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STUDIA MATHEMATICA BULGARICA

ПЛИСКА

БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

ON THE RATES OF CONVERGENCE OF TWO MODULI OF FUNCTIONS

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Relations between the moduli of continuity (smoothness) in $L_p[0, 2\pi](1 \leq p < \infty)\omega_k(f;\ \delta)_p$ and the averaged moduli $\tau_k(f;\ \delta)_p$ are investigated. For this purpose fractional order moduli of continuity and fractional order derivatives are used.

1. Preliminaries. Let $C=C[0, 2\pi]$ be the space of all continuous 2π periodic functions with supremum-norm $L_p=L_p[0, 2\pi]$ $(1 \leq p < \infty)$ the space of all measurable 2π periodic functions, for which the norm $\|f\|_p=[\int_0^{2\pi}|f(x)|^p dx]^{1/p}$ is finite. Let X stand for C or L_p.

For $f \in X$, $\int_0^{2\pi} f(x)dx=0$, the following definition of α integral $(\alpha>0)$ was introduced by H. Weyl [7]:

$$f_\alpha(x)=\frac{1}{2\pi}\int_0^{2\pi}\psi_\alpha(x-t)f(t)dt, \tag{1.1}$$

where $\psi_\alpha(t)=\sum_{k=-\infty, k\neq 0}^{\infty} e^{ikt}(ik)^{-\alpha}$. We shall use only the following properties of the kernel ψ_α (see Zygmund [8, ch. XII]):

(1.2) ψ_α is 2π periodic, $\psi_\alpha(x)$ is differentiable
for $x \neq 2k\pi$, $\psi_\alpha^{(i)}(t)=O(t^{\alpha-1-i})$ for $0<|t|\leq\pi$, $i=0, 1$, and $\int_{-\pi}^{\pi}\psi_\alpha(t)dt=0$.

The difference of f of fractional order α $(\alpha>0)$ is given by

$$\Delta_h^\alpha f(x)=\sum_{j=0}^{\infty}(-1)^j\binom{\alpha}{j}f(x-jh).$$

The function $f \in X$ is said to be the α derivative $(\alpha>0)$ of $F \in X(F^{(\alpha)}=f)$ if $\lim_{h\to 0}\|h^{-\alpha}\Delta_h^\alpha F-f\|_X=0$.

In the recent paper of Butzer, Dyckoff, Görlich & Stens [1] the following important relation between the above definitions of integral and derivative was given:

(1.3)
$F^{(\alpha)}=f$ iff $F(x)=f_\alpha(x)$ (a. e.) in case $X=L_p$ and
$F^{(\alpha)}=f$ iff $F(x)=f_\alpha(x)$ in case $X=C$.

The X modulus of continuity of fractional order α $(\alpha>0)$ of f is given by

$$\omega_\alpha(f;\ \delta)=\omega_\alpha(f;\ \delta)_X=\sup_{0<h\leq\delta}\|\Delta_h^\alpha f\|_X. \tag{1.4}$$

Let us list some properties of the fractional moduli of continuity (see [1, Lemma 6, Theorem 3]).

L e m m a 1. *For* $f, g \in X$, $0<\alpha, \beta$, *we have*

(1.5) $\omega_\alpha(f;\delta)$ *is a non-negative, increasing function of* $\delta \in (0,\infty)$, $\lim_{\delta\to 0+}\omega_\alpha(f;\delta)=0$;

(1.6) $$\omega_\alpha(f;\delta)\le c(\alpha,\beta)\,\omega_\beta(f;\delta) \quad (\beta\le\alpha);$$

(1.7) $$\omega_\alpha(f+g;\delta)\le\omega_\alpha(f;\delta)+\omega_\alpha(g;\delta);$$

(1.8) $$\omega_\alpha(f;\delta)_X\le c(\alpha)\delta^\alpha\,\|f^{(\alpha)}\|_X \textit{ if } f^{(\alpha)}\in X;$$

