromov-Leontiev's expansion.*

Proof. The proofs ot bilinearity, commutativity and associativity proceed much in one and the same way. Let us prove e. g. the associativity. If $k = (f \tilde{*} g) \tilde{*} h - f \tilde{*} (g \tilde{*} h)$, then by (39), we get $a_{l,m}(k) = 0$, $l, m = 1, 2, \ldots$ Hence, from theorem 7, $k \equiv 0$. The convolution $f * g$ is a coefficient convolution too (see (33)). Now, using (33) and (39), as in the proof of the associativity, we can check easily the identity $(1*1)*(f \tilde{*} g) = (1 \tilde{*} 1)*(f*g)$, i. e. $(l_1 l_2)^2 (f \tilde{*} g) = r*f*g$ with $r = 1 \tilde{*} 1$. Thus, we get (40). From explicit representation (40) the continuity of $f \tilde{*} g$ in $A(\bar{D})$ follows immediately.

Conversely, let $r \in A(\bar{D})$ be arbitrary. We need the following easy for verifying properties of the convolution (10):

1) For arbitrary $f, g \in A(\bar{D})$, the relations

(41) $\Phi_u\{(f*g)(u, v)\} = 0, \quad v \in \bar{V}$ and $\Psi_v\{(f*g)(u, v)\} = 0$, $u \in \bar{U}$ hold.

Indeed, if we denote shortly

$$h(u, \xi, v, \eta) \overset{\text{def}}{=} \int_{\xi}^{u} \int_{\eta}^{v} f(u + \xi - \sigma, v + \eta - \tau) g(\sigma, \tau) d\sigma \, d\tau$$

it is clear that $h(u, \xi, v, \eta) = -h(\xi, u, v, \eta)$ and from (10) we obtain

$$\Phi_u\{f*g(u, v)\} = \Phi_u \Phi_\xi \Psi_\eta\{h(u, \xi, v, \eta)\} = -\Phi_\xi \Phi_u \Psi_\eta\{h(\xi, u, v, \eta)\} = -\Phi_\xi\{f*g(\xi, v)\}.$$

Hence the first of identities (41) holds. The second follows in the same way.

2) In the same way as in [5] it is easy to prove the following two "operational" relations:

(42) $$a_{l,m}\left[\frac{\partial}{\partial u} f(u, v)\right] = \lambda_l a_{l,m}(f) \text{ when } \Phi_u\{f(u, v)\} = 0, \ v \in \bar{V}$$

and

(43) $$a_{l,m}\left[\frac{\partial}{\partial v} f(u, v)\right] = \mu_m a_{l,m}(f) \text{ when } \Psi_v\{f(u, v)\} = 0, \ u \in \bar{U}.$$

Hence, for each $f \in A(\overline{D})$ with $\Phi_u\{f(u, v)\}=0$, $v \in V$, $\Psi_v\{f(u, v)\}=0$, $u \in \overline{U}$ we have

$$a_{l,m}\left[\frac{\partial^2}{\partial u \partial v} f\right]=\lambda_l \ \mu_m a_{l,m}(f). \tag{44}$$

From (18), (40) and (41), we get at once

$$f \tilde{*} g=\frac{\partial^2}{\partial u \partial v}\left[r*\frac{\partial^2}{\partial u \partial v}(f*g)\right]. \tag{45}$$

At last, by some elementary algebra, using (33), (44) and (45), we get

$$a_{l,m}(f \tilde{*} g)=\{[\lambda_l L'(\lambda_l) M'(\mu_m)]^2 a_{l,m}(r)\} a_{l,m}(f) a_{l,m}(g). \tag{46}$$

Hence the operation $f \tilde{*} g$, defined by (40) with an arbitrary $r \in A(\overline{D})$ is a coefficient convolution of Gromov-Leontiev's expansion.

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