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ALGEBRAIC EXTENSIONS OF PRIME ALGEBRAS AND ALGEBRAS OF QUOTIENTS

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Let F be a field, $R\supset B$ be F-algebras such that for every element $x\in R$ there is a polynomial $f(t)$ with $f(x)\in B$. The following results are proved: If R is prime and has no nonzero algebraic one-sided ideals, then B is prime with no nonzero algebraic one-sided ideals; if moreover R is noncommutative, then the algebras of quotients of R and B coincide. In that case R is a right Goldie algebra iff B is such an algebra.

Introduction. In what follows R will denote an associative algebra with 1 over a field F and B will denote a subalgebra of R with the same unit 1. An algebra R is said to be radical (coradical) over B if for every $x\in R$ there is an integer $n(x)>1$ (resp. polynomial $f_x(t)\in F[t]$) such that $x^{n(x)}\in B$ (resp. $x+x^2f_x(x)\in B$). There have been several theorems proved in the last few years, which describe the relationship between the properties of R and B in the case when R is radical (coradical) over B. For example Herstein, Rowen and Zelmanov proved that if R has no nil ideals and R is radical over a PI-subalgebra B, then R satisfies all polynomial identities, satisfied by B. Chacron proved the same theorem in the case when R is coradical over B. Babkov [1] proved that if R is a prime noncommutative algebra with no nonzero nil one-sided ideals and R is radical over B, or R is prime, noncommutative and R is coradical over B, then the algebras of quotients of R and B coincide.

More generally following [3] we shall consider the relationship between R and B in the case when for every $x\in R$ there is a polynomial $f_x(t)\in F[t]$ such that $f_x(x)\in B$.

We shall make some specifications. Whenever we shall use the term "polynomial" it will be understood to be a polynomial with coefficients in F. If $x\in R$ when we write $f_x(t)$ we shall mean such a polynomial whose coefficients depend on x. When we say that an element $x\in R$ is algebraic, we shall mean that it is algebraic over F, i. e. there is a polynomial $f_x(t)$, such that $f_x(x)=0$. A subset of R is said to be algebraic if all its elements are algebraic. An element which is not algebraic will be called transcendental. We shall say that R is F-algebraic over B, if for every $x\in R$ there exists a polynomial $f_x(t)$, such that $f_x(x)\in B$. It is clear that we may assume that all polynomials are with zero constant term, i. e. that they are in $tF[t]$. Bergen and Herstein ([3]) proved that if R has no nonzero algebraic ideals and R is F-algebraic over the subalgebra B, which satisfies a polynomial identity, then R satisfies all polynomial identities, satisfied by B.

Our aim in this paper is to prove, that if R has no nonzero algebraic one-sided ideals and R is F-algebraic over B, then the algebras of quotients of R and B coincide.

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1. F-Algebraicity and Prime Algebras.

Lemma 1.1. *Every algebraic one-sided ideal of R is either nil, or has an idempotent* $e^2=e\neq 0$.

Proof. Let ρ is a right algebraic ideal of R. If ρ is not nil, then there is a $x\in\rho$, such that $x^k\neq 0$ for each integer $k\geq 1$. But x is an algebraic and hence there are $\alpha_1, \alpha_2, \ldots, \alpha_p \in F$ and an integer $s\geq 1$ for which:

$$x^s=\alpha_1\ x^{s+1}+\ \ldots\ +\alpha_p\, x^{s+p}\neq 0.$$

In that case:

$$x^s=\alpha_1\ x\,(\alpha_1\, x^{s+1}+\ \ldots\ +\alpha_p\ x^{s+p})+\ \ldots\ +\alpha_p\, x^{s+p}$$
$$=\beta_1\ x^{s+2}+\ \ldots\ +\beta_p\ x^{s+p+1}$$
$$=\gamma_1\ x^{2s}+\gamma_2\ x^{2s+1}+\ \ldots\ +\gamma_p\ x^{2s+p-1}=x^{2s}\ h\,(x),$$

where $0\neq h\,(t)\in F[t]$. From here: $e=x^s\,h\,(x)\neq 0$ and $e\in\rho$. By that $e^2=x^{2s}\ h^2\,(x)$ $=x^s h(x)=e$. Therefore e is an idempotent.

