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CONFORMAL FLAT AK_2-MANIFOLDS

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In this note we examine conformal flat AK_2-manifolds of dimension $2m \geqq 6$.

1. Introduction. Let M be a $2m$-dimensional almost Hermitian manifold with metric g and almost complex structure J. The Riemannian connection and the curvature tensor are denoted by ∇ and R, respectively. The manifold is said to be a Kähler or nearly Kähler, or almost Kähler manifold, if

$$\nabla J = 0 \quad \text{or} \quad (\nabla_X J)X = 0, \quad \text{or}$$

(1.1)

$$g((\nabla_X J)Y, Z) + g((\nabla_Y J)Z, X) + g((\nabla_Z J)X, Y) = 0,$$

respectively. The corresponding classes of manifolds are denoted by K, NK, AK, respectively. It is well known, that for these classes

$$(\nabla_X J)Y + (\nabla_{JX} J)JY = 0 \tag{1.2}$$

holds [2].

For a given class L of almost Hermitian manifolds its subclass L_i is defined by the identity (i), where

1) $R(X, Y, Z, U) = R(JX, JY, Z, U)$,

2) $R(X, Y, Z, U) = R(JX, JY, Z, U) + R(JX, Y, JZ, U) + R(JX, Y, Z, JU)$,

3) $R(X, Y, Z, U) = R(JX, JY, JZ, JU)$.

Then we have $L_1 \subset L_2 \subset L_3$ and $NK = NK_2$, $K = NK_1 = AK_1$, $K = NK \cap AK$ [2].

The Weyl conformal curvature tensor C for M is defined by

$$C(X, Y, Z, U) = R(X, Y, Z, U) - \frac{1}{2m-2}\{g(X, U)S(Y, Z)$$
$$- g(X, Z)S(Y, U) + g(Y, Z)S(X, U) - g(Y, U)S(X, Z)\}$$
$$+ \frac{\tau}{(2m-1)(2m-2)}\{g(X, U)g(Y, Z) - g(X, Z)g(Y, U)\},$$

where S and τ are the Ricci tensor and the scalar curvature, respectively. It is well known, that (if $m \geqq 2$) M is conformal flat if and only if $C = 0$.

Conformal flat Kähler and nearly Kähler manifolds are classifield in [4] and [5]. Here, we shall prove the following theorem:

Theorem. *Let $M \in AK_2$ be a $2m$-dimensional conformal flat manifold, $m > 2$. Then it is one of the following:*

a) *a flat Kähler manifold;*

b) *a 6-dimensional almost Kähler manifold of constant negative sectional curvature;*

c) *locally* $M_1 \times M_2$, *where* M_1 *(resp.* M_2*) is a 4-dimensional almost Kähler manifold of constant sectional curvature* $-c$ *(resp. a 2-dimensional Kähler manifold of constant sectional curvature c),* $c>0$;

d) *locally* $M_3 \times M_2$, *where* M_3 *is a 6-dimensional almost Kähler manifold of constant sectional curvature* $-c$.

Remark 1. We don't know whether there exists an almost Kähler manifold of constant negative sectional curvature of dimension 4 or 6.

Remark 2. If a conformal flat almost Hermitian manifold M satisfies the identity 3), then $S(X, Y)=S(JX, JY)$ and M satisfies also the identity 2).

2. Preliminaries. Let Q be a tensor of type (1.1). According to the Ricci identity,

$$(\nabla_X(\nabla_Y Q))Z-(\nabla_Y(\nabla_X Q))Z=R(X, Y)QZ-QR(X, Y)Z. \tag{2.1}$$

From the second Bianchi identity it follows

$$\sum_{i=1}^{2m}(\nabla_{E_i}R)(X, Y, Z, E_i)=(\nabla_X S)(Y, Z)-(\nabla_Y S)(X, Z), \tag{2.2}$$

$$\sum_{i=1}^{2m}(\nabla_{E_i}S)(X, E_i)=\frac{1}{2}X(\tau), \tag{2.3}$$

where $\{E_i;\ i=1, \ldots, 2m\}$ is a local orthonormal frame field. We shall assume that $E_{m+i}=JE_i$, $i=1, \ldots, m$.

