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БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

APPROXIMATION OF FUNCTIONS BY MEANS OF LINEAR SUMMATION BASKAKOV'S OPERATORS IN L_p

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We obtain an estimation for the approximation in L_p by means of Baskakov's operators using the moduli $\tau_k(f;\delta)_{L_p}$.

The problem of approximation of functions by means of linear summation operators in L_p-metric is considered in [1]. In the case when we approximate the p-integrable function by means of Bernstein's and Szasz-Mirakian's operators the estimations given in [1] are obtained by the following moduli:

$$\tau_k(f;\delta)_{L_p[a,b]}=\|\omega_k(f,x;\delta)\|_{L_p},\quad 1\leq p<\infty,$$

where

$$\omega_k(f,x;\delta)=\sup\{|\Delta_h^k f(t)|,\ t,\ t+kh\in[x-k\delta/2,\ x+k\delta/2]\cap[a,b]\},$$

$$\Delta_h^k f(t)=\sum_{n=0}^{k}(-1)^{k+n}\binom{k}{n}f(t+nh).$$

It is well known that the Bernstein's and the Szasz-Mirakian's operators are special case of Baskakov's operators [4]:

$$B_n^*(f;x)=\sum_{k=0}^{\infty}f\left(\frac{k}{n}\right)\frac{\varphi_n^{(k)}(x)}{k!}(-x)^k, \tag{1}$$

where the sequence of functions $\{\varphi_n(x)\}_{n=1}^{\infty}$ satisfies the following conditions:

a) every function $\varphi_n(x)$ is analytic in $|x-a|\leq a,\ a>0$;

b) $\varphi_n(0)=1,\quad n=1,2,3,\dots$;

c) $(-1)^k\varphi_n^{(k)}(x)\geq 0,\quad k=0,1,2,\dots,\quad 0\leq x\leq a$;

d) $-\varphi_n'(x)=n\varphi_{m_n},\quad m_n/n\to 1$ for $n\to\infty$;

Let us add a new condition on the functions $\{\varphi_n(x)\}_{n=1}^{\infty}$:

e) there exists a natural number μ such that $\varphi_{n+\mu}'(x)=-n\varphi_n(x),\ n=1,2,\dots$

The purpose of this paper is to find an estimation for the approximation in L_p by means of Baskakov's operators. This estimation will make use of the modulus $\tau_2(f;\delta)_{L_p}$, see [2], [3].

The most essential properties of $\tau_k(f;\delta)_{L_p}$ are:

a) $$\omega_k(f;\delta)_{L_p}\leq\tau_k(f;\delta)_{L_p}\leq\omega_k(f;\delta),$$

where $\omega_k(f;\delta)_{L_p}=\sup\{(\int_a^{b-kh}|\Delta_h^k f(x)|^p\,dx)^{1/p},\ 0<h\leq\delta\}$,

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(2) $\omega_k(f;\ \delta)=\sup\{\omega_k(f,\ x;\ \delta),\quad a\leqq x\leqq b\}$;

b) $\tau_k(f;\ \delta)_{L_p}\leqq c(k)\delta\tau_{k-1}(f';\ \delta)_{L_p}$;

c) $\tau_k(f;\ m\delta)_{L_p}\leqq(2m)^{k+1}\tau_k(f;\ \delta)_{L_p}$, $m>0$ is an integer;

d) $\tau_k(f;\ \delta)_{L_p}\leqq\|f'\|_{L_p}$;

e) $\tau_1(f;\ \delta)_{L_p}\leqq 2\delta\bigvee_a^b f'$,

where $\bigvee_a^b f$ denotes the variation of the function f in the interval $[a,\ b]$.

The following interpolation theorem [1] is basic for our considerations:

Theorem 1. *Let L_n be a linear operator which satisfies:*

a) *if the bounded function $f\in L_p$, then $L_n(f;\ x)\in L_p$, $x\in[a,\ b]$;*

b) $\|L_n(f;\ .)\|_{L_p}\leqq K\|f\|_{l^p_{\Sigma_n}}$, *where K is an absolute constant and*

$$\|f\|_{l^p_{\Sigma_n}}=\{\sum_{i=1}^{n}|f(x_i)|^p\Delta_i\}^{1/p},\ \Delta_i=x_{i+1}-x_{i-1},\ x_0=a,\ x_{n+1}=b;$$

c) *if $f\in W_p^r(f\in W_p^r$ if $f^{(r-1)}$ is absolutely continuous and $f^{(r)}\in L_p)$ then*

$$\|L_n(f;\ .)-f\|_{L_p}\leqq K_r d_n^s\|f^{(r)}\|_{L_p},\quad s\leqq r,$$

where $d_n=\max_{1\leqq i\leqq n}\Delta_i$ and K_r is an absolute constant which may depend only on r.