Corollary 1. 2. *If R is an algebraic domain, then R has no nonzero algebraic one-sided ideals.*

Lemma 1.3 ([3], Theorem 4.1). *Let R be prime with no nonzero algebraic one-sided ideals and λ be a left ideal of R. If $b\in R$ is such that for every $x\in\lambda$ there exists a polynomial $f_x(t)\in tF[t]$ for which $bf_x(x)=0$, then $b=0$.*

Lemma 1.4. *Let R be prime with no nonzero algebraic one-sided ideals. If $a\in R$ and for every $x\in R$ there exists a polynomial $f_x(t)\in tF\,[t]$ such that $af_x(x)\ a=0$, then $a=0$.*

Proof. First we show that $a^2=0$. Let $x\in R$. There exists a polynomial $f(t)\in tF[t]$ such that $af(ax)\ a=0$. For $g(t)=tf(t)$ we have $0=af(ax)\,ax=ag(ax)$ $=a^2f(xa)\,x$, and $a^2\,g(xa)=a^2f(xa)\ xa=0$. Thus for every $xa\in Ra$ there is a polynomial $g(t)\in tF[t]$ with $a^2\ g(xa)=0$. By lemma 3: $a^2=0$.

Let $r\in R$ with $r^2=0$. For every $x\in R$ and each integer k is held:

$$(axar+r)^k=(axar)^k+r.\ (axar)^{k-1}.$$

By our hypothesis there is a polynomial $f(t)=th(t)\in tF[t]$, for which $af(axar$ $+r)\,a=0$. By the above $f(axar+r)=f(axar)+r.h(axar)$. Since $a^2=0$, we get $a.f(axar)=0$. Thus $ar.h\,(axar)\ a=0$. On the other hand:

$$ar.(axar)^m\,a=arax.(arax)^{m-1}\ ara.$$

Hence

$$f\,(arax)=h\,(arax)\,arax=ar.h\,(axar)\ ax=0.$$

So we recieve that $araR$ is an algebraic right ideal of R and therefore $ara=0$.

Let $r,\ s\in R$ with $rs=0$. Then $(sxr)^2=0$ for every $x\in R$. Thus $asxra=0$, i. e. $asRra=0$. Since R is prime, $as=0$ or $ra=0$. So in any case we have $ras=0$.

At the end let $x\in R$. By our hypothesis there exists $\alpha_1, \alpha_2, \ldots, \alpha_k\in F$, such that

$$a.(\alpha_1\,x+\ \ldots\ +\alpha_k\ x^k)\ a=0,$$

i. e.

$$ax.\,(\alpha_1+\ \ldots\ +\alpha_k\ x^{k-1})\ a=0.$$

By the above we recieve. that

$$axa(\alpha_1 + \dots + \alpha_k x^{k-1})\, a = 0.$$

And since $a^2=0$:

$$axa \cdot (\alpha_2 x + \dots + \alpha_k x^{k-1})\, a = 0.$$

Continuing in this way we obtain $\alpha_k a(xa)^{k-1}(xa)=0$ and so $(ax)^{k+1}=0$. In other words aR is a nil and therefore an algebraic right ideal. Hence $a=0$.

Lemma 1.5. *Let R be prime with no nonzero algebraic one-sided ideals. If $a, b \in R$ and for every $x \in R$ there exists a polynomial $f_x(t) \in tF[t]$ such that $a.f_x(x)\, b=0$, then $a=0$ or $b=0$.*

Proof. Let $y \in R$. For every $x \in R$ there is a polynomial $f_x(t) \in tF[t]$ with $a.f_x(x).b=0$ and so $bya.f_x(x).bya=0$. By lemma 1.4. we obtain $bya=0$, i. e. $bRa=0$. Since R is prime, $a=0$ or $b=0$.