(1.9) $$\omega_\alpha(f;\delta)_X\le c(\alpha-\beta)\delta^{\alpha-\beta}\omega_\beta(f^{(\alpha-\beta)};\delta)_X \textit{ if } f^{(\alpha-\beta)}\in X.$$

(1.10) *For each* $0<\beta<\theta<\alpha$ *and each* $\gamma>\theta-\beta$ *we have*
$$\omega_\alpha(f;\delta)_X=O(\delta^\theta) \textit{ if } f^{(\beta)}\in X \textit{ and } \omega_\gamma(f^{(\beta)};\delta)_X=O(\delta^{\theta-\beta}).$$

In this paper $c(A, B, \ldots)$ denotes a positive constant depending only on the marked parameters. It may differ at each occurrence.

The X averaged modulus of integer order k of a measurable bounded function f is given by

(1.11) $$\tau_k(f;\delta)=\tau_k(f;\delta)_X=\|\omega_k(f,\cdot;\delta)\|_X,$$

where $\omega_k(f,x;\delta)=\sup\{|\Delta_h^k f(t)|: t, t+kh\in[x-k\delta/2,\ x+k\delta/2]\}$ is the loca modulus of continuity of f. For the history of τ_k see [5]. Here some proper ties of τ_k are collected.

L e m m a 2. *For f, g-bounded measurable functions we have*

(1.12) $\tau_k(f;\delta)$ *is a non-negative, increasing function of* $\delta\in(0,\infty)$;

(1.13) $$\tau_k(f;\delta)\le c(k)\tau_{k-1}(f;\delta);$$

(1.14) $$\tau_k(f+g;\delta)\le\tau_k(f;\delta)+\tau_k(g;\delta);$$

(1.15) $$\tau_k(f;\delta)_X\le\delta^k\,\|f^{(k)}\|_X \textit{ for } f^{(k)}\in X;$$

(1.16) $$\tau_1(f;\delta)_1\le\delta\,.\,V_0^{2\pi}f \textit{ for } f\in B.V.;$$

(1.17) $$\tau_k(f;\delta)_X\le c(k)\,\delta\,\omega_{k-1}(f';\delta)_X \textit{ for } f'\in X;$$

(1.18) $$\omega_k(f;\delta)_{C[0,2\pi]}=\tau_k(f;\delta)_{C[0,2\pi]};$$

(1.19) $$\omega_k(f;\delta)_p\le\tau_k(f;\delta)_p \ (1\le p<\infty).$$

Properties (1.12)—(1.15) and (1.18), (1.19) are a simple consequence of definitions (1.4) and (1.11) and properties of finite difference. (1.16) is proved in [9] and (1.17) — in [3].

R e m a r k. The derivatives in Lemma 2 should be considered in the usua sense. So the functions in (1.15) and (1.17) are continuous (cf. (1.3)).

Our purpose is to investigate the relation between the rates of convergence of ω_k and τ_k. In the case $X=C$ the situation is trivial because of (1.18) — then ω_k and τ_k coincide. In the following we shall consider only the case $X=L_p\,(1\le p<\infty)$. Here (1.19) gives that $(0<\alpha\le k)$

(1.20) $$\tau_k(f\ \delta)_p=O(\delta^\alpha)$$

implies

$$\omega_k(f;\ \delta)_p = O(\delta^\alpha). \tag{1.21}$$

So our aim will be to find out when (1.21) implies (1.20).

2. Main results. *For* $f \in L_p$ $(1<p<\infty)$ *and* $1/p<\alpha<1$ *we have*

$$\tau_1(f_\alpha;\ \delta)_p \leq c(\alpha, p)\delta^\alpha \|f\|_{L_p}. \tag{2.1}$$

Remark. We can consider (2.1) as an extension of (1.15) for $k=1$. An inequality with $\tau_k(k>1)$ instead of τ_1 in (2.1) follows from (2.1) and (1.13). Let us note that f_α is continuous and even satisfies some Lipschitz conditions (see e. g. [8, ch. XII]). This is of great importance for the validity of Theorem 1.

