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TWO INTERVAL METHODS FOR ALGEBRAIC EQUATIONS WITH REAL ROOTS

NIKOLAI V. KJURKCHIEV, SVETOSLAV M. MARKOV

Two interval (two-sided) algorithms for the solution of algebraic equations with real roots are considered. Both algorithms deliver on each step bounds for all roots simultaneously. The first algorithm has a quadratic convergence, and the second has a cubic convergence. A computer realization of the algorithms is proposed.

1. Introduction. We propose two interval (two-sided) algorithms for the simultaneous determination of all roots of an algebraic equation possessing only real roots. The first method has quadratic convergence whereas the second has cubic one. The proposed methods can be considered as two-sided analogues of the methods described correspondingly in [4, 5] and [6]. A machine realization of the methods is proposed in the spirit of the new computing conception, which assumes the utilization of a computer which executes the arithmetic operations with directed roundings in the sense of [1, 7]. We note the interesting fact that our realization involves computation of intervals with contraction (in the sense of [2]), which provides additional control on the round-off errors in the computations. Our considerations do not make use of interval-arithmetic.

2. A two-sided method with quadratic convergence. The two-sided method formulated in this section can be considered as a modification of the algorithm for simultaneous determination of all roots of an algebraic equation proposed by K. Dočev ([4, 5], see also the last section 6 of this paper).

We assume that the algebraic equation of degree n with given real coefficients

$$f(x)=x^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0=0 \tag{1}$$

has n simple (unknown) real roots $x_1<x_2<\cdots<x_n$ and therefore $f(x)=\Pi_{j=1}^{n}(x-x_j)$. We also assume that the roots are located in n nonintersecting (known) intervals $X_i^{(0)}=[\underline{x}_i^{(0)}, \overline{x}_i^{(0)}]$, $i=1,\ldots,n$, that is $X_i^{(0)}\cap X_j^{(0)}=\emptyset$ for $i\neq j$ and $x_i\in X_i^{(0)}$ for $i=1,\ldots,n$.

We propose the following two-sided iteration algorithm for simultaneous computation of the roots of (1):

$$\begin{cases} \overline{x}_i^{(k+1)}=\overline{x}_i^{(k)}-f(\overline{x}_i^{(k)})/(\Pi_{j=1}^{i-1}(\overline{x}_i^{(k)}-\underline{x}_j^{(k)})\Pi_{j=i+1}^{n}(\overline{x}_i^{(k)}-\overline{x}_j^{(k)})), \\ \underline{x}_i^{(k+1)}=\underline{x}_i^{(k)}-f(\underline{x}_i^{(k)})/(\Pi_{j=1}^{i-1}(\underline{x}_i^{(k)}-\underline{x}_j^{(k)})\Pi_{j=i+1}^{n}(\underline{x}_i^{(k)}-\overline{x}_j^{(k)})), \end{cases} \tag{2}$$

where the value of the symbols $\Pi_{j=1}^{0}(\overline{x}_i^{(k)}-\underline{x}_j^{(k)})$, $\Pi_{j=1}^{0}(\underline{x}_i^{(k)}-\underline{x}_j^{(k)})$, $\Pi_{j=n+1}^{n}(\overline{x}_i^{(k)}-\overline{x}_j^{(k)})$ and $\Pi_{j=n+1}^{n}(\underline{x}_i^{(k)}-\overline{x}_j^{(k)})$ is assumed to be equal to one.

In what follows we denote by d the number $d=\min_{i\neq j}|x_i-x_j|$. Since the roots $x_1<x_2<\cdots<x_n$ are assumed simple, we have $d>0$.

In these notations we can formulate the following assertion on the rate of convergence of (2):

Theorem 1. *Let q and c be such that $0<q<1$, $0<c<d/n$. If the two-sided initial approximations $\{[\underline{x}_i^{(0)}, \overline{x}_i^{(0)}]\}_{i=1}^n$ satisfy the inequalities*

$$(3)\qquad \begin{aligned} &0\leq \overline{x}_i^{(0)}-x_i\leq cq,\\ &0\leq x_i-\underline{x}_i^{(0)}\leq cq, \quad i=1,2,\ldots,n, \end{aligned}$$

then for the two-sided approximations produced by (2) *the inequalities*

$$(4)\qquad \begin{aligned} &0\leq \overline{x}_i^{(k)}-x_i\leq cq^{2^k},\\ &0\leq x_i-\underline{x}_i^{(k)}\leq cq^{2^k}, \quad i=1,2,\ldots,n, \end{aligned}$$

hold true for all $k=1,2,\ldots$.

Proof. Inequations (3) imply that (4) hold true for $k=0$. Assume that (4) hold true for some $k=m$, that is we have

$$(5a)\qquad 0\leq \overline{x}_i^{(m)}-x_i\leq cq^{2^m}, \quad i=1,2,\ldots,n,$$

$$(5b)\qquad 0\leq x_i-\underline{x}_i^{(m)}\leq cq^{2^m}, \quad i=1,2,\ldots,n.$$

Since $q<1$, the inequalities

$$(6)\qquad \begin{aligned} &0\leq \overline{x}_i^{(m)}-x_i\leq c,\\ &0\leq x_i-\underline{x}_j^{(m)}\leq c, \quad i=1,2,\ldots,n \end{aligned}$$

hold true as well. From (6) and the choice of c it follows that no two of the numbers $\underline{x}_1^{(m)}, \underline{x}_2^{(m)},\ldots,\underline{x}_n^{(m)}, \overline{x}_1^{(m)}, \overline{x}_2^{(m)},\ldots,\overline{x}_n^{(m)}$ can be equal (with a possible exclusion of the numbers of the pairs $(\underline{x}_i^{(m)}, \overline{x}_i^{(m)})$, $i=1,\ldots,n$) and therefore the denominators in (2) do not vanish.

We shall prove that (4) hold true for $k=m+1$, that is we have

$$(7a)\qquad 0\leq \overline{x}_i^{(m+1)}-x_i\leq cq^{2^{m+1}}, \quad i=1,\ldots,n,$$

$$(7b)\qquad 0\leq x_i-\underline{x}_i^{(m+1)}\leq cq^{2^{m+1}}, \quad i=1,\ldots,n.$$

We shall first prove the inequalities $0\leq\overline{x}_i^{(m+1)}-x_i$ and $0\leq x_i-\underline{x}_i^{(m+1)}$. Setting $f(\overline{x}_i^{(m)})=\Pi_{j=1}^n(\overline{x}_i^{(m)}-x_j)$ we obtain from the first equality of (2):

$$(8)\qquad \overline{x}_i^{(m+1)}-x_i=\overline{x}_i^{(m)}-x_i-\prod_{j=1}^{i-1}\frac{\overline{x}_i^{(m)}-x_j}{\overline{x}_i^{(m)}-\underline{x}_j^{(m)}}\prod_{j=i+1}^{n}\frac{\overline{x}_i^{(m)}-x_j}{\overline{x}_i^{(m)}-\overline{x}_j^{(m)}}\cdot(\overline{x}_i^{(m)}-x_i)$$

$$=(\overline{x}_i^{(m)}-x_i)\Big(1-\prod_{j=1}^{i-1}\frac{\overline{x}_i^{(m)}-x_j}{\overline{x}_i^{(m)}-\underline{x}_j^{(m)}}\prod_{j=i+1}^{n}\frac{\overline{x}_i^{(m)}-x_j}{\overline{x}_i^{(m)}-\overline{x}_j^{(m)}}\Big)=(\overline{x}_i^{(m)}-x_i)(1-\pi_1\pi_2),$$

wherein π_1 and π_2 denote the corresponding products above. According to (5b) we have $x_j-\underline{x}_j^{(m)}\geq 0$ for $j=1,2,\ldots,n$. Using this we obtain in the situation