Lemma 1.6 (Levitzky, [7, Lemma 1.1]). *Let ρ be a nonzero nil right ideal of R. Suppose that given $a \in \rho$, $a^n=0$ for a fixed integer n; then R has a nonzero nilpotent ideal.*

Remark 1.7. It is clear, that if R is F-algebraic over B and $r_1, \dots, r_k \in R$, there is a polynomial $f(t)=f_{r_1, \dots, r_k}(t)$ such that $f(r_1), f(r_2), \dots, f(r_k) \in B$.

Theorem 1.7. If R is *prime with no nonzero algebraic one-sided ideals and R* is *F-algebraic over B, then B is prime with no nonzero algebraic one-sided ideals.*

Proof. Let $a, b \in B$ and $aBb=0$. Since R is F-algebraic over B, then for every $x \in R$ there is a polynomial $f_x(t)$ for which $f_x(x) \in B$ and so $af_x(x)\, b=0$. By lemma 1.5. we recieve: $a=0$ or $b=0$. Therefore B is prime.

Obviously B contains transcendental elements. Let $0 \neq \rho \neq B$ be a right ideal of B. Suppose that ρ is an algebraic one. By lemma 1.1. there are two possibilities:

(i) There is an idempotent $1 \neq e \in \rho$, $0 \neq e$. By our hypothesis eR is not an algebraic right ideal of R and therefore there is an element $x \in R$ such that ex is a transcendental one. But R is F-algebraic over B and hence there is a polynomial $f_x(t)$ with $f_x(ex) \in B$. It is clear that $f_x(ex)$ is a transcendental element. On the other hand, it follows from $e^2=e \in \rho$ that $f_x(ex)=e\, f_x(ex) \in \rho$. Thus ρ is not algebraic — a contradiction.

(ii) ρ is a nil right ideal. Let $a \in \rho$ and $r \in R$ with $r^2=0$. There exists a polynomial $f(t)=th(t) \in tF[t]$, such that $f(ar) \in B$ and $f(ar+r) \in B$. But for every integer k is held: $(ar+r)^k=(ar)^k+r.(ar)^{k-1}$. Then we have $f(ar+r)=f(ar)+r.h(ar)$ and from here: $r.h(ar) \in B$. Hence $f(ar)=a.rh(ar) \in \rho$. Thus $f(ar)$ is nil and therefore ar is an algebraic element. Let $p(t) \in F[t]$ and $p(ar)=0$. Then for $q(t)=tp(t)$ we have $q(ra)=ra.p(ra)=r.p(ar).a=0$. Thus ra is an algebraic element, too.

Let $x \in R$. Then $(rxr)^2=0$. So $rxra$ is an algebraic element. Let $u(t) \in F[t]$ with $u(rxra)=0$. For $v(t)=tu(t)$ is held: $v(rarx)=ra.u(rxra)\, rx=0$. Thus $rarx$ is algebraic, i. e. $rarR$ is an algebraic right ideal of R. It follows by our hypothesis that $rar=0$, i. e. $r\rho r=0$.

On the other hand, B is a prime algebra and so by lemma 1.6. there exists $a \in \rho$, such that $a^k=0$, $a^{k-1} \neq 0$ and $k \geq 4$. But $(a^{k-2})^2=0$ and consequently $a^{k-2}\rho\, a^{k-2}=0$. Hence $a^{k-2}\rho$ is a nilpotent right ideal of B. Since B is prime we must have $a^{k-2}\rho=0$, and so $a^{k-1}=0$ — a contradiction.

The theorem is proved.