Proof. Let $t_1, t_2 \in [x-\delta/2, x+\delta/2]$. We set $h_i = t_i - x$, $i=1, 2$. Then $|h_i| \leq \delta/2$. From (1.1) and (1.2) we have

$$2\pi f_\alpha(t_i) = \int_0^{2\pi} \psi_\alpha(x+h_i-t) f(t)dt = \int_{-\pi}^{\pi} \psi_\alpha(t+h_i) f(x-t)dt$$

and therefore

$$2\pi[f_\alpha(t_1)-f_\alpha(t_2)] = \int_{-\pi}^{\pi} f(x-t)[\psi_2(t+h_1)-\psi_2(t+h_2)]dt. \tag{2.2}$$

We divide the integral in (2.2) into two parts $A(x)=\int_{|t|\leq\delta}$ and $B(x)=\int_{\delta\wedge|t|\leq\pi}$. For $A(x)$ (1.2) and Hölder's inequality $(1/p'+1/p=1)$ give

$$|A(x)| \leq [\int_{-\delta}^{\delta} |f(x-t)|^p dt]^{1/p} \{ [\int_{-\delta}^{\delta} |\psi_\alpha(t+h_1)|^{p'} dt]^{1/p'} \tag{2.3}$$

$$+[\int_{-\delta}^{\delta} |\psi_\alpha(t+h_2)|^{p'}dt]^{1/p'} \} \leq 2 [\int_{-\delta}^{\delta} |f(x-t)|^p dt]^{1/p} [\int_{-2\delta}^{2\delta} |\psi_\alpha(t)|^{p'}dt]^{1/p'}$$

$$\leq c(\alpha) [\int_{-\delta}^{\delta} |f(x-t)|^p dt] \int_{-2\delta}^{2\delta} t^{(\alpha-1)p'}dt]^{1/p'}$$

$$\leq c(\alpha;\ p) [\int_{-\delta}^{\delta} |f(x-t)|^p dt]^{1/p}\ \delta^{\alpha-1/p},$$

because $(\alpha-1)p' > (1/p-1)p' = -1$. For $B(x)$ (1.2) gives

$$B(x)| \leq \int_{\delta<|t|\leq\pi} |f(x-t)|\ |h_1-h_2|\ |\psi'_\alpha(t+\theta_t\, \delta/2)|\, dt \tag{2.4}$$

$$\leq c(\alpha)\delta \int_{\delta<|t|\leq\pi} |f(x-t)|\ (|t|-\delta/2)^{\alpha-2}dt.$$

From (2.2), (2.3) and (2.4) we obtain the following estimate for the local modulus of continuity:

$$\omega_1(f, x;\ \delta) \leq c(\alpha, p)\, \delta^{\alpha-1/p} [\int_{-\delta}^{\delta} |f(x-t)|^p dt]^{1/p} \tag{2.5}$$

$$+ c(\alpha)\delta \int_{\delta<|t|\leq\pi} |f(x-t)|\ (|t|-\delta/2)^{\alpha-2}dt.$$

Now (1.11), (2.5) and Minkowski's inequality give

$$\tau_1(f_\alpha;\delta)_p\le c(\alpha,p)\,\delta^{\alpha-1/p}\,\Big[\int_{-\delta}^{\delta}\int_0^{2\pi}|f(x-t)|^p dx\,dt\Big]^{1/p}$$

$$+c(\alpha)\,\delta\int_{\delta<|t|\le\pi}\|f(.-t)\|_p(|t|-\delta/2)^{\alpha-2}dt$$

$$\le c(\alpha,p)\,\delta^{\alpha-1/p}\,\|f\|_p\,\delta^{1/p}+c(\alpha)\,\delta\,\|f\|_p\int_{\delta/2}^{\pi}t^{\alpha-2}dt=c(\alpha,p)\delta^\alpha\,\|f\|_p.$$

This proves the theorem.