Then for every p-integrable bounded function f in $[a,\ b]$ the following estimation holds true for $d_n\leqq 1$:

$$\|L_n(f;\ .)-f\|_{L_p}\leqq C\tau_r(f;\ d_n^{s/r})_{L_p}.$$

The constant C depends only on r, K, K_r.

It is easy to see that Theorem 1 remains true by the following additional condition on the functions f:

f — is p-integrable bounded function on $[0,\ a]$ and $f(x)=0$ for $x\geqq a$.

We shall prove that the Baskakov's operators satisfy the assumptions of Theorem 1.

Lemma 1. *If f is a bounded function in* $[0,\ a]$, $a>0$, *then*

$$\|B_n^*(f;\ .)\|_{L_p[0,\ a]}\leqq\|f\|_{l^p_{\Sigma_n}},$$

$$\Sigma_n=\{i/n,\ i=0,\ 1,\ldots,\ k,\ k/n\leqq a<(k+1)/n\}.$$

Proof. We set $f(x)=0$ for $x\geqq a$. Then we have

$$\|B_n^*(f;\ .)\|_{L_p[0,\ a]}=$$

$$=\{\int_0^a|\sum_{i=0}^{k}f\Big(\frac{i}{n}\Big)\frac{\varphi_n^{(i)}(x)(-x)^i}{i!}|^p dx\}^{1/p}\leqq\{\int_0^a[\sum_{i=0}^{k}|f(\frac{i}{n})|\,|\frac{\varphi_n^{(i)}(x)(-x)^i}{i!}|]^p dx\}^{1/p}.$$

Using Jensen's inequality, we obtain

$$\|B_n^*(f;\ .)\|_{L_p[0,\ a]}\leqq\{\int_0^a[\sum_{i=0}^{k}|f(\frac{i}{n})|^p\frac{\varphi_n^{(i)}(x)(-x)^i}{i!}]dx\}^{1/p}$$

$$=\{\sum_{i=0}^{k}\int_0^a|f(\frac{i}{n})|^p\frac{\varphi_n^{(i)}(x)(-x)^i}{i!}dx\}^{1/p}=\{\sum_{i=0}^{k}|f(\frac{i}{n})|^p\int_0^a\frac{\varphi_n^{(i)}(x)(-x)^i}{i!}dx\}^{1/p}.$$

Further we have

$$J=\int_0^a \frac{\varphi_n^{(i)}(x)(-x)^i}{i!}\,dx=\int_0^a \frac{(-x)^i}{i!}\,d\varphi_n^{(i-1)}(x)$$

$$=\frac{\varphi_n^{(i-1)}(x)(-x)^i}{i!}\Big|_0^a-\int_0^a \frac{\varphi_n^{(i-1)}(x)}{i!}\,d(-x)^i=\frac{\varphi_n^{(i-1)}(a)\,(-a)^i}{i!}+\int_0^a \frac{\varphi_n^{(i-1)}(x)\,(-x)^{i-1}}{(i-1)!}\,dx$$

$$=\frac{\varphi_n^{(i-1)}(a)\,(-a)^i}{i!}+\frac{\varphi_n^{(i-2)}(a)\,(-a)^{i-1}}{(i-1)!}+\int_0^a \frac{\varphi_n^{(i-2)}(x)\,(-x)^{i-2}}{(i-2)!}\,dx$$

$$=\cdots=\sum_{k=1}^{i}\frac{\varphi_n^{(k-1)}(a)\,(-a)^k}{k!}+\int_0^a \varphi_n(x)dx.$$