2. Algebras of Quotients. Let $S \subset R$ be a subset of R. We shall use the following designations: $a^{-1} S = \{x \in R: ax \in S\}$; $l_R(S) = \{x \in R: xS = 0\}$ — the left annihilator of S in R; $r_R(S) = \{x \in R: Sx = 0\}$ — the right annihilator of S in R.

Following [2] and [8], we shall say that the right ideal ρ of R is a dense one if for every $x \in R$ is held: $l_R(x^{-1}\rho) = 0$. In other words ρ is dense iff for every $0 \neq r \in R$ and every $s \in R$ there exists an element $y \in \rho$ such that $sy \in \rho$ and $ry \neq 0$. We shall denote with D_R the set of all right dense ideals of R. The algebra

$$Q(R) = \varinjlim_{\rho \in D_R} \operatorname{Hom}(\rho_R, R_R)$$

is the algebra of quotients of R.

Theorem 2.1 ([2, 8]). *If R is a subalgebra of Q, then $Q = Q(R)$ is the algebra of quotients of R iff.*

(i) *For every $0 \neq q \in Q$: $q^{-1} R = \{a \in R: qa \in R\}$ is a right dense ideal of R.*

(ii) *For every $0 \neq q \in Q$: $q(q^{-1} R) \neq 0$.*

(iii) *If ρ is a right dense ideal of R and $f \in \operatorname{Hom}(\rho_R, R_R)$ then there exists an element $q \in Q$, such that $f(x) = qx$ for every $x \in \rho$.*

If B is a subalgebra of R, we shall say that B is an essential subalgebra of R, if for every one-sided ideal I of R is held: $B \cap I \neq (0)$.

Lemma. 2.2. *If R has no nonzero algebraic one-sided ideals, B is a subalgebra of R and R is F-algebraic over B, then B is an essential subalgebra of R.*

Proof. Let $(0) \neq I$ be a one-sided ideal of R. There exists a transcendental element $s \in I$. But there is a polynomial $f(t)$ such that $f(s) \in B$. It is clear that $0 \neq f(s) \in B \cap I$.

Lemma 2.3 ([1, Lemma 1]). *Let B be an essential subalgebra of R and for every $0 \neq r \in R$ there exists a right dense ideal ρ_r of B such that $r.\rho_r \subset B$. Then $Q(R) = Q(B)$.*

The center $C(R)$ of R and the algebraic hypercenter $A(R)$ of R are defined by

$$C(R) = \{x \in R: xr = rx, \text{ all } r \in R\},$$

$$A(R) = \{x \in R: xf(r) = f(r)x, \ f = f_{r,x}(t) \in tF[t], \text{ all } r \in R\}.$$

Theorem 2.4 ([3, Theorem 1.6]). *If R has no nonzero algebraic ideals then $C(R) = A(R)$.*

Lemma 2.5. *Let R be a noncommutative domain, which is not algebraic over F, R be F-algebraic over B, $Q = Q(R)$, $C = C(Q)$ and $b \in C(Q)$ be an invertible element for which $bR \subset R$ and $bB \subset B$. If $0 \neq x \in R$ and $xB \cap B = (0)$, then $x^2 + bx \in C(R)$.*

Proof. Let $R_b = R + bF \subset Q$ and $\rho = xR_b \cap (x+b)R$. Since $bR \subset R$, then ρ is a right ideal of R and $z = x(x+b) \in \rho$. If $a \in \rho$ then $a = xa_1 = (x+b)a_2$, where $a_1 \in R_b$ and $a_2 \in R$. It is clear that $a_1 x \in R$ and $a_2(x+b) \in R$. Therefore there is a polynomial $f(t) \in tF[t]$ with $f(a) \in B$, $y_1 = f(a_1 x) \in B$ and $y_2 = f(a_2(x+b)) \in B$. On the other hand:

$$xy_1 = x.f(a_1 x) = x.(\alpha_0 (a_1 x)^k + \dots + \alpha_p (a_1 x)^{k+p})$$
$$= (\alpha_0 (xa_1)^k + \dots + \alpha_p (xa_1)^{k+p}) x = f(xa_1) x = f(a) x$$
$$(x+b) y_2 = (x+b) f(a_1(x+b)) = f((x+b) a_2)(x+b) = f(a)(x+b).$$