$j<i$ that $\overline{x}_i^{(m)}-\underline{x}_j^{(m)}=\overline{x}_i^{(m)}-x_j+x_j-\underline{x}_j^{(m)}\geqq\overline{x}_i^{(m)}-x_j>0$ and consequently $0<(\overline{x}_i^{(m)}-x_j)/(\overline{x}_i^{(m)}-\underline{x}_j^{(m)})\leqq 1$, so that for the product π_1 in (8) we have

$$0<\pi_1=\prod_{j=1}^{i-1}(\overline{x}_i^{(m)}-x_j)/(\overline{x}_i^{(m)}-\underline{x}_j^{(m)})\leqq 1.$$

Analogously it can be seen that $0<\pi_2=\prod_{j=i+1}^{n}(\overline{x}_i^{(m)}-x_j)/(\overline{x}_i^{(m)}-\overline{x}_j^{(m)})\leqq 1$ and therefore the expression $1-\pi_1\pi_2$ in (8) is nonnegative. This implies that the signs of $\overline{x}_i^{(m+1)}-x_i$ and $\overline{x}_i^{(m)}-x_i$ are equal and hence $\overline{x}_i^{(m+1)}-x_i\geqq 0$.

We showed that $\overline{x}_i^{(0)}-x_i\geqq 0$ and $\overline{x}_i^{(m)}-x_i\geqq 0$ imply $\overline{x}_i^{(m+1)}-x_i\geqq 0$. It follows by induction that $\overline{x}_i^{(k)}\geqq x_i$ holds true for all $i=1,\dots,n$ and every $k=0$, $1,\dots$ Analogously it is proved that $\underline{x}_i^{(k)}\leqq x_i$ for all $i=1,\dots,n$ and every $k=0$, $1,\dots$ We thus proved that the process is two-sided.

We shall now prove that (2) has quadratic convergence, showing first that $\overline{x}_i^{(m+1)}\leqq cq^{2^{m+1}}$. To this we shall estimate the products π_1 and π_2 from below.

If $j<i$, we have

$$\overline{x}_i^{(m)}-\underline{x}_j^{(m)}=(x_i-x_j)+(\overline{x}_i^{(m)}-x_i)+(x_j-\underline{x}_j^{(m)})\geqq x_i-x_j\geqq d\geqq d-c, \tag{9a}$$

using the fact that $\overline{x}_i^{(m)}-x_i\geqq 0$ and $x_j-\underline{x}_j^{(m)}\geqq 0$. If $j>i$, then we obtain similarly for $\overline{x}_j^{(m)}-\overline{x}_i^{(m)}$

$$\overline{x}_j^{(m)}-\overline{x}_i^{(m)}=(x_j-x_i)+(\overline{x}_j^{(m)}-x_j)-(\overline{x}_i^{(m)}-x_i)\geqq d-c. \tag{9b}$$

We shall now estimate π_1 and π_2 from below. We have

$$\pi_1=\prod_{j=1}^{i-1}\frac{\overline{x}_i^{(m)}-x_j}{\overline{x}_i^{(m)}-\underline{x}_j^{(m)}}=\prod_{j=1}^{i-1}\left(1-\frac{x_j-\underline{x}_j^{(m)}}{\overline{x}_i^{(m)}-\underline{x}_j^{(m)}}\right)\geqq\prod_{j=1}^{i-1}\left(1-\frac{cq^{2^m}}{d-c}\right)=\left(1-\frac{cq^{2^m}}{d-c}\right)^{i-1},$$

where we have used the inequalities (5b) and (9a). Similarly, using (5a) and (9b) we obtain

$$\pi_2=\prod_{j=i+1}^{n}\frac{\overline{x}_i^{(m)}-x_j}{\overline{x}_i^{(m)}-\overline{x}_j^{(m)}}=\prod_{j=i+1}^{n}\left(1-\frac{\overline{x}_j^{(m)}-x_j}{\overline{x}_j^{(m)}-\overline{x}_i^{(m)}}\right)\geqq\left(1-\frac{cq^{2^m}}{d-c}\right)^{n-i}.$$

This implies

$$\pi_1\pi_2\geqq\left(1-\frac{cq^{2^m}}{d-c}\right)^{n-1}\geqq 1-(n-1)\frac{cq^{2^m}}{d-c}$$

and $1-\pi_1\pi_2\leqq(n-1)cq^{2^m}/(d-c)\leqq q^{2^m}$ since $c/(d-c)\leqq 1/(n-1)$ (which is equivalent to the assumption $c\leqq d/n$ in the formulation of the theorem).

Using this estimate in (8) and inequality (5a) we obtain

$$\overline{x}_i^{(m+1)}-x_i=(\overline{x}_i^{(m)}-x_i)(1-\pi_1\pi_2)<cq^{2^m}(1-(1-q^{2^m}))=cq^{2^{m+1}}.$$

Thereby (7a) are proved. The second inequality of (7b) remains to be proved. It is proved by similar arguments: we shall only note that by the estimation of the products from below the inequality

(9c) $$\underline{x}_i^{(m)}-\underline{x}_j^{(m)}=(x_i-x_j)+(x_j-\underline{x}_j)-(x_i-\underline{x}_i)\geq d-c \quad \text{for } j<i$$

is used. The theorem is proved.

Remark 1. One can give better estimates of the differences in the denominators in the products π_1 and π_2, relaxing thereby the restrictive condition $c\leq d/n$ in the formulation of the theorem to the condition $c\leq d/(2\ln n+1)$. To this end instead of (9a) and (9b) we may use the better estimates

$$\bar{x}_i^{(m)}-\underline{x}_j^{(m)}\geq x_i-x_j\geq(i-j)d\geq(i-j)(d-c) \quad \text{for} \quad j<i,$$

$$\bar{x}_j^{(m)}-\bar{x}_i^{(m)}\geq(x_j-x_i)-(\bar{x}_i^{(m)}-x_i)\geq(j-i)d-c\geq(j-i)(d-c) \quad \text{for} \quad j>i,$$

and obtain

$$\pi_1=\prod_{j=1}^{i-1}\left(1-\frac{x_j-\underline{x}_j^{(m)}}{\bar{x}_i^{(m)}-\underline{x}_j^{(m)}}\right)\geq\prod_{j=1}^{i-1}\left(1-\frac{cq^{2^m}}{(i-j)(d-c)}\right)=\prod_{j=1}^{i-1}\left(1-\frac{\alpha}{j}\right), \quad \alpha=cq^{2^m}/(d-c).$$

For $0\leq\alpha\leq1$, we have $\Pi_{j=1}^{k}(1-\alpha/j)\geq1-h_k\alpha$, where $h_k=\Sigma_{j=1}^{k}(1/j)$. Using this inequality we obtain $\pi_1\geq\Pi_{j=1}^{i-1}(1-\alpha/j)\geq1-h_{i-1}\alpha$. Similarly we obtain $\pi_2\geq1-h_{n-i}\alpha$. Further we have $1-\pi_1\pi_2\leq1-(1-h_{i-1}\alpha)(1-h_{n-i}\alpha)=(h_{i-1}+h_{n-i})\alpha+h_{i-1}h_{n-i}\alpha^2\leq(h_{i-1}+h_{n-i})\alpha\leq h_n^*\alpha$, where

$$h_n^*=\max_{1\leq i\leq n}\{h_{i-1}+h_{n-i}\}=\begin{cases}h_m+h_{m-1}, & \text{if} \quad n=2m,\\ 2h_m, & \text{if} \quad n=2m+1.\end{cases}$$

