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ПЛИСКА

БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

PARTIALLY MONOTONE APROXIMATION OF DIFFERENTIABLE FUNCTIONS

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Let $\Delta=[-1, 1]$, $C_\Delta^{(k)}$ be the class of k-times differentiable in Δ functions with a continuous k-th derivative, let H_n be the set of algebraic polynomials of degree not greater than n, $H_n^1=\{P\colon P\in H_n,\ P'(x)\geqq 0 \text{ for } x\in\Delta\}$, $H_n^2=\{P\colon P\in H_n,\ P'(x)\leqq 0 \text{ for } x\in[-1, 0],\ P'(x)\geqq 0 \text{ for } x\in[0, 1]\}$. The following result [1] is due to G. Lorentz and K. Zeller:

Theorem A. *If $f\in C_\Delta^{(0)}=C_\Delta$ and $f(x)$ is monotonely increasing for $x\in\Delta$, then for any positive integer n there exists $P\in H_n^1$ such that*

$$\max_{x\in\Delta}|f(x)-P(x)|=\|f-P\|\leqq c_1\omega(f;\ n^{-1}),$$

where $\omega(f;\ \delta)$, $(\delta>0)$, is the modulus of continuity of the function f

$$\omega(f, \delta)=\sup_{|x_1-x_2|\leqq\delta}|f(x_1)-f(x_2)|;\quad x_1, x_2\in\Delta.$$

Remark. Everywhere further $c_i(*)$, $i=1, \dots,$ will denote the positive constants depending only on the parameters pointed between brackets. The denotation c_i defines an absolute constant.

In [2] the following generalization of Theorem A has been obtained:

Theorem B. *If $f\in C_\Delta$ and $f(x)$ monotonely decreasing for $x\in[-1, 0]$ and monotonely increasing for $x\in[0, 1]$, then for any positive integer n there exists $P\in H_n^2$ such that $\|f-P\|\leqq c_2\omega(f;\ n^{-1})$.*

In [3] DeVore proves:

Theorem C. *If $f\in C_\Delta^{(k)}$ and $f(x)$ is monotonely increasing for $x\in\Delta$, then for any positive integer n there exists $P\in H_n^1$ such that*

$$\|f-P\|\leqq c_3(k)\omega(f^{(k)};\ n^{-1})/n^k,\quad k=1, 2, \dots$$

The aim of the present paper is the proof of:

Theorem 1. *Let $f\in C_\Delta^{(2)}$, $f''\in \mathrm{Lip}_M 1$, $f''(0)\neq 0$, $f(x)$ be monotonely decreasing for $x\in[-1, 0]$ and monotonely increasing for $x\in[0, 1]$. Then for every positive integer n there exists $P\in H_n^2$ such that $\|f-P\|\leqq c_4M/n^3$.*

In [4] the following theorem has been proved:

Theorem D. *If $f\in C_\Delta^{(k)}$, $f(0)=f^{(1)}(0)=\dots=f^{(k)}(0)$, $f(x)$ is monotonely decreasing for $x\in[-1, 0]$ and $f(x)$ is monotonely increasing for $x\in[0,1]$, then for any positive integer n there exists $P\in H_n^2$ such that*

$$\|f-P\|\leqq c_5(k)\omega(f^{(k)};\, n^{-1})/n^k, \quad k=1,\, 2, \dots$$

Theorem D is an analogue of Theorem C for the partially monotone approximation under the additional conditions $f(0)=f^{(1)}(0)=\dots=f^{(k)}(0)=0$. In [4] an idea for the proof of Theorem D has been suggested without the conditions $f(0)=f^{(1)}(0)=\dots=f^{(k)}(0)=0$. Actually, the condition $f(0)=0$ is removed in a trivial way, the condition $f^{(1)}(0)=0$ is imposed by the nature of the problem. That is why in the present paper a proof of Theorem 1 has been accomplished, Theorem 1 being Theorem D for $k=2$ and without the additional condition $f^{(2)}(0)=0$. The Theorems A, B, C, D and 1 for the monotone and partially monotone approximation are analogues of the well-known theorems of Jackson for approximation of continuous and differentiable functions by algebraic polynomials.

Analogous problems concerning the monotone and partially monotone approximation of functions by splines have been considered by De Vore, D. Leviatan, H. Mhaskar, R. Beatson and others.

As a result of discussion with D. Leviatan, it was found out that the result of Theorem D has been obtained by him independently of the paper [4].