It follows from here that

$$x\,.\,(y_2-y_1)=f(a)\,b-by_2=b\,(f(a)-y_2).$$

But $b\,(f(a)-y_2)\in bB\subset B$ and $x\,(y_2-y_1)\in xB$. Therefore $x\,(y_2-y_1)\in xB\cap B=(0)$ and since R is a domain and $x\neq 0$, then $y_1=y_2=f(a)$.

So we recieve that $xf(a)-f(a)x=xy_1-f(a)\,x=0$, i. e.

$$f(a)\,z-zf(a)=f(a)\,x\,(x+b)-x\,(x+b)\,f(a)=0.$$

Therefore $z\in A\,(\rho)$. But by Corollary 1.2. ρ contains a transcendental element and thus it has no nonzero algebraic ideals. By theorem 2.4: $A\,(\rho)=C(\rho)$. It follows from here that $z\in C\,(\rho)$.

At the end let $s\in R$ and $0\neq r\in\rho$. We have:

$$r\,.\,zs=rz\,.\,s=zr\,.\,s=z\,.\,rs=rs\,.\,z=r\,.\,sz,$$

i. e. $r.\,(zs-sz)=0$ and since R is a domain, then $zs=sz$. Therefore $z=x^2+bx\in C\,(R)$.

Lemma 2.6. *Let R and Q be as in Lemma* 1.5. *Then* $C\,(R)=R\cap C\,(Q)$ *and every element of* $C\,(R)$ *is an invertible one in* Q.

Proof. It is clear that $R\cap C(Q)\subset C(R)$. Let $r\in C\,(R)$ and $q\in Q$, $q\neq 0$. Then by Theorem 2.1: $q^{-1}R$ is a dense right ideal of R and for every $0\neq t\in q^{-1}R$ we have $qt\in R$. It follows from here:

$$(qr-rq)\,t=q\,.\,rt-r\,.\,qt=q\,.\,tr-qt\,.\,r=qtr-qtr=0.$$

But $0\neq t$ and therefore $qr=rq$, i. e. $r\in C\,(Q)\cap R$ and $C\,(Q)\cap R=C\,(R)$.

Recall that if $x\in R$, then in $Q\,(R)$, x is equal to $\mathrm{cl}\,(R_R\to R_R,\ t\to xt)$. Since R is a domain, then $x^{-1}=\mathrm{cl}\,(xR_R\to R_R,\ xr\to r)$ and $x^{-1}\in C\,(Q)$.

From now on, we shall use the following designation:

$$H=\{0\neq x\in R:\ xB\cap B=(0)\}.$$

Lemma 2.7. *Let R and B be as Lemma 2.5. If $H\neq\emptyset$, then $H\not\subset C(R)$ and $B\cap C(R)=\{0,\ 1\}$.*

Proof. Suppose that $H\subset C(R)$. Let $0\neq x\in H\subset C(R)$. For every $a\in B$ we have $xaB\cap B=(0)$. (If $y\in xaB\cap B$, then $y\in xB\cap B=(0)$, since $xaB\subset xB$, and $y=0$). In other words, for every $a\in B$: $xa\in C\,(R)$. Then for every $a\in B$ is held: $a=x^{-1}\,ax\in C\,(Q)\cap R=C\,(R)$, i. e. $B\subset C\,(R)$. So we recieve that B is a commutative algebra. It follows by [3, Theorem 2.6] that R is commutative algebra, too — a contradiction. Therefore $H\not\subset C\,(R)$.