However, $h_n=\Sigma_{j=1}^{n}(1/j)=\ln n+C+\gamma_n$, where the sequence $\gamma_n>0$ converges monotonically to zero with $n\to\infty$, and $C=0{,}5772156\ldots$ is the Enler's constant. Therefore h_n^* has the asymptotics of

$$2(\ln(n/2)+C)=2\ln n-2\ln2+2C=2\ln n-0{,}2318632\ldots.$$

Computing h_n^* and $2\ln n$ for the first several values of n, we conclude that the inequality $h_n^*<2\ln n$ holds true for all n. Therefore by the new assumption $c\leq d/(2\ln n+1)$, which is equvalent to $2c\ln n\leq d-c$, we have $1-\pi_1\pi_2\leq2(\ln n)\alpha=2(\ln n)cq^{2^m}/(d-c)\leq q^{2^m}$ and the inequality $\bar{x}_i^{(m+1)}-x_i=(\bar{x}_i^{(m)}-x_i)(1-\pi_1\pi_2)<cq^{2^m}\cdot q^{2^m}=cq^{2^{m+1}}$ shows that a quadratic convergence takes place. Similarly the quadratic convergence of the sequence $\underline{x}_i^{(m)}$ can be shown. We note that the condition $c\leq d/(2\ln n+1)$ is less restrictive than the condition $c\leq d/n$, when $n\geq4$.

Remark 2. The rate of convergence of the iteration process (2) can be accelerated if the already computed approximations are used immediately in the next computations as follows:

(2) $$\begin{cases}\bar{x}_i^{(k+1)}=\bar{x}_i^{(k)}-f(\bar{x}_i^{(k)})/\left(\prod_{j=1}^{i-1}(\bar{x}_i^{(k)}-\underline{x}_j^{(k+1)})\prod_{j=i+1}^{n}(\bar{x}_i^{(k)}-\bar{x}_j^{(k)})\right),\\[2ex] \underline{x}_i^{(k+1)}=\underline{x}_i^{(k)}-f(\underline{x}_i^{(k)})/\left(\prod_{j=1}^{i-1}(\underline{x}_i^{(k)}-\underline{x}_j^{(k+1)})\prod_{j=i+1}^{n}(\underline{x}_i^{(k)}-\bar{x}_j^{(k)})\right),\end{cases}$$

$$i=1,\ldots,n;\quad k=0,1,2,\ldots.$$

Remark 3. The above algorithm can be used also in the situation, when the roots of (1) are complex, but satisfy a partial ordering $z_1 < z_2 < \cdots < z_n$.

Remark 4. The equality

$$\frac{1}{2}\Big(\sum_{i=1}^{m} \underline{x}_i^{(k)} + \sum_{i=1}^{n} \overline{x}_j^{(k)}\Big) = \sum_{i=1}^{n} x_i = -a_1, \quad k=1, 2, \ldots$$

holds true independently of the initial approximations $\underline{x}_i^{(0)}, \overline{x}_i^{(0)}, i=1, \ldots, n$. It can be used for a control over the computations.

3. A two-sided method with cubic convergence. In this section we consider a numerical method for a simultaneous bounding of all roots of (1), which can be considered as a two-sided analogue of the method proposed by L. Ehrlich [6].

It is assumed again that the equation (1): $f(x) \equiv x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ possesses n simple real roots $x_1 < x_2 < \cdots < x_n$ located in nonintersecting initial intervals $X_i^{(0)} = [\underline{x}_i^{(0)}, \overline{x}_i^{(0)}]$. Consider the iteration procedure

$$(10) \quad \begin{cases} \overline{x}_i^{(k+1)} = \overline{x}_i^{(k)} - f(\overline{x}_i^{(k)}) / \Big(f'(\overline{x}_i^{(k)}) - f(\overline{x}_i^{(k)}) \sum\limits_{\substack{i=1 \\ j \neq 1}}^{n} (\overline{x}_i^{(k)} - \underline{x}_j^{(k)})^{-1}\Big), \\ \underline{x}_i^{(k+1)} = \underline{x}_i^{(k)} - f(\underline{x}_i^{(k)}) / \Big(f'(\underline{x}_i^{(k)}) - f(\underline{x}_i^{(k)}) \sum\limits_{\substack{j=1 \\ i \neq 1}}^{n} (\underline{x}_i^{(k)} - \overline{x}_j^{(k)})^{-1}\Big), \end{cases}$$

$$i=1, \ldots, n; \quad k=0, 1, 2, \ldots$$

Denote as above $d = \min_{i \neq j} |x_i - x_j| > 0$. The following assertion on cubic convergence of (10) holds true:

Theorem 2. *Let q and c be such that $0 < q < 1$, $0 < c \leq d/(2+\sqrt{n})$. If the two-sided initial approximations $\{\underline{x}_i^{(0)}, \overline{x}_i^{(0)}\}_{i=1}^n$ satisfy the inequalities*

$$(11) \quad \begin{aligned} &0 \leq \overline{x}_i^{(0)} - x_i \leq cq, \\ &0 \leq x_i - \underline{x}_i^{(0)} \leq cq, \ i=1, 2, \ldots, n, \end{aligned}$$

then for the two-sided approximations produces by (10) *the inequalities*

$$(12) \quad \begin{aligned} &0 \leq \overline{x}_i^{(k)} - x_i \leq cq^{3^k}, \\ &0 \leq x_i - \underline{x}_i^{(k)} \leq cq^{3^k}, \quad i=1, \ldots, n, \end{aligned}$$

hold true for all $k=0, 1, 2, \ldots$.

Proof. The inequalities (12) hold true for $k=0$ in view of (11). Assume that (12) hold true for some $k=m$, that is

$$(13a) \quad 0 \leq \overline{x}_i^{(m)} - x_i \leq cq^{3^m}, \quad i=1, \ldots, n;$$

$$(13b) \quad 0 \leq x_i - \underline{x}_i^{(m)} \leq cq^{3^m}, \quad i=1, \ldots, n.$$

We shall prove that (12) hold true for $k=m+1$, that is

$$(14a) \quad 0 \leq \overline{x}_i^{(m+1)} - x_i \leq cq^{3^{m+1}}, \quad i=1, \ldots, n,$$

$$(14b) \quad 0 \leq x_i - \underline{x}_i^{(m+1)} \leq cq^{3^{m+1}}, \quad i=1, \ldots, n.$$

Ws shall first prove that $0 \leq \bar{x}_i^{(m+1)} - x_i$ and $0 \leq x_i - \underline{x}_i^{(m+1)}$, that is the process (10) is two-sided.