Further we will need the following theorem of V. N. Malozemov [5]:

Theorem E. *For every function* $\xi(x)\in C_\Delta^{(k)}$ *and every positive integer* n *there exists* $Q\in H_n$ *such that*

$$|\xi^{(i)}(x)-Q^{(i)}(x)|\leqq c_6(k)\Big(\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\Big)^{k-i}\omega\Big(\xi^{(k)};\,\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\Big)$$

$$x\in\Delta, \quad i=0,\, 1, \dots,\, k.$$

As a corollary from the result in [6] the following lemma is easily obtained:

Lemma A. *If the spline function* φ *of order* $[c_7(k)]$ *and degree* $[c_7(k)]$ *is such that* $\varphi\in C_\Delta^{(1)}$ *and* $\varphi'\in \mathrm{Lip}_M 1$, *then for any natural number* n *a polynomial* $R\in H_n$ *exists, for which*

$$\int_{-1}^{1}|\varphi(x)-R(x)|\,dx\leqq c_8(k)\,\frac{M}{n^3}$$

and $R(x)\geqq\varphi(x)$, $x\in\Delta$, *are fulfilled.*

In [4] the following lemma is proved:

Lemma B. *For arbitrary non-negative integers* n *and* k, $k\leqq n$, $n\geqq 1$, *there exists a polynomial* $T\in H_n$, *monotonely increasing and odd in* Δ, *such that* $-1\leqq T(x)\leqq 1$ *for* $x\in\Delta$ *and*

$$\int_{-1}^{1}\frac{|x|^k\,|\sigma(x)-T(x)|}{\sqrt{1-x^2}}\,dx\leqq c_9(k)/n^{k+1},$$

where

$$\sigma(x)=\begin{cases}-1 \ \ for\ x\in[-1,\, 0);\\ \ \ 1\ \ for\ x\in[0,\, 1].\end{cases}$$

Proof of theorem 1. Without loss of generality it might be assumed that $f(0)=0$. In view of the conditions

$$f\in C_\Delta^{(2)},$$
(1) $$f'(x)\leq 0 \text{ for } x\in[-1,\ 0],$$
$$f'(x)\geq 0 \text{ for } x\in[0,\ 1],$$

it follows that $f'(0)=0$, i. e. $f(0)=f'(0)=0$.

Expand $f(x)$ into a MacLoren series:

$$f(x)=f(0)+\frac{x}{1!}f'(0)+\frac{x^2}{2!}f''(\zeta(x))=F(x)+\frac{x^2}{2!}f''(0),$$

where

$$F(x)=\frac{x^2}{2!}[f''(\zeta(x))-f''(0)]=f(x)-\frac{x^2}{2!}f''(0).$$

Obviously, $F\in C_\Delta^{(2)}$, $F''\in \mathrm{Lip}_{c_{10}M}\,1$ and $F(0)=F'(0)=F''(0)=0$.

Let

$$\bar F(x)=\begin{cases}-F(x) & \text{for } x\in[-1,\ 0],\\ F(x) & \text{for } x\in[0,\ 1].\end{cases}$$

Then

(2) $$\bar F\in C_\Delta^{(2)},\quad \bar F''\in \mathrm{Lip}_{c_{10}M}\,1,\quad \bar F(0)=\bar F'(0)=\bar F''(0)=0$$

and

(3) $$\bar F'(x)\geq -f''(0)|x|,\quad x\in\Delta,$$

which follows immediately from (1).

Moreover,

(4) $$f''(0)>0,$$

as is easily seen from (1) and from the assumption that $f''(0)\neq 0$.

First, we will prove the existence of a polynomial $P_1\in H_n$ such that

$$\|\bar F-P_1\|\leq c_{11}M/n^3,\quad P_1'(x)\geq -f''(0)\,|x|,\quad x\in\Delta.$$

It can be proved that for every function satisfying conditions (2), (3) and (4), for $\bar F(x)$, respectively, there exists a spline function φ of order and degree bounded by an absolute constant such that

(5) $$\varphi\in C_\Delta^{(1)},\quad \varphi'\in \mathrm{Lip}_{c_{12}M}\,1$$

and

(6) $$-f''(0)\,|x|\leq\varphi(x)\leq\bar F'(x),\quad x\in\Delta.$$

For example the partially polynomial function

$$\varphi(x)=\begin{cases} f''(0)x & \text{for } x\in(-\infty, -\frac{2f''(0)}{M}], \\ \frac{M}{4}x^2+2f''(0)x+\frac{[f''(0)]^2}{M} & \text{for } x\in[-\frac{2f''(0)}{M}, -\frac{f''(0)}{M}], \\ -\frac{3}{4}Mx^2 & \text{for } x\in[-\frac{f''(0)}{M}, \frac{f''(0)}{M}], \\ \frac{M}{4}x^2-2f''(0)x+\frac{[f''(0)]^2}{M} & \text{for } x\in\left[\frac{f''(0)}{M}, \frac{2f''(0)}{M}\right], \\ -f''(0)x & \text{for } x\in[\frac{2f''(0)}{M}, \infty) \end{cases}$$

satisfies the above conditions.