We use that $\varphi'_{n+\mu}(x)=-n\varphi_n(x)$, $n=1, 2, \dots$ or $\varphi_n(x)=-\varphi'_{n+\mu}(x)/n$. Then $\varphi_n^{(k-1)}(x)=-\varphi_{n+\mu}^{(k)}(x)/n$ and

$$J=-\frac{1}{n}\sum_{k=1}^{i}\frac{\varphi_{n+\mu}^{(k)}(a)\,(-a)^k}{k!}-\frac{1}{n}\int_0^a \varphi'_{n+\mu}(x)dx$$

$$=-\frac{1}{n}\sum_{k=1}^{i}\frac{\varphi_{n+\mu}^{(k)}(a)\,(-a)^k}{k!}-\frac{1}{n}\,\varphi_{n+\mu}(a)+\frac{1}{n}\,\varphi_{n+\mu}(0)$$

$$=-\frac{1}{n}\sum_{k=0}^{i}\frac{\varphi_{n+\mu}^{(k)}(a)\,(-a)^k}{k!}+\frac{1}{n}\,\varphi_{n+\mu}(0).$$

It is easy to see that $\varphi_{n+\mu}(0)=1$ and

$$\sum_{k=0}^{i}\frac{\varphi_{n+\mu}^{(k)}(a)(-a)^k}{k!}\leqq\sum_{k=0}^{\infty}\frac{\varphi_{n+\mu}^{(k)}(a)(-a)^k}{k!}=B_n^*(1\,;\;x)=1.$$

We thus obtain $|J|\leqq|-1/n|+|1/n|=2/n$ and

$$\|B_n^*(f\,;\;.\,)\|_{L_p[0,\,a]}\leqq\Big\{\sum_{i=0}^{k}|f(\tfrac{i}{n})|^p\,\frac{2}{n}\Big\}^{1/p}\leqq\|f\|_{l^p_{\Sigma_n}}.$$

Lemma 2. *Let f be a bounded function in $[0, \infty)$ and $f\in W_p^2$ in $[0, \infty)$. Let $f(x)=0$ for $x\geqq a$. Then*

$$\|B_n^*(f;\,.\,)-f\|_{L_p[0,\,a]}\leqq c(a)\|f''\|_{L_p[0,\,a]}\,n^{-1}.$$

Proof. The Taylor's formula gives for t, $x\in[0, a]$:

$$f(t)=f(x)+(t-x)f'(x)+\int_x^a (t-\theta)_+f''(\theta)d\theta+\int_0^x(\theta-t)_+f''(\theta)d\theta,$$

$$B_n^*(f(t)\,;\;x)=B_n^*(f(x)\,;\;x)+B_n^*(tf'(x)\,;\;x)+B_n^*(-xf'(x)\,;\;x)$$

$$+\int_x^a B_n^*((t-\theta)_+\,;\;x)f''(\theta)\,d\theta+\int_0^x B_n^*((\theta-t)_+\,;\;x)f''(\theta)d\theta \tag{3}$$

$$=f(x)B_n^*(1\,;\;x)+f'(x)B_n^*(t\,;\;x)-xf'(x)B_n^*(1\,;\;x)+\int_0^a N(x\,;\;\theta)f''(\theta)d\theta,$$

where

$$N(x;\ \theta)=\begin{cases} B_n^*((t-\theta)_+;\ x), & \theta\geqq x,\\ B_n^*((\theta-t)_+;\ x), & \theta<x.\end{cases}$$

Using that $B_n^*(1;\ x)=1$ and $B_n^*(t;\ x)=x$ we obtain $B_n^*(f(t);\ x)-f(x)=\int_0^a N(x;\ \theta)f''(\theta)d\theta$. Then

$$\|B_n^*(f;\ .)-f\|_{L[0,\ a]}=\int_0^a |B_n^*(f;\ .)-f|dx$$

(4) $$=\int_0^a \Big|\int_0^a N(x;\ \theta)f''(\theta)d\theta\Big|dx\leqq\int_0^a |f''(\theta)|\int_0^a N(x;\ \theta)dxd\theta$$

$$\leqq \max_{0\leqq\theta\leqq a}\int_0^a N(x;\ \theta)dx\int_0^a |f''(\theta)|d\theta=\|f''\|_{L[0,\ a]}\max_\theta\int_0^a N(x;\ \theta)\,dx.$$

We shall estimate $\int_0^a N(x;\ \theta)dx$. Setting $q_{n,k}(x)=\varphi_n^{(k)}(x)(-x)^k/k!$ we have

$$\int_0^a N(x;\ \theta)\,dx=\int_0^\theta B_n^*((t-\theta)_+;\ x)dx+\int_\theta^a B_n^*((\theta-t)_+;\ x)dx$$