Let now $b\in B\cap C(R)$ and suppose that $b\neq 0,\ 1$. It follows by Lemma 2.5 that for every $x\in H$ is held: $x^2+bx\in C\,(R)$ and $x^2+x\in C\,(R)$. Then $(x^2+bx)-(x^2+x)=(b-1)\,x\in C\,(R)$. By Lemma 2.6: $0\neq b-1\in C\,(R)$ is an invertible element and thus $x\in C\,(Q)\cap R=C\,(R)$, i. e. $H\subset C\,(R)$ — a contradiction. Therefore $C\,(R)\cap B=\{0,\ 1\}$.

Corollary 2.8. *If R and B are as in Lemma 2.5, then $C\,(R)$ is a field and* char $C\,(R)=2$.

Proof. Since $F\subset C\,(R)\cap B=\{0,\ 1\}$, then $F=GF(2)$ and it is clear that if every element in $C\,(R)$ is an invertible one, then $C\,(R)$ will be a field and char $C\,(R)=2$.

Let $b\in C\,(R)$. There is a polynomial $f(t)=th\,(t)\in F[t]$ such that $f(b)\in C\,(R)\cap B=GF(2)$. Thus either $f(b)=1$, or $f(b)=0$. If $f(b)=1$, then $b\,.\,h\,(b)=1$ and $h\,(b)\in C\,(R)$, i. e. b is an invertible element. If $f(b)=0$, then $0=b^n-b^{n+1}\,g(b)$

$=b^n.(1-bg(b))$ and since R is a domain, then $1-b.g(g)=0$, i. e. in this case b is an invertiable element, too.

Lemma 2.9. *If R and B are as in Lemma 2.5 and $H\neq\emptyset$, then R and B are division algebras.*

Proof. First let $0\neq x\in H$. Then $x^2+x\neq 0$ (since $x\neq 0, -1$) and by Lemma 2.5: $x^2+x\in C(R)$. By Corollary 2.8 there is a $y\in C(R)$ such that $y.(x^2+x)=(x^2+x)y=1$, i. e. $x.y(x+1)=y(x+1)x=1$. Therefore x is an invertible element and $x^{-1}\in H$, because if $u\in x^{-1}B\cap B$, then $xu\in B\cap xB=(0)$ and thus $u=0$ (R is a domain).

Let now $0\neq a\in B$ and $0\neq x\in H$. In that case $xaB\cap B=(0)$ and consequently there exists $(xa)^{-1}\in R$. Let $b=(xa)^{-1}x$. Then $ba=1$. Moreover, $x.(ab-1)=xa.(xa)^{-1}x-x=x-x=0$ and since R is a domain, $ab=1=ba$, i. e. every element in B is an invertible one in R.

We shall show that if $0\neq a\in B$, then $a^{-1}\in B$. Really there is a polynomial $f(t)\in tF[t]$ such that

$$f(a^{-1})=a^{-n}+\alpha_1 a^{-(n+1)}+\ldots+\alpha_k a^{-(n+k)}\in B.$$

Multiplying by $a^{n+k-1}\in B$ we recieve:

$$a^{k-1}+\alpha_1 a^{k-2}+\ldots+\alpha_{k-1}+\alpha_k a^{-1}\in B.$$

Hence $\alpha_{k-1}+\alpha_k a^{-1}\in B$ and since $\alpha_{k-1}\in F\subset B$, then $\alpha_k a^{-1}\in B$. Therefore B is a division algebra.

At the end let $0\neq r\in R$. If $r\in H$, then r^{-1} exists. If $r\notin H$, then $rB\cap B\neq(0)$ and consequently there are elements $a, b\in B$, such that $rb=a\in B$. It follows from here that $r=rb.b^{-1}=ab^{-1}\in B$ and thus r is an invertible element.

