# PLISKA

## STUDIA MATHEMATICA BULGARICA

# ПЛИСКА

## БЪЛГАРСКИ МАТЕМАТИЧЕСКИ СТУДИИ

# $AH_3$-MANIFOLDS OF CONSTANT ANTIHOLOMORPHIC SECTIONAL CURVATURE

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The purpose of this paper is to prove that an $AH_3$-manifold of constant antiholomorphic sectional curvature is a real space form or a complex space form.

**1. Introduction.** Let $M$ be a $2m$-dimensional almost Hermitian manifold with metric tensor $g$ and almost complex structure $J$. The Riemannian connection and the curvature tensor are denoted by $\nabla$ and $R$, respectively.

If $\nabla J = 0$ or $(\nabla_X J)X = 0$, or

$$g((\nabla_X J)Y, Z) + g((\nabla_Y J)Z, X) + g((\nabla_Z J)X, Y) = 0,$$

then $M$ is said to be a Kähler, or nearly Kähler, or almost Kähler manifold, respectively. The corresponding classes of manifolds are denoted by $K$, $NK$, $AK$. The general class of all almost Hermitian manifolds is denoted by $AH$. If $L$ is a class of almost Hermitian manifolds, its subclass of $L_i$-manifolds is defined by the identity i), where

1) $R(X, Y, Z, U) = R(X, Y, JZ, JU)$;

2) $R(X, Y, Z, U) = R(X, Y, JZ, JU) + R(X, JY, Z, JU) + R(JX, Y, Z, JU)$;

3) $R(X, Y, Z, U) = R(JX, JY, JZ, JU)$.

It is well known, that

$$K = K_1 \subset NK = NK_2, \quad K \subset AK_2,$$

$$K = NK \cap AK, \quad AH_1 \subset AH_2 \subset AH_3,$$

see e. g. [4].

A plane $\alpha$ in $T_p(M)$ is said to be holomorphic (resp. antiholomorphic) if $\alpha = J\alpha$ (resp. $\alpha \perp J\alpha$). The manifold $M$ is said to be of pointwise constant holomorphic (resp. antiholomorphic) sectional curvature $v$, if for each point $p \in M$ the curvature of an arbitrary holomorphic (resp. antiholomorphic) plane $\alpha$ in $T_p(M)$ doesn't depend on $\alpha$: $K(\alpha) = v(p)$.

For Kähler manifolds the requirements for constant holomorphic and constant antiholomorphic sectional curvature are equivalent [2]. In [3] is proved a classification theorem for nearly Kähler manifolds of constant holomorphic sectional curvature.

If $M$ is a $2m$-dimensional $AH_3$-manifold of pointwise constant antiholomorphic sectional curvature $v$, and if $m > 2$, then $v$ is a global constant [5]. In [1] is proved a classification theorem for nearly Kähler manifolds of constant antiholomorphic sectional curvature and a corresponding result for $AK_3$-manifolds is obtained in [6].

In section 3 we shall prove the following theorem:

T h e o r e m. *Let $M$ be a $2m$-dimensional $AH_3$-manifold, $m > 2$. If $M$ is of pointwise constant antiholomorphic sectional curvature, then $M$ is a real space form or a complex space form.*

Here a real space form means a Riemannian manifold of constant sectional curvature and a complex space form means a Kähler manifold of constant holomorphic sectional curvature.

**2. Basic formulas.** If $M$ is an $AH_3$-manifold, its Ricci tensor $S$ satisfies

$$S(X, Y) = S(Y, X) = S(JX, JY).$$

If moreover $M$ has pointwise constant antiholomorphic sectional curvature $\nu$, its curvature tensor has the form

(2.1)
$$R = \frac{1}{6}\, \psi(S) + \nu\pi_1 - \frac{2m-1}{3}\, \nu\pi_2,$$

where

$$\psi(Q)(x, y, z, u) = g(x, Ju)Q(y, Jz) - g(x, Jz)Q(y, Ju) - 2g(x, Jy)Q(z, Ju)$$
$$+ g(y, Jz)Q(x, Ju) - g(v, Ju)Q(x, Jz) - 2g(z, Ju)Q(x, Jy)$$

for an arbitrary tensor $Q$ of type $(0, 2)$ and

$$\pi_1(x, y, z, u) = g(x, u)g(y, z) - g(x, z)g(y, u),$$

$$\pi_2 = \frac{1}{2}\, \psi(g),$$

see [1]. According to (2.1), $M$ is an $AH_2$-manifold.

On the other hand, it is known, that if $M$ is an $AK_3$-manifold,

(2.2)
$$R(x, y, z, u) - R(x, y, Jz, Ju) = \frac{1}{2}\, g((\nabla_x J)y - (\nabla_y J)x, (\nabla_z J)u - (\nabla_u J)z)$$

holds good [4].