Applying the cited above Lemma A for the function φ, we obtain a polynomial $R\in H_n$, for which

(7) $$R(x)\geqq\varphi(x), \quad x\in\Delta,$$

and

(8) $$\int_{-1}^{1}|\varphi(x)-R(x)|dx\leqq c_{13}\frac{M}{n^3}$$

hold.

Form the function $\psi(x)=\int_0^x\varphi(y)dy$.

(5) implies that $\psi\in C_\Delta^{(2)}$, $\psi''\in \mathrm{Lip}_{c_{12}M}\,1$, whence (having in mind (6)) we obtain that the function $p(x)=\bar{F}(x)-\psi(x)$ satisfies the following conditions $p\in C_\Delta^{(2)}$, $p''\in\mathrm{Lip}_{c_{13}M}\,1$ and

$$p'(x)=\bar{F}'(x)-\psi'(x)=\bar{F}'(x)-\varphi(x)\geqq 0, \quad x\in\Delta.$$

Then the theorem of DeVore, applied to the function $p(x)$, assures the existence of a polynomial $S\in H_n$ such that

(9) $$S'(x)\geqq 0, \quad x\in\Delta$$

and

(10) $$\|p-S\|\leqq c_{14}M/n^3.$$

Form the polynomial $P_1(x)=S(x)+\int_0^x R(y)dy$, which obviously belongs to the class H_n.

(8) and (9) yield

$$|\bar{F}(x)-P_1(x)|\leqq|\bar{F}(x)-\psi(x)+S(x)|+|\psi(x)-\int_0^x R(y)dy|$$

$$\leqq\|p-S\|+\int_0^x|\varphi(y)-R(y)|dy\leqq c_{15}\frac{M}{n^3},$$

i. e.

(11) $$\|\bar{F}-P_1\|\leqq c_{15}M/n^3.$$

In view of (6), (7) and (9) it follows that

(12) $$P_1'(x)=S'(x)+R(x)\geqq 0+\varphi(x)\geqq -f''(0)|x|, \quad x\in\Delta.$$

Form the polynomial $\tilde{P}(x)=\int_0^x P_1'(y)T(y)dy$, where T is the polynomial from Lemma B. Obviously, $\tilde{P}\in H_{2n-1}$.

By $\Delta(z)$ denote the difference $\sigma(z)-T(z)$, where, as was already mentioned,

$$\sigma(z)=\begin{cases}-1 & \text{for } z\in[-1,0)\\ 1 & \text{for } z\in[0,1].\end{cases}$$

Let $x\leqq 0$. Then

$$\tilde{P}(x)=\int_0^x P_1'(y)T(y)dy=\int_0^x P_1'(y)[-1-\Delta(y)]dy$$

$$=-P_1(x)+P_1(0)-\int_0^x P_1'(y)\Delta(y)\,dy,$$

whence

$$|F(x)-\tilde{P}(x)|=|F(x)+P_1(x)-P_1(0)+\int_0^x P_1'(y)\Delta(y)\,dy|$$

$$\leqq|F(x)+P_1(x)|+|P_1(0)|+\int_{-1}^1|P_1'(y)|\Delta(y)|dy;\quad x\in[-1,0].$$

(11), (2) and $\bar{F}(x)=-F(x)$ for $x\in[-1,0]$ imply

$$|F(x)+P_1(x)|=|-\bar{F}(x)+P_1(x)|\leqq\|\bar{F}-P_1\|\leqq c_{16}M/n^3,$$

$$|P_1(0)|=|\bar{F}(0)-P_1(0)|\leqq c_{16}M/n^3,$$

whence for $x\in[-1,0]$ we obtain

$$|F(x)-\tilde{P}(x)|\leqq c_{17}\frac{M}{n^3}+\int_{-1}^1|P_1'(y)|\,|\Delta(y)|dy. \tag{13}$$