Lemma 2.10 ([5, Corollary from Theorem 1]). *Let R be a division algebra over a finite field F, and let B be a subalgebra $\neq R$, such that R is F-algebraic over B. Then R is a field.*

Theorem 2.11. *Let R be a noncommutative, nonalgebraic domain, B be a F-subalgebra of R and R be F-algebraic over B. Then $Q(R)=Q(B)$.*

Proof. By Corollary 1.2 R has no nonzero algebraic one-sided ideals. By Lemma 2.2 B is an essential subalgebra of R.

Suppose that $H\neq\emptyset$. Since R and B are as in Lemma 2.9, then they are division algebras and by Lemma 2.10 R is a field — a contradiction. Therefore $H=\emptyset$.

Let $0\neq r\in R$ and $\rho_r=\{a\in B\colon ra\in B\}$. It is clear that ρ_r is a right ideal of B, $\rho_r\neq(0)$ (since $H=\emptyset$, then $rB\cap B\neq(0)$) and $r.\rho_r\subset B$. Let $0\neq a\in B$. Then $ra\neq 0$ (R is a domain) and thus $raB\cap B\neq(0)$. Hence there exists an element $0\neq y\in B$, such that $ray\in B$, i. e. $ay\in\rho_r$ and $0\neq y\in a^{-1}\rho_r$. Therefore $a^{-1}, \rho_r\neq(0)$ is a right ideal of B and $l_B(a^{-1}\rho_r)=(0)$, since B is a domain. It follows from here that ρ_r is a dense right ideal of B and by Lemma 2.3: $Q(R)=Q(B)$.

The theorem is proved.

In the following lemmas let R be a prime noncommutative F-algebra (with unit 1), which has no nonzero algebraic one-sided ideals, B be a subalgebra of R (with the same unit 1) and R be F-algebraic over B. Let B_0 be the set of all $r\in R$ for which there is a right dense ideal ρ_r of B, such that $r.\rho_r\subset B$.

Lemma 2.12 ([1, Lemma 4]) *B_0 is a subalgebra of R and $B\subset B_0$.*

Lemma 2.13. *All zero divisors of R are in B_0.*

Proof. By Theorem 1.7 B is a prime algebra with no nonzero algebraic one-sided ideals.

Let $0 \neq r \in R$ is a right zero divisor, i. e. $l_R(r) \neq (0)$. We replace $\rho = \{b \in B: rb \in B\}$. It is clear that ρ is a right ideal of B and $r.\rho \subset B$. Suppose that ρ is not a dense right ideal of B. There are elements $a, b \in B$, $b \neq 0$, such that if $y \in B$ and $ay \in \rho$, then $by = 0$. In other words, it follows from $y \in B$ and $ray \in B$ that $by = 0$.

At that if $ra = 0$, then $bB = 0$—a contradiction. Hence $ra \neq 0$. Let $0 \neq x \in l_R(r)$. There is a polynomial $f(t) \in tF[t]$ with $f(x) \in B$ and $f(rax + x) \in B$. But $xr = 0$ and for each integer k we have:

$$(rax + x)^k = ra \, . \, x^k + x^k.$$

Thus $f(rax + x) = raf(x) + f(x)$ and consequently $raf(x) \in B$. It follows from here that $bf(x) = 0$. So we recieve that for every $x \in l_R(r)$ there is a polynomial $f(t) \in tF[t]$ such that $bf(x) = 0$. By lemma 1.3: $b = 0$—a contradiction. Therefore ρ is a dense right ideal of B and thus $r \in B_0$.