Corollary 1. *For* $f\in L_p(1<p<\infty)$, $1/p<\alpha<1$, $F=f_\alpha$ *we have*

(2.6) $$\tau_k(F:\delta)_p\le c(\alpha,p)\delta^\alpha\omega_{k-\alpha}(f;\delta)_p\,.$$

Proof. Let G be the k-th modified Steklov function of F with a step δ, i. e.

$$G(x)=(-1)^{k-1}\delta^{-k}\sum_{r=1}^{k}(-1)^{k-r}\binom{k}{r}\int_0^\delta\cdots\int_0^\delta f(x+(t_1+\cdots+t_k)r/k)dt_1\ldots dt_k\,.$$

Then

(2.7) $$(G-F)^{(\alpha)}(x)=(-1)^{k-1}\delta^{-k}\Big(\int_0^\delta\cdots\int_0^\delta\Delta^k_{(t_1+\dots+t_k)/k}F(x)dt_1\ldots dt_k\Big)^{(\alpha)}$$

$$=(-1)^{k-1}\,\delta^{-k}\int_0^\delta\cdots\int_0^\delta\Delta^k_{(t_1+\dots+t_k)/k}f(x)dt_1\ldots dt_k$$

and

(2.8) $$G^{(k)}(x)=\sum_{r=1}^{k}(-1)^{r-1}\binom{k}{r}\Big(\frac{k}{r}\Big)^k\,\delta^{-k}\Delta^k_{r\delta/k}F(x)\quad\text{(a. e.)}$$

(see [4]). Then (1.14), (1.15), (1.13), (2.8), (1.4), Theorem 1, (1.9), (2.7) and (1.6) give

$$\tau_k(F;\delta)_p\le\tau_k(G;\delta)_p+\tau_k(F-G;\delta)_p$$

$$\le c(k)\,\delta^k\,\|G^{(k)}\|_p+c(k)\tau_1(F-G;\delta)_p\le c(k)\omega_k(F;\delta)_p+c(k,\alpha)\delta^\alpha\,\|(F-G)^{(\alpha)}\|_p$$

$$\le c(k)\delta^\alpha\omega_{k-\alpha}(f;\delta)_p+c(k,\alpha)\delta^\alpha\omega_k(f;\delta)_p\le c(k,\alpha)\,\delta^\alpha\omega_{k-\alpha}(f;\delta)_p$$

Theorem 2. *For* $f\in L_p(1\le p<\infty)$, $k\ge1$, $1/p<\alpha<k$ *for* $p>1$ *and* $1\le\alpha<k$ *for* $p=1$, $F=f_\alpha$, *we have*

(2.9) $$\tau_k(F;\delta)_p\le c(\alpha,p)\,\delta^\alpha\omega_{k-\alpha}(f;\delta)_p.$$

Proof. For $p>1$ we prove (2.9) by Corollary 1 and (1.9). For $p=1$ we obtain (2.9) from (1.17) and (1.9).

Theorem 3. *If* k *is natural,* $1/p<\theta\le k$ *for* $1<p<\infty$, $\theta=1$ *for* $p=1$ *and* $k=1$, $1<\theta\le k$ *for* $p=1$ *and* $k>1$, *then (1.21) implies*

(2.10) $$f(x)=F(x)\ \text{(a. e.)}\ \textit{and}\ \tau_k(F;\delta)_p=O(\delta^\theta).$$

Remark. Under the conditions of Theorem 3 we cannot simply state that (1.21) implies (1.20), because ω_k does not depend on the values of f for all $x\in[0, 2\pi]$, but for x belongs to a full-measured set. In (2.10) F is this function in the equivalent class of f which is continuous (see the remark after Theorem 1) with the exception of the case $p=1$, $k=1$, where F has a bounded variation.

Proof. a) The saturation case $\theta=k$.