Using the identity $f'(x)/(x) = \Sigma_{j=1}^n (x-x_j)^{-1}$ for $x \neq x_1, \dots, x_n$ and the first equality of (10) we obtain

$$\bar{x}_i^{(m+1)} - x_i = \bar{x}_i^{(m)} - x_i - f(\bar{x}_i^{(m)})/(f'(\bar{x}_i^{(m)}) - f(\bar{x}_i^{(m)}) \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{\bar{x}_i^{(m)} - \underline{x}_j^{(m)}})$$

$$= (\bar{x}_i^{(m)} - x_i)(1 - 1/((\bar{x}_i^{(m)} - x_i)(\sum_{j=1}^{n} \frac{1}{\bar{x}_i^{(m)} - x_j} - \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{1}{\bar{x}_i^{(m)} - \underline{x}_j^{(m)}})))$$

$$= (\bar{x}_i^{(m)} - x_i)(1 - 1/(1 + (\bar{x}_i^{(m)} - x_i) \sum_{\substack{j=1\\ j\neq i}}^{n} (\frac{1}{\bar{x}_i^{(m)} - x_j} - \frac{1}{\bar{x}_i^{(m)} - \underline{x}_j^{(m)}})))$$

$$= (\bar{x}_i^{(m)} - x_i)(1 - 1/(1 + (\bar{x}_i^{(m)} - x_i) \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{x_j - \underline{x}_j^{(m)}}{(\bar{x}_i^{(m)} - x_j)(\bar{x}_i^{(m)} - \underline{x}_j^{(m)})})).$$

We thus obtained

$$\bar{x}_i^{(m+1)} - x_i = (\bar{x}_i^{(m)} - x_i) \frac{s^{(m,i)}}{1 + s^{(m,i)}}, \tag{15}$$

wherein

$$s^{(m,i)} = (\bar{x}_i^{(m)} - x_i) \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{x_j - \underline{x}_j^{(m)}}{(\bar{x}_i^{(m)} - x_j)(\bar{x}_i^{(m)} - \underline{x}_j^{(m)})}.$$

Using that, according to (13a, b), $0 \leq \bar{x}_i^{(m)} - x_i$, $0 \leq x_i - \underline{x}_i^{(m)}$, and the fact that $\bar{x}_i^{(m)} \leq \underline{x}_{i+1}^{(m)}$ (which follows from (11) and the choice of q and c), we see that $s^{(m,i)} \geq 0$ and therefore $s^{(m,i)}/(1+s^{(m,i)} \geq 0$ as well. Together with (15) this implies $0 \leq \bar{x}_i^{(m+1)} - x_i$. Similarly the inequality $0 \leq x_i - \underline{x}_i^{(m+1)}$ is obtained. We thus showed that the iteration process (10) is two-sided.

We shall now prove that (10) has a cubic convergence. To this end we shall estimate $s^{(m,i)}$ from above. Using assumptions (13) and the inequalities $|x_i - x_j| \geq d - c > d - 2c$, $|\bar{x}_i^{(m)} - \underline{x}_j^{(m)}| \geq d - 2c$ we obtain

$$\bar{x}_i^{(m+1)} - x_i = (\bar{x}_i^{(m)} - x_i) \frac{s^{(m,i)}}{1 + s^{(m,i)}} \leq (\bar{x}_i^{(m)} - x_i) s^{(m,i)}$$

$$= (\bar{x}_i^{(m)} - x_i)(\bar{x}_i^{(m)} - x_i) \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{x_j - \underline{x}_j^{(m)}}{(\bar{x}_i^{(m)} - x_j)(\bar{x}_i^{(m)} - \underline{x}_j^{(m)})}$$

$$\leq cq^{3^m} \cdot cq^{3^m} \cdot \sum_{j=1}^{n} \frac{cq^{3^m}}{(d-2c)^2} = cd^{3^{m+1}} \frac{nc^2}{(d-2c)^2} \leq cq^{3^{m+1}},$$

since the inequalities $c \leq d/(2+\sqrt{n})$ and $nc^2/(d-2c)^2 \leq 1$ are equivalent.

4. Computer realization of the algorithms. We shall make use of the notations and computer realization technique described in [3], sec. 2A. In addition, we shall make use of the following notations:

$$\widehat{\Sigma}_{i=1}^{n} a_i = a_1 \widehat{+} a_2 \widehat{+} \cdots \widehat{+} a_n, \quad \check{\Sigma}_{i=1}^{n} a_i = a_1 \check{+} a_2 \check{+} \cdots + a_n;$$

$$\widehat{\Pi}_{i=1}^{n} a_i = a_1 \widehat{\times} a_2 \widehat{\times} \cdots \widehat{\times} a_n, \quad \check{\Pi}_{i=1}^{n} a_i = a_1 \check{\times} a_2 \check{\times} \cdots \check{\times} a_n,$$

where $a_1, a_2, \ldots, a_n$ are machine numbers. We note that the values of the above sums and products depend in general on the order of execution of the operations; for the sake of definiteness let us assume that the operations are executed from left to right.

Consider the computer realization of method (2), which we shall rewrite in the following form:

$$(16) \quad \begin{aligned} &\bar{x}_i^{(k+1)} = \bar{x}_i^{(k)} - f(\bar{x}_i^{(k)})/p^{(k,i)}, \\ &\underline{x}_i^{(k+1)} = \underline{x}_i^{(k)} - f(\underline{x}_i^{(k)})/q^{(k,i)}, \quad i=1, \ldots, n; \quad k=0, 1, \ldots, \end{aligned}$$

where

$$(16') \quad \begin{aligned} p^{(k,i)} &= \prod_{j=1}^{i-1} (\bar{x}_i^{(k)} - \underline{x}_j^{(k)}) \prod_{j=i+1}^{n} (\bar{x}_i^{(k)} - \bar{x}_j^{(k)}), \\ q^{(k,i)} &= \prod_{j=1}^{i-1} (\underline{x}_i^{(k)} - \underline{x}_j^{(k)}) \prod_{j=i+1}^{k} (\underline{x}_i^{(k)} - \bar{x}_j^{(k)}). \end{aligned}$$

Using the equality notation "=" in the meaning described in [3] (see the example in [3], sec 2A) and formulas (1) from [3] we may write

$$\bar{x}_i^{(k+1)} = \uparrow(\bar{x}_i^{(k)} - f(\bar{x}_i^{(k)})/p^{(k,i)}) = \bar{x}_i^{(k)} \hat{-} \downarrow(f(\bar{x}_i^{(k)})/p^{(k,i)}).$$

In order to find a computable lower bound for $f(\bar{x}_i^{(k)})/p^{(k,i)}$ we observe that, in the situation when $i=n, n-2, n-4, \ldots,$ the value of $f(\bar{x}_i^{(k)})$ is positive (this follows from geometrical arguments with respect to the polynomial f). On the other hand, the value of the denominator:

$$p^{(k,i)} = \prod_{j=1}^{i-1} (\bar{x}_i^{(k)} - \underline{x}_j^{(k)}) \prod_{j=i+1}^{n} (\bar{x}_i^{(k)} - \bar{x}_j^{(k)}) = (-1)^{n-i} \prod_{j=1}^{i-1} (\bar{x}_i^{(k)} - \underline{x}_j^{(k)}) \prod_{j=i+1}^{n} (\bar{x}_j^{(k)} - \bar{x}_i^{(k)})$$

has the sign of $(-1)^{n-i}$ and is therefore also positive for $i=n, n-2, \ldots$. We thus have $\downarrow(f(\bar{x}_i^{(k)})/p^{(k,i)}) \geq \downarrow f(\bar{x}_i^{(k)}) \check{:} \uparrow p^{(k,i)}$ and therefore the computer realization of $\bar{x}_i^{(k+1)}$ may be given by

$$\bar{x}_i^{(k+1)} = \bar{x}_i^{(k)} \hat{-} (\downarrow f(\bar{x}_i^{(k)}) \check{:} \uparrow p^{(k,i)}), \quad \text{when} \quad i=n, n-2, n-4, \ldots$$