Let now $0 \neq s \in R$ be a left zero divisor, i. e. $r_R(s) \neq (0)$. As we have just proved, we see that there exists a left dense ideal λ of B such that $\lambda . s \subset B$. Let $\rho = \{b \in B: sb \in B\}$. Obviously ρ is a right ideal of B and $s.\rho \subset B$. Suppose that ρ is not a dense right ideal of B. There exists $a, b \in B$, $b \neq 0$ such that it follows from $y \in B$, $say \in B$, that $by = 0$. It is clear that $sa \neq 0$, $r_B(\lambda) = (0)$ (B is prime). But B is an essential subalgebra of R and $r_B(\lambda) = B \cap r_R(\lambda)$. Consequently $r_R(\lambda) = (0)$ and $\lambda . sa \neq (0)$ is a left ideal of B. For every $l \in \lambda$ there is a polynomial $f(t) = th(t) \in tF[t]$, with $f(sal) = sa \, . \, h(lsa) \, l \in B$ and since $\lambda . s \subset B$, $a \in B$, $l \in B$, then $h(lsa) . l \in B$. Hence $b \, . \, h(lsa) \, l = 0$ and $b \, . \, f(lsa) = 0$. By Lemma 1.3: $b = 0$—a contradiction. Therefore ρ is a dense right ideal of B and $s \in B_0$.

Theorem 2.14. *Let R be a prime, noncommutative F-algebra with no nonzero algebraic one-sided ideals, B be a subalgebra of R and R be F-algebraic over B. Then $Q(R) = Q(B)$.*

Proof. By Lemma 2.2 B is an essential subalgebra of R and by Lemma 2.3 it is enough to prove that $B_0 = R$. By that we may assume that there are zero divisors in R. Let $0 \neq u \in R$, $0 \neq v \in R$ and $uv = 0$. By Lemma 2.13: $Ru \subset B_0$ and $vR \subset B_0$. Then $I = RuBvR \subset B_0$ (since $B \subset B_0$) and I is an ideal of R. However B is prime. Hence $uBv \neq 0$ and $I \neq (0)$.

Let now $0 \neq r \in R$ and $\rho = \{y \in B: ry \in B\}$. Obviously ρ is a right ideal of B and $r.\rho \subset B$. We shall show that ρ is a dense right ideal of B. Let $a, b \in B$ and $b \neq 0$. Since R is prime and B is an essential subalgebra of R, then $bI \neq (0)$ and $bI \cap B \neq (0)$. Consequently there is $y \in I \subset B_0$, such that $by \in B$ and $by \neq 0$. On the other hand, $ray \in I \subset B_0$. Then there exists a dense right ideal σ of B such that $ray.\sigma \subset B$ and $y . \sigma \subset B$. It follows from here that there is an element $s \in \sigma$ such that $bys \neq 0$ (since $0 \neq by \in B$). We have $ys \in B$, $ra.ys \in B$ and $b.ys \neq 0$. Hence $ys \in B$, $a.ys \in \rho$ and $b.ys \neq 0$, i. e. ρ is a dense right ideal of B. Therefore $B_0 = R$.

The theorem is proved.

3. Some Corollaries. Recall that a right singular ideal of R is

$$Z(R) = \{y \in R: r_R(y) \text{ is essential}\}.$$

If B is a subalgebra of R we shall say ([1]) that R is a rational extension of B if $Q(R) = Q(B)$.

Lemma 3.1 ([1, Lemma 8]). *If R is a rational extension of B, then $Z(B) = B \cap Z(R)$.*

Corollary 3.2. *Let R be a prime, noncommutative F-algebra with no nonzero algebraic one-sided ideals, B be a subalgebra of R and R be F-algebraic over B. Then $Z(B)=B\cap Z(R)$.*

If M_R is a right R-module, we shall denote with $\dim_G(M_R)$ — the Goldie dimension of M_R.

Lemma 3.3 ([1, Lemma 9]). *If R is rational extension of B, then*

$$\dim_G(B_B)=\dim_G(R_B)=\dim_G(R_R).$$

Theorem 3.4. *Let R be a prime, noncommutative F-algebra with no nonzero algebraic one-sided ideals, B be a subalgebra of R and R be F-algebraic over B. R is a right Goldie algebra iff B is a right Goldie algebra. In that case R and B are orders in one and the same simple Artinian algebra.*

Proof. Analogous to the proof of [1, Corollary 2].

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