Let $x>0$. Then

$$\tilde{P}(x)=\int_0^x P_1'(y)\Delta(y)\,dy=\int_0^x P_1'(y)\,[1-\Delta(y)]dy$$

$$=P_1(x)-P_1(0)-\int_0^x P_1'(y)\Delta(y)dy,$$

whence

$$|F(x)-\tilde{P}(x)|=|F(x)-P_1(x)+P_1(0)+\int_0^x P_1'(y)\Delta(y)dy|$$

$$\leqq|F(x)-P_1(x)|+|P_1(0)|+\int_{-1}^1|P_1(y)|\;|\Delta(y)|dy$$

$$=|\bar{F}(x)-P_1(x)|+|\bar{F}(0)-P_1(0)|+\int_{-1}^1|P_1'(y)|(\Delta(y)|dy.$$

Therefore, for $x \in [0, 1]$

(14) $$|F(x)-\tilde{P}(x)| \leq c_{18}\frac{M}{n^3}+\int_{-1}^{1}|P_1'(y)|\,|\Delta(y)|dy.$$

It remains to estimate from above $\int_{-1}^{1}|P_1'(y)|\,|\Delta(y)|dy$.

In view of Malozemov theorem it follows that a polynomial $Q \in H_n$ exists, for which

(15) $$\|\bar{F}-Q\| \leq c_{19}M/n^3$$

and

(16) $$|\bar{F}'(y)-Q'(y)| \leq c_{20}M/n^2, \quad y \in \Delta.$$

(11) and (15) imply that $|P_1(y)-Q(y)| \leq c_{21}M/n^3$, $y \in \Delta$, whence, having applied the second inequality of Bernstein, we get

$$\sqrt{1-y^2}\,|P_1'(y)-Q'(y)| \leq c_{22}M/n^2, \quad y \in \Delta.$$

From (16) and from the inequality $|\bar{F}'(y)| \leq c_{23}My^2$, $y \in \Delta$, which follows easily from (2), we get

$$|P_1'(y)| \leq c_{24}My^2+c_{24}\frac{M}{n^2}+c_{24}\frac{M}{n^2}\cdot\frac{1}{\sqrt{1-y^2}}, \quad y \in \Delta.$$

Applying Lemma A and using the inequality $1 \leq 1/\sqrt{1-y^2}$, $y \in \Delta$, we obtain

(17) $$\int_{-1}^{1}|P_1'(y)|\,|\Delta(y)|\,dy \leq c_{25}\frac{M}{n^3}.$$

(13), (14) and (17) imply $\|F-\tilde{P}\| \leq c_{26}M/n^3$.

Now we will check that $\tilde{P}'(x) \leq -f''(0)x$ for $x \in [-1, 0]$, $\tilde{P}'(x) \geq -f''(0)x$ for $x \in [0, 1]$.

Let x_0 be an arbitrary point from $[-1, 0]$. Then $T(x_0)=-1-\Delta(x_0)$, and $\Delta(x_0) \leq 0$. If $P_1'(x_0) \leq 0$, then $P_1'(x_0)\Delta(x_0) \geq 0$.

From (12) and from the last inequality it follows

(18) $$\tilde{P}'(x_0)=P_1'(x_0)T(x_0)=P_1'(x_0)[-1-\Delta(x_0)]$$
$$=-P_1'(x_0)-P_1'(x_0)\Delta(x_0) \leq -P_1'(x_0) \leq -f''(0)x_0.$$

If $P_1'(x_0) \geq 0$, then $P_1'(x_0)\,T(x_0) \leq 0$. However, (4) implies that $f''(0)x_0 \leq 0$, therefore $-f''(0) \geq 0$, whence we get

(19) $$\tilde{P}'(x_0)=P_1'(x_0)T(x_0) \leq 0 \leq -f''(0)x_0.$$

From (18) and (19) we obtain $\tilde{P}'(x) \leq -f''(0)x$ for $x \in [-1, 0]$.

It is analogously verified that $\tilde{P}'(x) \geq -f''(0)x$ for $x \in [0, 1]$.

Therefore, for any natural number n a polynomial $\hat{P} \in H_n$ can be found such that

$$\|F-\hat{P}\| \leq c_{27}M/n^3,$$

$$\hat{P}'(x) \leq -f''(0)x \text{ for } x \in [-1, 0], \quad \hat{P}'(x) \geq -f''(0)x \text{ for } x \in [0, 1].$$

Then the polynomial $P(x)=\widehat{P}(x)+\frac{x^2}{2!}f''(0)$ will satisfy the conditions of the Theorem. Thus the proof of the theorem is accomplished.

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