From Theorem 2.2.26 in [2, p. 110] follows that (1.21) implies

$$f(x)=F(x) \text{ (a. e.) and}$$

$$\text{(2.11)} \qquad F, F', \ldots, F^{(k-2)}\in A.C., \ F^{(k-1)}\in B.V. \text{ if } p=1 \text{ and}$$

$$F, F', \ldots, F^{(k-1)}\in A.C., \ F^{(k)}\in L_p \text{ if } p>1$$

(the case $p=1$, $k=1$ is the famous Hardy-Littlewood's result). Now (2.11) (1.15) and (1.16) give (2.10) if $p>1$ or $p=1$, $k=1$. If $p=1$, $k>1$ using (2.11) (1.17), (1.9), (1.19) and (1.16) we have

$$\tau_k(F;\ \delta)_1\leqq c(k)\delta\omega_{k-1}(F';\ \delta)_1\leqq c(k)\delta^{k-1}\omega_1(F^{(k-1)};\ \delta)_1$$

$$\leqq c(k)\delta^{k-1}\tau_1(F^{(k-1)};\ \delta)_1\leqq c(k)\delta^k\cdot \overset{2\pi}{\underset{0}{\mathrm{V}}} F^{(k-1)}.$$

b) The non-saturation case $1/p<\theta<k$.

If $1/p<\theta\leqq 1$ ($k>1$ if $\theta=1$!) then (1.10) with $\alpha=k, \beta=(1/p+\theta)/2, \gamma=k-\beta$, states that (1.21) implies

$$\text{(2.12)} \qquad \omega_{k-\beta}(f^{(\beta)};\ \delta)_p=O(\delta^{\theta-\beta}).$$

We set $F=(f^{(\beta)})_\beta$. Then (1.3) gives $F(x)=f(x)$ (a. e.) and Corollary 1 and (2.12) give

$$\tau_k(F;\ \delta)_p\leqq c(\theta, p)\delta^\beta\omega_{k-\beta}(f^{(\beta)};\ \delta)_p=O(\delta^\theta).$$

If $1<\theta<k$ then (1.10) with $\alpha=k$, $\beta=1$, $\gamma=k-1$ states that (1.21) implies $\omega_{k-1}(f';\ \delta)_p=O(\delta^{\theta-1})$ and we prove (2.10) as in the above case $1/p<\theta<1$. This completes the proof.

Let us note that the only place where we really need fractional order derivatives and fractional order moduli of continuity in the above proof is the case $1/p<\theta\leqq 1$ in the non-saturation case. But this is the delicate place in our investigation which needs a special treatment.

As an immediate consequence of Theorem 3 and the two facts:

1) $\tau_k(f;\ n^{-1})_p$ can be used to characterize the best one-sided trigonometrical approximation $\tilde{E}_n(f)_p$ of $f\in L_p$ (see [5; 6]);

2) $\omega_k(f;\ n^{-1})_p$ can be used to characterize the best trigonometrical approximation $E_n(f)_p$ of $f\in L_p$ (see [10]), we get the following eguivalence ($\theta>1/p$):

$$E_n(f)_p=O(n^{-\theta})\Leftrightarrow f(x)=F(x) \text{ (a. e.) and } \tilde{E}_n(F)_p=O(n^{-\theta}).$$

Let us note that the left side of the above equivalence implies that F is continuous.

For another application of the proved connection between the two moduli see [11].

The results of this paper can be also applied in numerical analysis wherever coefficients and solutions are fractional integrals and estimates are given by the modulus τ_k.

3. An important example. The following example shows that the theorems in point 2 cannot be extended to $\alpha\leq 1/p$ or $\theta\leq 1/p$ for $p>1, k\geq 1$ and/or $p=1$, $k>1$ and to $\alpha<1$ or $\theta<1$ for $p=1$, $k=1$. The constructed function f is continuous and therefore each equivalent to f function f_1 will have the same modulus ω_k but a worse modulus τ_k.

Example. *Given p and θ, $1\leq p<\infty$, $0<\theta<1/p$. We set $x_k=k^{-\alpha}$, $y_k=k^{-\beta}$, $a_k=e^{-k}$, where $\alpha=\theta p/(2-2\theta p)$, $\beta=\theta/2$ and $k\geq K=K(\theta, p)$, where K is chosen such that $x_k-a_k>x_{k+1}+a_{k+1}$ for each $k>K$. We define a continuous function f as follows:*

$f(x_k)=y_k$, $f(x_k+a_k)=f(x_k-a_k)=0$, f is linear in $[x_k, x_k+a_k]$, $[x_k-a_k, x_k]$, $[x_{k+1}+a_{k+1}, x_k-a_k]$, $f(0)=f(\pi)=0$, $f(x)=0$ for $x\in[x_K+a_K, \pi]$, f is π-periodic.