Let the tensor S' be defined by

$$S'(X, Y)=\sum_{i=1}^{2m}R(X, E_i, JE_i, JY).$$

For an AK_2-manifold the following identities [1, 2] hold:

$$2(\nabla_X(S-S'))(Y, Z)=(S-S')((\nabla_X J)Y, JZ)+(S-S')(JY, (\nabla_X J)Z, \tag{2.4}$$

$$\sum_{i=1}^{2m}(\nabla_{E_i}(\nabla_{E_i}J))Y=\sum_{i=1}^{2m}J(\nabla_{E_i}J)(\nabla_{E_i}J)Y, \tag{2.5}$$

$$R(X, Y, Z, U)-R(X, Y, JZ, JU)=\frac{1}{2}g(K(X, Y), K(Z, U)), \tag{2.6}$$

where $K(X, Y)=(\nabla_X J)Y-(\nabla_Y J)X$.

A 2-dimensional almost Hermitian manifold is a Kähler manifold. It follows easily from (2.6), that if M is an almost Kähler manifold of constant curvature c and if $\dim M\geq 4$, then $c\leq 0$ and $c=0$ if and only if M is a Kähler manifold. On the other hand, an almost Kähler manifold of dimension $2m\geq 8$ is automatically a Kähler manifold [3].

3. Proof of the theorem. From $C=0$, (2.2) and (2.3) it follows

$$(\nabla_X S)(Y, Z)-(\nabla_Y S)(X, Z)=\frac{1}{2(2m-1)}\{g(Y, Z)X(\tau)-g(X, Z)Y(\tau)\}. \tag{3.1}$$

Since $C=0$ we have

$$S'=\frac{1}{m-1}S-\frac{\tau}{2(m-1)(2m-1)}g.$$

Hence, using (2.4), we find

$$(3.2)\qquad 2(\nabla_X S)(Y, Z)=S((\nabla_X J)Y, JZ)+S(JY, (\nabla_X J)Z)-\frac{X(\tau)}{(m-2)(2m-1)}g(Y, Z).$$

Let $X\perp Y, JY$. According to (3.2) and (1.2),

$$(\nabla_X S)(Y, Y)+(\nabla_X S)(JY, JY)-(\nabla_Y S)(X, Y)-(\nabla_{JY} S)(X, JY)=-\frac{X(\tau)}{(2m-1)(m-2)}g(Y, Y).$$

The last equality and (3.1) give $X(\tau)=0$. From $X(\tau)=0$, (3.1) and (3.2) we obtain

$$(3.3)\qquad (\nabla_X S)(Y, Z)=(\nabla_Y S)(X, Z),$$

$$(3.4)\qquad 2(\nabla_X S)(Y,Z)=S((\nabla_X J)Y, JZ)+S(JY, (\nabla_X JZ).$$

Now let $p\in M$ and $\{e_i;\ i=1,\dots,2m\}$ be an orthonormal basis of $T_p(M)$, such that $e_{i+m}=Je_i$ and $Se_i=\lambda e_i$ for $i=1,\dots, m$. Let $\{E_i;\ i=1,\dots, 2m\}$ be a local orthonormal frame field, such that $E_{i/p}=e_i$ for $i=1,\dots, 2m$. We have

$$\sum_{i=1}^{2m}(\nabla_{e_i}(\nabla_{e_i}S))(e_j, e_j)$$

$$=\sum_{i=1}^{2m}\{\nabla_{E_i}(\nabla_{E_i}S)(E_j, E_j)-(\nabla_{\nabla_{E_i}E_i}S)(E_j, E_j)-2(\nabla_{E_i}S)(\nabla_{E_i}E_j, E_j)\}_p \quad \text{using (3.4)}$$

$$=\sum_{i=1}^{2m}\{(\nabla_{E_i}S)((\nabla_{E_i}J)E_j, JE_j)+S((\nabla_{E_i}(\nabla_{E_i}J)E_j, JE_j)$$