(5) $$=\int_0^\theta \Big[\sum_{k=0}^\infty \Big(\frac{k}{n}-\theta\Big)_+ q_{n,k}(x)\Big]dx+\int_\theta^a \Big[\sum_{k=0}^\infty \Big(\theta-\frac{k}{n}\Big)_+ q_{n,k}(x)\Big]dx$$

$$=\sum_{k=0}^\infty \Big\{\Big(\frac{k}{n}-\theta\Big)_+\int_0^\theta q_{n,k}(x)dx+\Big(\theta-\frac{k}{n}\Big)_+\int_\theta^a q_{n,k}(x)dx\Big\}.$$

We shall prove that:

(6) $$\int_0^\theta q_{n,k}(x)dx=\frac{1}{n}\sum_{i=k+1}^\infty q_{n,i}(\theta),\quad \int_\theta^a q_{n,k}(x)dx\leqq\frac{1}{n}\sum_{i=0}^k q_{n,i}(\theta).$$

Indeed, we have

$$\int_0^\theta q_{n,k}(x)dx=\int_0^\theta \frac{\varphi_n^{(k)}(x)(-x)^k}{k!}dx=-\frac{1}{(k+1)!}\int_0^\theta \varphi_n^{(k)}(x)d(-x)^{k+1}$$

$$=-\frac{\varphi_n^{(k)}(x)(-x)^{k+1}}{(k+1)!}\Big|_0^\theta+\int_0^\theta \frac{(-x)^{k+1}\varphi_n^{(k+1)}(x)}{(k+1)!}dx=-\frac{\varphi_n^{(k)}(\theta)(-\theta)^{k+1}}{(k+1)!}-\int_0^\theta \frac{\varphi_n^{(k+1)}(x)d(-x)^{k+2}}{(k+2)!}$$

$$=-\frac{\varphi_n^{(k)}(\theta)(-\theta)^{k+1}}{(k+1)!}-\frac{\varphi_n^{(k+1)}(\theta)(-\theta)^{k+2}}{(k+2)!}+\int_0^\theta \frac{\varphi_n^{(k+2)}(x)(-x)^{k+2}}{(k+2)!}dx$$

$$=\cdots=-\sum_{\nu=k}^\infty \frac{\varphi_n^{(\nu)}(\theta)(-\theta)^{\nu+1}}{(\nu+1)!}=-\sum_{\nu=k}^\infty \Big(-\frac{1}{n}\Big)\varphi_{n+\mu}^{(\nu+1)}(\theta)(-\theta)^{\nu+1}/(\nu+1)!$$

$$=\frac{1}{n}\sum_{i=k+1}^\infty q_{n,i}(\theta).$$

Similarly,

$$\int_\theta^a \frac{\varphi_n^{(k)}(x)(-x)^k}{k!}dx = \frac{(-x)^k\varphi_n^{(k-1)}(x)}{k!}\Big|_\theta^a + \int_\theta^a \frac{\varphi_n^{(k-1)}(x)(-x)^{k-1}}{(k-1)!}dx$$

$$= \cdots = \sum_{\nu=1}^{k} \frac{\varphi_n^{(\nu-1)}(x)\,(-x)^\nu}{\nu!}\Big|_\theta^a + \int_\theta^a \varphi_n(x)dx$$

$$= \sum_{\nu=1}^{k}(-\frac{1}{n})\varphi_{n+\mu}^{(\nu)}(x)(-x)^\nu/\nu!\,\Big|_\theta^a - \frac{1}{n}\int_\theta^a \varphi'_{n+\mu}(x)dx$$

$$= -\frac{1}{n}\sum_{\nu=1}^{k}\frac{\varphi_{n+\mu}^{(\nu)}(x)(-x)^\nu}{\nu!} - \frac{1}{n}[\varphi_{n+\mu}(a)-\varphi_{n+\mu}(\theta)]$$

$$= -\frac{1}{n}\Big[\sum_{\nu=0}^{k}\frac{\varphi_{n+\mu}^{(\nu)}(a)\,(-a)^\nu}{\nu!} - \sum_{\nu=0}^{k}\frac{\varphi_{n+\mu}^{(\nu)}(\theta)\,(-\theta)^\nu}{\nu!}\Big]$$

$$= \frac{1}{n}\sum_{i=0}^{k} q_{n,i}(\theta) - \frac{1}{n}\sum_{i=0}^{k} q_{n,i}(a) \leq \frac{1}{n}\sum_{i=0}^{k} q_{n,i}(\theta).$$