Then for $t\in(0, 1]$ we have

1) $\tau_k(f;t)_p \asymp t^{\theta}$ *for* $p\geq 1$, $k\geq 1$;

2) $\omega_k(f;t)_p \asymp t^{1/p}\ln^{-\theta/2}t^{-1}$

for $p>1$, $k\geq 1$ *and/or* $p=1$, $k>1$;

3) $\omega_1(f;t)_1 \asymp t\ln^{1-\theta/2}t^{-1}$.

Remark. We use the following denotations:

a) $f(x)\ll g(x)$ means that there is a positive constant c such that $f(x)\leq cg(x)$;

b) $f(x)\asymp g(x)$ means that $f(x)\ll g(x)$ and $g(x)\ll f(x)$;

c) $[x]$ will denote the biggest integer less than or equal to x.

Proof. We shall prove the above only for $k=1$. Calculation are similar if $k>1$.

Let $0<t<1$. We set $M=[-\ln t]$ and $N=\max\{k: x_k-a_k-(x_{k+1}+a_{k+1})>t\}$ Then $N\asymp t^{-1/(\alpha+1)}$ for $t\leq c(\theta,p)$ ($c(\theta,p)$ properly chosen). We set

$$\bar{\omega}(x)=\begin{cases} ty_k/a_k & \text{if } |x-x_k|\leq 3a_k/2 \text{ and } k<M;\\ y_k & \text{if } |x-x_k|<3t/2 \text{ and } M\leq k\leq N;\\ 0 & \text{otherwise;}\end{cases}$$

$$\underline{\omega}(x)=\begin{cases} ty_k/(2a_k) & \text{if } |x-x_k|<a_k \text{ and } k<M;\\ y_k/2 & \text{if } |x-x_k|<t \text{ aud } M\leq k\leq N;\\ 0 & \text{otherwise.}\end{cases}$$

Then we have

$$\underline{\omega}(x)\leq\omega_1(f,x;t)\leq\bar{\omega}(x) \text{ if } x_N\leq x\leq\pi \text{ and}$$

$$0\leq\omega_1(f,x;t)\leq y_N \quad \text{if } 0\leq x\leq x_N.$$

Therefore

$$\Big(\sum_{k=M}^{N} y_k^p t\Big)^{1/p}\ll\tau_1(f;t)_p\ll\Big(y_N^p x_N+\sum_{k=M}^{N} y_k^p t+\sum_{k=K}^{M-1}(ty_k/a_k)^p a_k\Big)^{1/p}.$$

Using given values of parameters we get

a) $t \sum_{k=M}^{N} y_k^p = t \sum_{k=M}^{N} k^{-\beta p} \asymp t \sum_{k=1}^{t^{-1/(\alpha+1)}} k^{-\beta p} \asymp t \, . \, t^{-(1-\beta p/(\alpha+1)} = t^{(\alpha+\beta p)/(\alpha+1)} = t^{\theta p}$;

b) $y_N^p x_N = N^{-(\beta p+\alpha)} \ll t^{(\alpha+\beta p)/(\alpha+1)} = t^{\theta p}$;

c) $\sum_{k=K}^{M} t^p y_k^p a_k^{1-p} = t^p \sum_{k=K}^{M} k^{-\theta p/2} e^{(p-1)k}$

$$\ll \begin{cases} t^p(-\ln t)^{-\theta p/2} e^{(p-1)(-\ln t)} = t(-\ln t)^{-\theta p/2} \ll t^{\theta p} & \text{if } p>1, \\ t(-\ln t)^{1-\theta p/2} \ll t^{\theta} & \text{if } p=1. \end{cases}$$

Therefore $\tau_1(f;\, t)_p \asymp t^{\theta}$.