$$+S((\nabla_{E_i}J)E_j, (\nabla_{E_i}J)E_j)\}_p \quad \text{using (2.5) and (3.4)}$$

$$=-\sum_{i=1}^{2m}(\nabla_{e_i}S)((\nabla_{e_i}J)e_j, J_{e_j}) \quad \text{using (3.3)}$$

$$=\sum_{i=1}^{2m}(\nabla_{(\nabla_{e_j}J)e_j}S)(e_i, Je_j), \quad \text{using (3.4)}$$

$$=\frac{1}{2}\sum_{i=1}^{2m}(\lambda_j-\lambda_i)g((\nabla_{\nabla_{e_i}Je_j}J)e_i, e_j)$$

and using (1.1), we obtain

$$(3.5)\qquad \sum_{i=1}^{2m}(\nabla_{e_i}(\nabla_{e_i}S))(e_j, e_j)$$

$$=\frac{1}{2}\sum_{i=1}^{2m}(\lambda_j-\lambda_i)\{g((\nabla_{e_i}J)e_j, (\nabla_{e_j}J)e_i)-g((\nabla_{e_i}J)e_j, (\nabla_{e_i}J)e_j)\}.$$

Because of $X(\tau)=0$ and (3.3) we have

$$\sum_{i=1}^{2m}(\nabla_{E_j}(\nabla_{E_i}S))(E_i, E_j)=0.$$

Using (3.3), we obtain also

$$\sum_{i=1}^{2m}(\nabla_{E_i}(\nabla_{E_i}S))(E_j, E_j)=\sum_{i=1}^{2m}(\nabla_{E_i}(\nabla_{E_j}S))(E_i, E_j).$$

From the last two equalities and (2.1) it follows

(3.6) $$\sum_{i=1}^{2m}(\nabla_{e_i}(\nabla_{e_i}S))(e_j, e_j)=\sum_{i=1}^{2m}(\lambda_j-\lambda_i)R(e_i, e_j, e_j, e_i)$$

Now we compute

$$(\nabla_{e_i}(\Delta_{e_j}S))(e_i, e_j)-(\nabla_{e_j}(\nabla_{e_i}S))(e_i, e_j) \quad \text{using (3.4)}$$

$$=\frac{1}{2}\{(\nabla_{e_i}S)((\nabla_{e_j}J)e_i, J_{e_j})+(\nabla_{e_i}S)(Je_j, (\nabla_{e_j}J)e_j)$$

$$+S((\nabla_{e_i}(\nabla_{e_j}J))e_i, Je_j)+S(Je_i, (\nabla_{e_j}(\nabla_{e_i}J))e_j)$$

$$-(\nabla_{e_i}S)(\nabla_{e_i}J)e_i, Je_j)-(\nabla_{e_j}S)(Je_i, (\nabla_{e_i}J)e_j)$$

$$-S((\nabla_{e_j}(\nabla_{e_i}J))e_i, Je_j)-S(Je_i(\nabla_{e_j}(\nabla_{e_i}J))e_j)\} \quad \text{using (2.1)}$$

$$=\frac{1}{2}\{(\nabla_{e_i}S)((\nabla_{e_j}J)e_i, Je_j)+(\nabla_{e_i}S)(Je_i, (\nabla_{e_j}J)e_j)$$

$$-(\nabla_{e_j}S)((\nabla_{e_i}J)e_i, Je_j)-(\nabla_{e_j}S)(Je_i, (\nabla_{e_i}J)e_j)+(\lambda_j-\lambda_i)R(e_i, e_j, e_j, e_i)\}$$

and using (3.4), we obtain

$$(\nabla_{e_i}(\nabla_{e_j}S))(e_i, e_j)-(\nabla_{e_j}(\nabla_{e_i}S))(e_i, e_j)=\frac{1}{2}(\lambda_j-\lambda_i)R(e_i, e_j, e_j, e_i)$$

$$+\frac{1}{4}(\lambda_j-\lambda_i)\{g((\nabla_{e_i}J)e_j, (\nabla_{e_j}J)e_i)-g((\nabla_{e_i}J)e_i, (\nabla_{e_j}J)e_j)\}.$$