Thus

$$\int_\theta^a q_{n,k}(x)dx \leqq \frac{1}{n}\sum_{i=0}^{k} q_{n,i}(\theta).$$

If we denote

$$r_k(\theta) = \frac{1}{n}\sum_{i=0}^{k}(\frac{i}{n}-\theta)_+,\quad r_{-1}(\theta)=0,\quad t_k(\theta)=\frac{1}{n}\sum_{i=k}^{\infty}(\theta-\frac{i}{n})_+$$

and use (5) and (6), we obtain

$$\int_0^a N(x\,;\,\theta)dx \leqq \frac{1}{n}\sum_{k=0}^{\infty}\{(\frac{k}{n}-\theta)_+\sum_{i=k+1}^{\infty} q_{n,i}(\theta) + (\theta-\frac{k}{n})_+\sum_{i=0}^{k} q_{n,i}(\theta)\} \tag{7}$$

$$= \sum_{k=0}^{\infty}\{(r_k(\theta)-r_{k-1}(\theta))\sum_{i=k+1}^{\infty} q_{n,i}(\theta) + (t_k(\theta)-t_{k+1}(\theta))\sum_{k=0}^{\infty} q_{n,i}(\theta)\}$$

$$= \sum_{k=0}^{\infty} r_k(\theta)q_{n,k+1}(\theta) + \sum_{k=0}^{\infty} t_k(\theta)q_{n,k}(\theta).$$

It is easy to see, that

$$\begin{aligned} &r_k(\theta)=0 \text{ for } k\leq n\theta,\\ &r_k(\theta)\leq[(k+1)/n-\theta]^2/2+3a/2n \text{ for } k>n\theta,\\ &t_k(\theta)=0 \text{ for } k\geq n\theta,\\ &t_k(\theta)\leqq(k/n-\theta)^2/2+3a/2n \text{ for } k<n\theta.\end{aligned} \tag{8}$$

Indeed, we have

$$r_k(\theta)=\frac{1}{n}\sum_{i=1}^{k}\left(\frac{i}{n}-\theta\right)_+=\frac{1}{n}\sum_{i=[n\theta]+1}^{k}\left(\frac{i}{n}-\theta\right)$$

$$=\frac{1}{n^2}\left(\frac{k(k+1)}{2}-\frac{[n\theta]([n\theta]+1)}{2}\right)-\frac{\theta}{n}(k-[n\theta])$$

$$+\frac{1}{2}\left(\frac{k+1}{n}-\theta\right)^2-\frac{1}{2}\left(\frac{k+1}{n}-\theta\right)^2\leq\frac{1}{2}\left(\frac{k+1}{n}-\theta\right)^2+\frac{3a}{2n},$$

$$t_k(\theta)=\frac{1}{n}\sum_{i=k}^{\infty}\left(\theta-\frac{i}{n}\right)_+=\frac{1}{n}\sum_{i=k}^{[n\theta]}\left(\theta-\frac{1}{n}\right)$$

$$=\frac{\theta}{n}([n\theta]-k+1)-\frac{1}{n^2}\left(\frac{[n\theta]([n\theta]+1)}{2}-\frac{k(k-1)}{2}\right)$$

$$+\frac{1}{2}\left(\frac{k}{n}-\theta\right)^2-\frac{1}{2}\left(\frac{k}{n}-\theta\right)^2\leq\frac{1}{2}\left(\frac{k}{n}-\theta\right)^2+\frac{3a}{2n}.$$

From (7) and (8) we obtain

$$\int_0^a N(x;\ \theta)dx\leq\sum_{k=0}^{\infty}(r_{k-1}(\theta)+t_k(\theta)q_{n,k}(\theta)$$

$$\leq\sum_{k=0}^{\infty}\left(\frac{1}{2}\left(\frac{k}{n}-\theta\right)^2+\frac{3a}{2n}\right)q_{n,k}(\theta)$$

$$=\frac{1}{2}\sum_{k=0}^{\infty}\frac{k^2}{n^2}q_{n,k}(\theta)-\theta\sum_{k=0}^{\infty}\frac{k}{n}q_{n,k}(\theta)+\frac{1}{2}\theta^2\sum_{k=0}^{\infty}q_{n,k}(\theta)+\frac{3a}{2n}\sum_{k=0}^{\infty}q_{n,k}(\theta)$$