In the situation when $i=n-1, n-3, n-5, \ldots$ the value of $f(\bar{x}_i^{(k)})$ is negative. Noticing that sign $p^{(k,i)} = \operatorname{sign} f(\bar{x})$ for all $i=1, 2, \ldots, n,$ we conclude that the value of $p^{(k,i)}$ is also negative in this situation. Thus we have

$$\bar{x}_i^{(k+1)} = \bar{x}_i^{(k)} \hat{-} (\uparrow f(\bar{x}_i^{(k)}) \check{:} \downarrow p^{(k,i)}), \quad \text{when} \quad i=n-1, n-3, n-4, \ldots$$

We can unify both situations by utilizing the roundings $\square$, $\square_0$ in the following formula

$$(17)\qquad \overline{x}_i^{(k+1)}=\overline{x}_i^{(k)}\mathbin{\hat{-}}(\Box f(\overline{x}_i^{(k)})\mathbin{\check{:}}\Box_0 p^{(k,i)}),\quad i=1,2,\dots,n.$$

For $p^{(k,i)}$ we have $\Box_0 p^{(k,i)}=(-1)^{n-i}\uparrow(\Pi_{j=1}^{i-1}(\overline{x}_i^{(k)}-\underline{x}_j^{(k)})\Pi_{j=i+1}^{n}(\overline{x}_j^{(k)}-\overline{x}_i^{(k)}))$ and therefore

$$(17')\qquad \Box_0 p^{(k,i)}=(-1)^{n-i}\hat{\Pi}_{j=1}^{i-1}(\overline{x}_i^{(k)}\mathbin{\hat{-}}\underline{x}_j^{(k)})\mathbin{\hat{\times}}\hat{\Pi}_{j=i+1}^{n}(\overline{x}_j^{(k)}\mathbin{\hat{-}}\overline{x}_i^{(k)}).$$

Similarly, for the computer realization of $\underline{x}_i^{(k+1)}$ we have

$$\underline{x}_i^{(k+1)}=\downarrow(\underline{x}_i^{(k)}-f(\underline{x}_i^{(k)})/q^{(k,i)})=\underline{x}_i^{(k)}\mathbin{\check{-}}\uparrow(f(\underline{x}_i^{(k)})/q^{(k,i)}).$$

Let $i=n, n-2, n-4, \dots$ Then $f(\underline{x}_i^{(k)})<0$ and, in view of $\operatorname{sign} q^{(k,i)}=-\operatorname{sign} f(\underline{x})_i^{(k)})$, $q^{(k,i)}>0$. Hence $\uparrow(f(\underline{x}_i^{(k)})/q^{(k,i)})\leqq\uparrow f(\underline{x}_i^{(k)})\mathbin{\hat{:}}\uparrow q^{(k,i)}$. Thus we can realize $\underline{x}_i^{(k+1)}$ in the computer by $\underline{x}_i^{(k+1)}=\underline{x}_i^{(k)}\mathbin{\check{-}}(\uparrow f(\underline{x}_i^{(k)})\mathbin{\hat{:}}\uparrow q^{(k,i)})$, when $i=n, n-2, \dots$

If $i=n-1, n-3, \dots$, we have $f(\underline{x}_i^{(k)})>0$, $q^{(k,i)}<0$. Therefore $\uparrow(f(\underline{x}_i^{(k)})/q^{(k,i)})\leqq\downarrow f(\underline{x}_i^{(k)})\mathbin{\hat{:}}\downarrow q^{(k,i)}$ and the computer realization formula for $\underline{x}_i^{(k+1)}$ obtains the form $\underline{x}_i^{(k+1)}=\underline{x}_i^{(k)}\mathbin{\check{-}}(\downarrow f(\underline{x}_i^{(k)})\mathbin{\hat{:}}\downarrow q^{(k,i)})$, when $i=n-1, n-3, \dots$

Both situations can be summarized as follows:

$$(18)\qquad \underline{x}_i^{(k+1)}=\underline{x}_i^{(k)}\mathbin{\check{-}}(\Box f(\underline{x}_i^{(k)}\mathbin{\hat{:}}\Box_0 q^{(k,i)}),$$

where

$$(18')\qquad \Box_0 q^{(k,i)}=(-1)^{n-i}\hat{\Pi}_{j=1}^{i-1}(\underline{x}_i^{(k)}\mathbin{\hat{-}}\underline{x}_j^{(k)})\mathbin{\hat{\times}}\hat{\Pi}_{j=i+1}^{n}(\overline{x}_j^{(k)}\mathbin{\hat{-}}\underline{x}_i^{(k)}).$$

Note that the computer realization (17), (17′), (18), (18′) of the algorithm (16)–(16′) requires a computation of the values of the polynomial f with rounding towards zero. This property of the algorithm can be used for automatic control over the roundoff error and a stopping criteria (see [2]). Recall that for a particular i we have either $\Box f(\underline{x}_i^{(k)})\leqq 0\leqq\Box f(\overline{x}_i^{(k)})$ or $\Box f(\underline{x}_i^{(k)})\geqq 0\geqq\Box f(\overline{x}_i^{(k)})$ for all $k=0, 1, 2, \dots$ If some of these inequalities is violated because of roundoff errors, further computations should be stopped.

We shall now consider the computer realization of our second method (10). We shall first rewrite it in the following way:

$$(19)\qquad \begin{aligned}&\overline{x}_i^{(k+1)}=\overline{x}_i^{(k)}-f(\overline{x}_i^{(k)})/p^{(k,i)},\\&\underline{x}_i^{(k+1)}=\underline{x}_i^{(k)}-f(\underline{x}_i^{(k)})/q^{(k,i)},\quad i=1,\dots,n;\ k=0,1,\dots,\end{aligned}$$

where

$$(19')\qquad \begin{aligned}&p^{(k,i)}=f'(\overline{x}_i^{(k)})-f(\overline{x}_i^{(k)})\Sigma_{j=1(j\neq i)}^{n}(\overline{x}_i^{(k)}-\underline{x}_j^{(k)})^{-1},\\&q^{(k,i)}=f'(\underline{x}_i^{(k)})-f(\underline{x}_i^{(k)})\Sigma_{j=1(j\neq i)}^{n}(\underline{x}_i^{(k)}-\overline{x}_j^{(k)})^{-1}.\end{aligned}$$