On the other hand, (2.1) implies

$$(\nabla_{e_i}(\nabla_{e_j}S))(e_i, e_j)-(\nabla_{e_j}(\nabla_{e_i}S))(e_i, e_j)=(\lambda_j-\lambda_i)R(e_i, e_j, e_j, e_i)$$

and hence we find

(3.7) $$(\lambda_j-\lambda_i)R(e_i, e_j, e_j, e_i)$$

$$=\frac{1}{2}(\lambda_j-\lambda_i)\{g((\nabla_{e_i}J)e_j, (\nabla_{e_j}J)e_i)-g((\nabla_{e_i}J)e_i, (\nabla_{e_j}J)e_j)\}$$

for all $i, j=1, \ldots, 2m$. If $e_i\neq e_j$, Je_j we have $R(e_i, Je_j, Je_j, e_i)=R(e_i, e_j, e_j, e_i)$ because of $C=0$. Consequently from (3.7) and (1.2) we derive

(3.8) $$(\lambda_j-\lambda_i)R(e_i, e_j, e_j, e_i)=\frac{1}{2}(\lambda_j-\lambda_i)g((\nabla_{e_i}J)e_j, (\nabla_{e_j}J)e_i)$$

and this is true also for $e_i=e_j$ or $e_i=Je_j$.

From (3.5), (3.6) and (3.8) we obtain

(3.9) $$\sum_{i=1}^{2m}(\lambda_j-\lambda_i)g((\nabla_{e_i}J)e_j, (\nabla_{e_i}J)e_j)=0$$

for any $j=1,\ldots, 2m$. Let $\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_m$. Using (3.9), we find

$$\lambda_i=\lambda_1 \quad \text{or} \quad (\nabla_{e_i}J)e_1=0 \tag{3.10}$$

$$\text{and} \quad \lambda_i=\lambda_m \quad \text{or} \quad (\nabla_{e_i}J)e_m=0$$

for each $i=1, \dots, m$. If there exists j, such that $\lambda_1<\lambda_j<\lambda_m$, then from (3.8) and (3.10) we derive

$$R(e_1, e_j, e_j, e_1)=0, \quad R(e_m, e_j, e_j, e_m)=0$$

and because of $C=0$ this implies $\lambda_1=\lambda_m$, which is a contradiction. Consequently we have the following two cases:

1) $\lambda_i=\lambda_j$ for all $i, j=1, \dots, m$;
2) $\lambda_i=\lambda$ for $i=1, \dots, n$, $\lambda_i=\mu$ for $i=n+1, \dots, m$, $\lambda\neq\mu$, $1\leq n<m$.

In both cases using (3.4) and (3.10), we obtain $\nabla S=0$ in p. Consequently the Ricci tensor is parallel.

If M is irreducible, it is an Einsteinnian manifold and since M is conformal flat, it is of constant sectional curvature. Then the theorem follows from the results in the end of Preliminaries.

Let M is reducible non Einsteinnian manifold. Then we have the case 2) for each $p\in M$. Now M is locally a product $M_1\times M_2$ where M_1 and M_2 are almost Kähler manifolds. Let for example $\dim M_1\geq 4$. Let x, y be orthogonal unit vectors in a point of M_1 and z be a unit vector on M_2. Because of $C=0$ we have

$$R(x-z, y+Jz, y-Jz, x+z)=0$$

or

$$R(x, y, y, x)+R(z, Jz, Jz, z)=0. \tag{3.11}$$

Hence M_1 is of constant sectional curvature, say $-c$ and consequently $c\geq 0$. If $\dim M_2=2$, it follows from (3.11) that M_2 is of constant sectional curvature c. If $\dim M_2\geq 4$, then M_2 is of constant sectional curvature, say k and from (3.11) we obtain $k=c$. If $c>0$, this is impossible, because of $\dim M_2\geq 4$ and if $c=0$ M is Einsteinnian, which is a contradiction.

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