To calculate $\omega_1(f;\, t)_p$ we proceed as follows:

$$\omega_1(f;\, t)_{L_p[0,\pi]} \leqq \omega_1(f;\, t)_{L_p[0,x_M]} + \omega_1(f;\, t)_{L_p[x_M,\pi]}$$

$$\leqq 2 \, \| f \|_{L_p[0,x_M]} + t \, \| f' \|_{L_p[x_M,\pi]}$$

$$\leqq \Big(\sum_{k=M}^{\infty} y_k^p \, 2a_k \Big)^{1/p} + t \Big(\sum_{k=1}^{M} (y_k/a_k)^p \, 2a_k \Big)^{1/p}$$

$$= \Big(2 \sum_{k=M}^{\infty} k^{-\beta p} e^{-k}\Big)^{1/p} + t \Big(2 \sum_{k=1}^{M} k^{-\beta p} e^{(p-1)k}\Big)^{1/p}.$$

Now for $p>1$ we have

$$\omega_1(f;\, t)_{L_p[0,\pi]} \ll M^{-\beta} e^{-M/p} + tM^{-\beta} e^{(1-1/p)M} \ll t^{1/p} \ln^{-\theta/2} t^{-1}.$$

If $p=1$ we have

$$\omega_1(f;\, t)_{L_1[0,\pi]} \ll M^{-\beta} e^{-M} + tM^{1-\beta} \ll t \ln^{1-\theta/2} t^{-1}.$$

To get an estimate for $\omega_1(f;\, t)_p$ from below we observe that for $2t \leqq a_k$ the equality $|f(x-t/2)-f(x+t/2)| = y_k t/a_k$ holds true for $t/2 < |x-x_k| < a_k - t/2$. We set $M_1 = [-\ln(2t)]$ and then $a_{M_1+1} < 2t \leqq a_{M_1}$. For $p>1$ we get

$$\omega_1(f;\, t)_p \gg \Big(\sum_{k=1}^{M_1} (y_k t/a_k)^p \, a_k \Big)^{1/p} \gg t^{1/p} \ln^{-\theta/2} t^{-1}$$

and for $p=1$ we get

$$\omega_1(f;\, t)_1 \gg \sum_{k=1}^{M_1} y_k t \gg t \ln^{1-\theta/2} t^{-1}.$$

This completes the proof for $k=1$.

Last we shall consider the following problem:

Let φ, ψ be continuous increasing functions in $[0, 1]$ $\varphi(0)=\psi(0)=0$. To find a function $\varphi(t)=\varphi(k, p;\, t)$ such that;

A) If $\psi(t)=O(\varphi(t)$ (or $\psi(t)=o(\varphi(t))$ then $\omega_k(f;\, t)_p = O(\psi(t))$ implies that there exists $F(x)=f(x)$ (a. e.) and $\tau_k(F;\, t)_p = O(\psi(t))$.

B) If $\psi(t) \neq O(\varphi(t))$ (or $\psi(t) \neq o(\varphi(t))$ then there exists f_ψ such that $\omega_k(f_\psi;\, t)_p = O(\psi(t)$ but for each equivalent to f_ψ function F we have $\tau_k(F;\, t)_p \neq O(\psi(t))$.

Point 2 states that $\varphi(t) \gg t$ for $p=k=1$ and $\varphi(t) \gg t^{1/p-\varepsilon}$ for each $\varepsilon>0$ in the opposite case.

The example in 3 states (if we set $a_k = \exp\{-k^{1/\gamma}\}$ for $\gamma>0$) that $\varphi(t) \ll t \ln^{\varepsilon} t$ for each $\varepsilon>0$ if $p=k=1$ and $\varphi(t) \ll (t . \ln^{-1} t^{-1})^{1/p}$ in the opposite case.

We make the conjecture that $\varphi(t)=t$ if $p=k=1$ and $\varphi(t)=(t, \ln^{-1} t^{-1})^{1/p}$ in the opposite case.

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