By means of the identity $f'(x)=f(x)\Sigma_{j=1}^{n}(x-x_j)^{-1}$, which is valid for $x\neq x_1,\dots,x_n$ we obtain

$$p^{(k,i)}=f(\overline{x}_i^{(k)})\Big(\frac{1}{\overline{x}_i^{(k)}-x_i}+\sum_{j=1(j\neq i)}^{n}\frac{x_j-\underline{x}_j^{(k)}}{(\overline{x}_i^{(k)}-x_j)(\overline{x}_i^{(k)}-\underline{x}_j^{(k)})}\Big),$$

$$q^{(k,i)}=f(\underline{x}_i^{(k)})\Big(\frac{1}{\underline{x}_i^{(k)}-x_i}+\sum_{j=1(j\neq i)}^{n}\frac{x_j-\overline{x}_j^{(k)}}{(\underline{x}_i^{(k)}-x_j)(\underline{x}_i^{(k)}-\overline{x}_j^{(k)})}\Big),$$

showing that the equalities sign $p^{(k,i)}=\text{sign}\, f(\overline{x}_i^{(k)})$ and $\text{sign}\, q^{(k,i)}=-\text{sign}\, f(\underline{x}_i^{(k)})$ hold true. The same situation took place by the first method (16)–(16′). Using this we see that the formulas for the computer computation of $\overline{x}_i^{(k+1)}$ and $\underline{x}_i^{(k+1)}$ have the same form as formulas (17) and (18) do, but with $p^{(k,i)}$ and $q^{(k,i)}$ defined by (19′), that is we have

$$\text{(20)}\quad \begin{aligned}&\overline{x}_i^{(k+1)}=\overline{x}_i^{(k)}\,\hat{-}\,(\Box f(\overline{x}_i^{(k)})\,\check{:}\,\Box_0 p^{(k,i)}),\\ &\underline{x}_i^{(k+1)}=\underline{x}_i^{(k)}\,\check{-}\,(\Box f(\overline{x}_i^{(k)})\,\hat{:}\,\Box_0 q^{(k,i)}),\quad i=1,\dots,n;\ k=0,1,\dots,\end{aligned}$$

wherein $p^{(k,i)}$ and $q^{(k,i)}$ are given by (19′).

We shall now discuss the computer realization of $\Box_0 p^{(k,i)}$, $\Box_0 q^{(k,i)}$. We first consider the computation of $\Box_0 p^{(k,i)}$.

i) For $i=n, n-2, n-1, \dots$ we have $p^{(k,i)}>0$, $f(\overline{x}_i^{(k)})>0$, $f'(\overline{x}_i^{(k)})>0$. Using these inequalities and denoting $\alpha^{(k,i)}=\Sigma_{j=1}^{i-1}(\overline{x}_i^{(k)}-\underline{x}_j^{(k)})^{-1}>0$, $\beta^{(k,i)}=\Sigma_{j=i+1}^{n}(\underline{x}_j^{(k)}-\overline{x}_i^{(k)})^{-1}>0$, so that $p^{(k,i)}=f'(\overline{x}_i^{(k)})-f(\overline{x}_i^{(k)})(\alpha^{(k,i)}-\beta^{(k,i)})$, we obtain

$$\begin{aligned}\Box_0 p^{(k,i)}&=\uparrow p^{(k,i)}=\uparrow(f'(\overline{x}_i^{(k)})-f(\overline{x}_i^{(k)})(\alpha^{(k,i)}-\beta^{(k,i)}))\\ &=\uparrow f'(\overline{x}_i^{(k)})\,\hat{-}\downarrow(f(\overline{x}_i^{(k)})\alpha^{(k,i)}-f(\overline{x}_i^{(k)})\beta^{(k,i)})\\ &=\uparrow f'(\overline{x}_i^{(k)})\,\hat{-}(\downarrow(f(\overline{x}_i^{(k)})\alpha^{(k,i)})\,\check{-}\uparrow(f(\overline{x}_i^{(k)})\beta^{(k,i)}))\\ &=\uparrow f'(\overline{x}_i^{(k)})\,\hat{-}(\downarrow f(\overline{x}_i^{(k)})\,\check{\times}\downarrow\alpha^{(k,i)}\,\check{-}\uparrow f(\overline{x}_i^{(k)})\,\hat{\times}\uparrow\beta^{(k,i)}).\end{aligned}$$

ii) For $i=n-1, n-3, n-5, \dots$ we have $p^{(k,i)}<0$, $f(\overline{x}_i^{(k)})<0$, $f'(\overline{x}_i^{(k)})<0$. Using these we obtain

$$\begin{aligned}\Box_0 p^{(k,i)}&=\downarrow p^{(k,i)}=\downarrow(f'(\overline{x}_i^{(k)})-f(\overline{x}_i^{(k)})(\alpha^{(k,i)}-\beta^{(k,i)}))\\ &=\downarrow f'(\overline{x}_i^{(k)})\,\check{-}(\uparrow(f(\overline{x}_i^{(k)})\alpha^{(k,i)})\,\hat{-}\downarrow(f(\overline{x}_i^{(k)})\beta^{(k,i)}))\\ &=\downarrow f'(\overline{x}_i^{(k)})\,\check{-}(\uparrow f(\overline{x}_i^{(k)})\,\hat{\times}\downarrow\alpha^{(k,i)}\,\hat{-}\downarrow f(\overline{x}_i^{(k)})\,\hat{\times}\uparrow\beta^{(k,i)}).\end{aligned}$$

Both cases i) and ii) can be summarized in a common formula as follows:

$$\text{(20′)}\quad \begin{aligned}&\Box_0 p^{(k,i)}=\Box_0 f'(\overline{x}_i^{(k)})[-]_0(\Box f(\overline{x}_i^{(k)})[\times]\downarrow\alpha^{(k,i)}[-]\Box_0 f(\overline{x}_i^{(k)})[\times]_0\uparrow\beta^{(k,i)}),\\ &\downarrow\alpha^{(k,i)}=\downarrow\Sigma_{j=1}^{i-1}(\overline{x}_i^{(k)}-\underline{x}_j^{(k)})^{-1}=\check{\Sigma}_{j=1}^{i-1}(1\,\check{:}\,(\overline{x}_i^{(k)}\,\hat{-}\,\underline{x}_j^{(k)})),\\ &\uparrow\beta^{(k,i)}=\uparrow\Sigma_{j=i+1}^{n}(\underline{x}_j^{(k)}-\overline{x}_i^{(k)})^{-1}=\hat{\Sigma}_{j=i+1}^{n}(1\,\hat{:}\,(\underline{x}_j^{(k)}\,\check{-}\,\overline{x}_i^{(k)})).\end{aligned}$$

The computer realization of $q^{(k,i)}$ is obtained similarly. The final form of $\Box_0 q^{(k,i)}$ is

$$\text{(20″)}\quad \begin{aligned}&\Box_0 q^{(k,i)}=\Box_0 f'(\underline{x}_i^{(k)})[-]_0(\Box_0 f(\underline{x}_i^{(k)})[\times]_0\uparrow\gamma^{(k,i)}[-]_0\Box f(\underline{x}_i^{(k)})[\times]\downarrow\delta^{(k,i)}),\\ &\uparrow\gamma^{(k,i)}=\hat{\Sigma}_{j=1}^{i-1}(1\,\hat{:}\,(x_i^{(k)}\,\check{-}\,\overline{x}_j^{(k)})),\\ &\downarrow\delta^{(k,i)}=\check{\Sigma}_{j=i+1}^{n}(1\,\check{:}\,(\overline{x}_j^{(k)}\,\hat{-}\,\underline{x}_i^{(k)})).\end{aligned}$$

Formulas (20), (20′), (20″) can be used for the computer realization of the method (10).

The above computer realization requires (as was the case with the first method) computation of the values of f with rounding towards zero, which again can be used for additional control over the computational errors as this is suggested in [2].

5. Numerical experiments. We give below the numerical solution of a problem, taken from [1] by means of algorithm (2). A computer realization of (2) given by formulas (17), (17′), (18), (18′) was used thereby. The necessary package for arithmetic operations with directed roundings was provided by N. Dushkov.