$$=\frac{1}{2}\left(x^2\frac{m_n}{n}+\frac{x}{n}-2\theta x+\theta^2\right)+\frac{3a}{2n}$$

$$=\frac{1}{2}(x-\theta)^2+\frac{x}{n}+\frac{3a}{2n}<\frac{a}{2n}+\frac{3a}{2n}<\frac{2a}{n},$$

where we have used, that

$$B_n^*(1;\ x)=1,\ B_n^*(t;\ x)=x,\ B_n^*(t^2;\ x)=x^2m_n/n+x/n,\ m_n/n\to 1.$$

We thus have

$$\max_{0\leq\theta\leq a}\int_0^a N(x;\ \theta)\,dx=\frac{2a}{n}$$

and

(9) $$\|B_n^*(f;\ .)-f\|_{L[0,\ a]}\leq C(a)\|f''\|_{L[0,\ a]}/n.$$

Using (3) we obtain

$$|B_n^*(f;\ x)-f(x)|\leq B_n^*\left(\left(\int_x^a(t-\theta)_+f''(\theta)d\theta+\int_0^x(\theta-t)_+f''(\theta)d\theta\right);\ x\right)$$

$$\leq\|f''\|_C B_n^*\left(\left(\int_x^a(t-\theta)_+d\theta+\int_t^x(\theta-t)_+d\theta\right);\ x\right)$$

$$\leq \| f'' \|_C B_n^*\Big(\Big(\int_x^t (t-\theta)d\theta + \int_t^x (\theta - t)d\theta\Big);\ x\Big)$$

$$= \| f'' \|_C B_n^*((t-x)^2;\ x) = \| f'' \|_C \{B_n^*(t^2;\ x) - 2xB_n^*(t;\ x) + x^2 B_n^*(1;\ x)\}$$

$$= \| f'' \|_C \Big(x^2 \frac{m_n}{n} + \frac{x}{n} - 2x^2 + x^2\Big) = \frac{x}{n} \| f'' \|_{C[0,\ a]} \leq \frac{a}{n} \| f'' \|_{C[0,\ a]}$$

and

$$|B_n^*(f;\ x) - f(x)| \leq a \| f'' \|_{C[0,\ a]}/n. \tag{10}$$

Using interpolation theory arguments [8] and noticing, that the space L_p is an intermediate space between L and C, we obtain from (9) and (10): $\| B_n^*(f;.) - f \|_{L_p[0,\ a]} \leq C(a) \| f'' \|_{L_p[0,\ a]}/n$, where the constant $C(a)$ depends only on a.

Lemma 1, Lemma 2 and Theorem 1 imply the following

Theorem 2. *If the bounded function f is p-integrable on $[0, \infty)$, and $f(x)=0$ for $x \geq a$, then*

$$\| B_n^*(f;\ .) - f \|_{L_p[0,\ a]} \leq C(a) \tau_2(f;\ n^{-1/2}),$$

where the constant $C(a)$ depends only on a.

Corollary 1. *If f is a function of bounded variation on $[0, \infty)$ and $f(x)=0$ for $x \geq a$, then*

$$\| B_n^*(f;\ .) - f \|_{L[0,\ a]} \leq C_1(a) \bigvee_0^a f \,.\, n^{-1/2}.$$

Corollary 2. *If f' is a function of bounded variation on $[0, \infty)$ and $(x)=0$ for $x \geq a$, then*

$$\| B_n^*(f;\ .) - f \|_{L[0,\ a]} \leq C_2(a) \bigvee_0^a f' \,.\, n^{-1}.$$

These corollaries follow from Theorem 2 and the properties b) and e) of the moduli $\tau_k(f;\ \delta)_{L_p}$.

In the periodical case the following property of $\tau_2(f;\ \delta)_{L_p}$ is satisfied [6]: $\tau_2(f;\ \delta)_{L_p} \leq 16\delta\omega(f';\ \delta)_{L_p}$.

In the general case it is proved [7], that: $\tau_k(f;\ \delta)_{L_p} \leq C(k)\delta\omega_{k-1}(f';\ \delta)_{L_p}$. From this and Theorem 2 we obtain:

Corollary 3. *If the function f has a derivative $f' \in L_p[0, \infty)$ and $f(x)=0$ for $x \geq a$, then*

$$\| B_n^*(f;\ .) - f \|_{L_p[0,\ a]} \leq C_3(a) \omega(f';\ n^{-1/2})_{L_p[0,\ a]}.$$

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