P r o b l e m (see [1, p. 133]). The eigenvalues of the matrix

$$A=\begin{pmatrix} 12 & 1 & 0 & 0 & 0 \\ 1 & 9 & 1 & 0 & 0 \\ 0 & 1 & 6 & 1 & 0 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

are to be determined.

N u m e r i c a l s o l u t i o n. The values of the function $f(x)=\det(A-xE)$ can be obtained by means of the iteration procedure

$$\text{(P)}\qquad \begin{aligned} &f_0(x)=1, \quad f_1(x)=x-a_1, \\ &f_k(x)=(x-a_k)f_{k-1}(x)-f_{k-2}(x), \quad k=2, \dots, 5, \\ &f(x)=f_5(x), \end{aligned}$$

wherein $a_1=12$, $a_2=9$, $a_3=6$, $a_4=3$, $a_5=0$ are the diagonal elements of A. The Gerschgorin's theorem produces the following initial intervals for the roots of $f(x)=0$:

$$\begin{aligned} &\underline{x}_1^{(0)}=-1.0 && \overline{x}_1^{(0)}=1.0 \\ &\underline{x}_2^{(0)}=1.0 && \overline{x}_2^{(0)}=5.0 \\ &\underline{x}_3^{(0)}=4.0 && \overline{x}_3^{(0)}=8.0 \\ &\underline{x}_4^{(0)}=7.0 && \overline{x}_4^{(0)}=11.0 \\ &\underline{x}_5^{(0)}=11.0 && \overline{x}_5^{(0)}=13.0 \end{aligned}$$

We note that the algorithm (2) turns out to be convergent, despite of the fact, that the initial intervals do not satisfy the conditions of Theorem 1.

The following results are obtained by means of formulas (17), (17′), (18) and (18′). We remark that the necessary expression for $\square f(x)$ can be easily obtained on the basis of formulas (P).

$$\begin{aligned} &\underline{x}_1^{(1)}=-0.719907407407407407 4\times 10^0 && \overline{x}_1^{(1)}=0.6473214285714287\times 10^0 \\ &\underline{x}_2^{(1)}=0.1820226879446260\times 10^1 && \overline{x}_2^{(1)}=0.4617563739376772\times 10^1 \end{aligned}$$

$\underline{x}_3^{(1)} = 0.4564671364076611 \times 10^1$	$\overline{x}_3^{(1)} = 0.7547201038706979 \times 10^1$
$\underline{x}_4^{(1)} = 0.7539111875953470 \times 10^1$	$\overline{x}_4^{(1)} = 0.1014422125380727 \times 10^2$
$\underline{x}_5^{(1)} = 0.1149454285461972 \times 10^2$	$\overline{x}_5^{(1)} = 0.1264037058382799 \times 10^2$
$\underline{x}_1^{(2)} = -0.5211150132880801 \times 10^0$	$\overline{x}_1^{(2)} = 0,2698718059591119 \times 10^0$
$\underline{x}_2^{(2)} = 0.2470322263560911 \times 10^1$	$\overline{x}_2^{(2)} = 0.4052152078430105 \times 10^1$
$\underline{x}_3^{(2)} = 0.5361068299164786 \times 10^1$	$\overline{x}_3^{(2)} = 0.6829199528827650 \times 10^1$
$\underline{x}_4^{(2)} = 0.8395360657675520 \times 10^1$	$\overline{x}_4^{(2)} = 0.9365625676545370 \times 10^1$
$\underline{x}_5^{(2)} = 0.1204068694431710 \times 10^2$	$\overline{x}_5^{(2)} = 0.1240485912575107 \times 10^2$
$\underline{x}_1^{(3)} = -0.3880642333754079 \times 10^0$	$\overline{x}_1^{(3)} = -0.8123150524417208 \times 10^{-}$
$\underline{x}_2^{(3)} = 0.2851853164017119 \times 10^1$	$\overline{x}_2^{(3)} = 0.3370608047750820 \times 10^1$
$\underline{x}_3^{(3)} = 0.5899901017430998 \times 10^1$	$\overline{x}_3^{(3)} = 0.6155600921679866 \times 10^1$
$\underline{x}_4^{(3)} = 0.8959670263685125 \times 10^1$	$\overline{x}_4^{(3)} = 0.9045091486413074 \times 10^1$
$\underline{x}_5^{(3)} = 0.1230205734618671 \times 10^2$	$\overline{x}_5^{(3)} = 0.1232129478228744 \times 10^2$
$\underline{x}_1^{(4)} = -0.3259323686327761 = 10^0$	$\overline{x}_1^{(4)} = -0.2845342295060234 \times 10^0$
$\underline{x}_2^{(4)} = 0.2976643022829831 \times 10^1$	$\overline{x}_2^{(4)} = 0.3008403665908639 \times 10^1$
$\underline{x}_3^{(4)} = 0.5998621602155305 \times 10^1$	$\overline{x}_3^{(4)} = 0.6002233886891408 \times 10^1$
$\underline{x}_4^{(4)} = 0.9015912894699321 \times 10^1$	$\overline{x}_4^{(4)} = 0.9016250730604659 \times 10^1$
$\underline{x}_5^{(4)} = 0.1231684961799483 \times 10^2$	$\overline{x}_5^{(4)} = 0.1231688378892840 \times 10^2$
$\underline{x}_1^{(5)} = -0.3169458949086399 = 10^0$	$\overline{x}_1^{(5)} = -0.3166231111746032 \times 10^0$
$\underline{x}_2^{(5)} = 0.2983858069614485 \times 10^1$	$\overline{x}_2^{(5)} = 0.2983883011113905 \times 10^1$
$\underline{x}_3^{(5)} = 0.5999999928184809 \times 10^1$	$\overline{x}_3^{(5)} = 0.6000000116470558 \times 10^1$
$\underline{x}_4^{(5)} = 0.901613630074 3464 \times 10^1$	$\overline{x}_4^{(5)} = 0.9016136304400465 \times 10^1$
$\underline{x}_5^{(5)} = 0.1231687595245489 \times 10^2$	$\overline{x}_5^{(5)} = 0.1231687595266509 \times 10^2$
$\underline{x}_1^{(6)} = -0.3168759530274313 \times 10^0$	$\overline{x}_1^{(6)} = -0.3168759511325731 \times 10^0$
$\underline{x}_2^{(6)} = 0.2983863696837963 \times 10^1$	$\overline{x}_2^{(6)} = 0.2983863696838936 \times 10^1$
$\underline{x}_3^{(6)} = 0.5999999999999999 \times 10^1$	$\overline{x}_3^{(6)} = 0.6000000000000001 \times 10^1$

$\underline{x}_4^{(6)}= \quad 0.9016136303161817\times10^1$ $\quad \bar{x}_4^{(6)}= \quad 0.9016136303161819\times10^1$

$\underline{x}_5^{(6)}= \quad 0.1231687595261686\times10^2$ $\quad \bar{x}_5^{(6)}= \quad 0.1231687595261688\times10^2$

$\underline{x}_1^{(7)}=-0.3168759526168761\times10^0$ $\quad \bar{x}_1^{(7)}=-0.3168759526168759\times10^0$

$\underline{x}_2^{(7)}= \quad 0.2983863696838181\times10^1$ $\quad \bar{x}_2^{(7)}= \quad 0.2983863696838183\times10^1$

$\underline{x}_3^{(7)}=\underline{x}_3^{(6)}$ $\quad \bar{x}_3^{(7)}=\bar{x}_3^{(6)}$

$\underline{x}_4^{(7)}=\underline{x}_4^{(6)}$ $\quad \bar{x}_4^{(7)}=\bar{x}_4^{(6)}$

$\underline{x}_5^{(7)}=\underline{x}_5^{(6)}$ $\quad \bar{x}_5^{(7)}=\bar{x}_5^{(6)}$

The computations are performed on an IBM/360 computer using double precision arithmetic with directed roundings. The results show that the round-off error in the above example is negligible.

6. Appendix: The Dočev's method with a new proof of convergence. In this section we shall formulate the method of K. Dočev [4, 5] and shall give a proof for convergence involving a less restrictive assumption of closeness of the initial approximations to the roots, as the assumptions used in the assertions for convergence known to us.

Consider the algebraic equation of nth degree, $n\geqq2$:

$$f(z)=z^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0=0 \tag{21}$$

with given coefficients $a_0, a_1, \ldots, a_{n-1}$. We assume that (21) possesses n simple roots $z_1, z_2, \ldots, z_n$. Thus we have $f(z)=\Pi_{i=1}^n(z-z_i)$. Denote $d=\min_{i\neq j}|z_i-z_j|$; since $z_1, z_2, \ldots, z_n$ are simple, $d>0$.

Assume next that n numbers $z_1^{(0)}, z_2^{(0)}, \ldots, z_n^{(0)}$ are given, which will be considered as initial approximations of the corresponding roots $z_1, z_2, \ldots, z_n$.

The method of Dočev is

$$z_i^{(k+1)}=z_i^{(k)}-f(z_i^{(k)})/\Pi_{j=1(j\neq i)}^n(z_i^{(k)}-z_j^{(k)}),\ i=1,\ldots,n;\ k=0, 1, \ldots \tag{22}$$

The following assertion for quadratic convergence of (22) holds true

Theorem 3. *Let* $0<q<1$ *and* $0<c\leqq d/(1+\alpha n)$, *where* $\alpha=1,7632283\ldots$ *is determined from the equality* $\alpha=\exp(\alpha^{-1})$. *If the initial approximations* $z_1^{(0)}, z_2^{(0)}, \ldots, z_n^{(0)}$ *of the roots* $z_1, z_2, \ldots, z_n$ *of* (21) *satisfy the inequalities*

$$|z_i^{(0)}-z_i|\leqq cq,\ i=1,\ldots,n, \tag{23}$$

then for the approximations delivered by (22) *the inequalities*

$$|z_i^{(k)}-z_i|\leqq cq^{2^k},\ i=1,\ldots,n, \tag{24}$$

hold true for all $k=1, 2, \ldots$

Remark. The proof of convergence of (22) given in [8], p. 206, requires the substantially more restrictive assumption for closeness of the initial approximations to the roots: $c<d/(2+2^n)$ (in [8] this is written in the equivalent form $2^nc/(d-2c)<1$).

Proof. By induction. The assumption (23) shows that (24) holds true for $k=0$. Let (24) be satisfied for some $k=m$, that is

(25) $$|z_i^{(m)}-z_i|\leqq cq^{2^m},\ i=1, 2, \ldots, n.$$

This implies, in view of $q<1$,

(26) $$|z_i^{(m)}-z_i|\leqq c,\ i=1, 2, \ldots, n.$$

From (26) and the choice of c it follows that $z_i^{(m)}\neq z_j^{(m)}$ for $i\neq j$. Moreover, for $i\neq j$ we have

(27) $$|z_i^{(m)}-z_j^{(m)}|\geqq|z_i-z_j|-|z_i^m-z_i|-|z_j^{(m)}-z_j|\geqq d-2c\geqq c\alpha(n-1),$$

the last inequality following from the assumption $c\leqq d/(1+\alpha n)$.

Using (22), we obtain for the difference $z_i^{(m+1)}-z_i$:

$$z_i^{(m+1)}-z_i=z_i^{(m)}-\prod_{j=1}^{n}(z_i^{(m)}-z_j)/\prod_{\substack{j=1\\ j\neq i}}^{n}(z_i^{(m)}-z_j^{(m)})-z_i=(z_i^{(m)}-z_i)(1-\prod_{\substack{i=1\\ j\neq i}}^{n}\frac{z_i^{(m)}-z_j}{z_i^{(m)}-z_j^{(m)}}).$$

We shall transform the last expression by means of the identity (see [9]):

$$\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{u_i-z_j}{u_i-u_j}-1=\sum_{\substack{s=1\\ s\neq i}}^{n}\frac{u_s-z_s}{u_i-u_s}\prod_{\substack{j=1\\ j\neq i}}^{s-1}\frac{u_i-z_j}{u_i-u_j},$$

which holds true for arbitrary $2n$ numbers $z_1, z_2, \ldots, z_n$; $u_1, u_2, \ldots, u_n$, such that $u_i\neq u_j$ for $i\neq j$. Using this identity with $u_i=z_i^{(m)}$ (which can be done because of $z_i^{(m)}\neq z_j^{(m)}$ for $i\neq j$), we obtain

$$|z_i^{(m+1)}-z_i|=|z_i^{(m)}-z_i|\,|\sum_{\substack{s=1\\ s\neq i}}^{n}\frac{z_s^{(m)}-z_s}{z_i^{(m)}-z_s^{(m)}}\prod_{\substack{j=1\\ j\neq i}}^{s-1}\frac{z_i^{(m)}-z_j}{z_i^{(m)}-z_j^{(m)}}|$$

$$\leqq|z_i^{(m)}-z_i|\sum_{\substack{s=1\\ s\neq i}}^{n}|\frac{z_s^{(m)}-z_s}{z_i^{(m)}-z_s^{(m)}}|\prod_{\substack{j=1\\ j\neq i}}^{s-1}|1+\frac{z_j^{(m)}-z_j}{z_i^{(m)}-z_j^{(m)}}|$$

$$\leqq|z_i^{(m)}-z_i|\sum_{\substack{s=1\\ s\neq i}}^{n}\frac{|z_s^{(m)}-z_s|}{|z_i^{(m)}-z_s^{(m)}|}\prod_{\substack{j=1\\ j\neq i}}^{s-1}(1+\frac{|z_j^{(m)}-z_j|}{|z_i^{(m)}-z_j^{(m)}|}).$$

Using inequalities (25)—(27) in the above expression, we obtain

$$|z_i^{(m+1)}-z_i|\leqq cq^{2^m}\sum_{\substack{s=1\\ s\neq i}}^{n}\frac{cq^{2^m}}{|z_i^{(m)}-z_s^{(m)}|}\prod_{\substack{j=1\\ j\neq i}}^{s-1}(1+\frac{c}{|z_i^{(m)}-z_j^{(m)}|})$$

$$\leqq cq^{2^m}(n-1)\frac{cq^{2^m}}{c\alpha(n-1)}(1+\frac{c}{c\alpha(n-1)})^{n-1}$$

$$=cq^{2^{m+1}}\frac{1}{\alpha}(1+\frac{1}{\alpha(n-1)})^{n-1}\leqq cq^{2^{m+1}}\frac{1}{\alpha}e^{1/\alpha}=cq^{2^{m+1}},$$

since α is such, that $\alpha=e^{1/\alpha}$.

This implies that (24) holds true for $k=m+1$, which proves the theorem.

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Centre for Mathematics and Mechanics
1090 Sofia P. O. Box 373

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