We shall use also the second Bianchi identity

(2.3)
$$(\nabla_x R)(y, z, u, v) + (\nabla_y R)(z, x, u, v) + (\nabla_z R)(x, y, u, v) = 0.$$

**3. Proof of the theorem**

Lemma. *The conditions of the theorem imply that $M$ is an Einsteinian manifold.*

Proof of Lemma. Let $p$ be an arbitrary point of $M$ and let $x, y \in T_p(M)$. According to the second Bianchi identity,

(3.1)
$$(\nabla_x R)(Jx, y, y, Jx) + (\nabla_{Jx} R)(y, x, y, Jx) + (\nabla_y R)(x, Jx, y, Jx) = 0.$$

Let $\{e_i, Je_i;\ i = 1, \ldots, m\}$ be an orthonormal basis of $T_p(M)$, such that $Se_i = \lambda_i e_i$, $i = 1, \ldots, m$. Putting in (3.1) $x = e_i$, $y = e_j$ or $x = e_k$, $y = e_i + e_j$ for $i \neq j \neq k \neq i$ and using (2.1), we obtain

(3.2)
$$(\nabla_{e_j} S)(e_i, e_j) + \{\lambda_i + \lambda_j - 2(2m-1)\nu\}g(Je_i, (\nabla_{e_j} J)e_j) = 0;$$

(3.3)
$$(\nabla_{e_i} S)(e_j, e_k) + \{\lambda_i + \lambda_k - 2(2m-1)\nu\}g(Je_k, (\nabla_{e_j} J)e_i)$$
$$+ (\nabla_{e_j} S)(e_i, e_k) + \{\lambda_j + \lambda_k - 2(2m-1)\nu\}g(Je_k, (\nabla_{e_i} J)e_j) = 0,$$

respectively. Analogously from

$$(\nabla_{e_j} R)(Je_j, e_j, e_j, Je_k) + (\nabla_{Je_j} R)(e_j, e_i, e_j, Je_k) + (\nabla_{e_j} R)(e_i, Je_j, e_j, Je_k) = 0$$

we find

$$3(\nabla_{e_i}S)(e_j,\ e_k)+6\{\lambda_j-(2m-1)\nu\}g((\nabla_{e_j}R)e_j,\ Je_k)$$

(3.4)

$$-(\nabla_{e_j}S)(e_i,\ e_k)-\{\lambda_i+\lambda_j-2(2m-1)\nu\}g((\nabla_{e_j}J)e_i,\ Je_k)=0$$

and hence

(3.5)

$$8(\nabla_{e_i}S)(e_j,\ e_k)+\{17\lambda_j-\lambda_i-16(2m-1)\nu\}g((\nabla_{e_i}J)e_j,\ Je_k)$$

$$+3(\lambda_i-\lambda_j)g((\nabla_{e_j}J)e_i,\ Je_k)=0.$$

In (3.5) we change $j$ and $k$ and add the result with (3.5)

(3.6)

$$16(\nabla_{e_i}S)(e_j,\ e_k)+17(\lambda_j-\lambda_k)g((\nabla_{e_i}J)e_j,\ Je_k)$$

$$+3(\lambda_i-\lambda_j)g((\nabla_{e_j}J)e_i,\ Je_k)+3(\lambda_i-\lambda_k)g((\nabla_{e_k}J)e_i,\ Je_j)=0.$$

On the other hand, (3.3) and (3.4) imply

(3.7) $\quad (3\lambda_j-\lambda_i-2\lambda_k)g((\nabla_{e_i}J)e_j,\ Je_k)+(3\lambda_i-\lambda_j-2\lambda_k)g((\nabla_{e_j}J)e_i,\ Je_k)=0.$

Hence it is not difficult to find

$$3(\lambda_j-\lambda_k)g((\nabla_{e_i}J)e_j,\ Je_k)+(\lambda_i-\lambda_j)g((\nabla_{e_j}J)e_i,\ Je_k)+(\lambda_i-\lambda_k)g((\nabla_{e_k}J)e_i,\ Je_j)=0$$

and by using (3.6) this implies

(3.8) $\qquad\qquad 2(\nabla_{e_i}S)(e_j,\ e_k)=(\lambda_k-\lambda_j)g((\nabla_{e_i}J)e_j,\ Je_k).$

Let us first assume that $g((\nabla_{e_i}J)e_j,\ Je_k)\neq0$. Using three times (3.7), we obtain

$$(3\lambda_i-\lambda_k-2\lambda_j)(3\lambda_j-\lambda_i-2\lambda_k)(3\lambda_k-\lambda_j-2\lambda_i)$$

$$-(3\lambda_i-\lambda_j-2\lambda_k)(3\lambda_j-\lambda_k-2\lambda_i)(3\lambda_k-\lambda_i-2\lambda_j)=0$$

or equivalently

$$(\lambda_i-\lambda_j)(\lambda_j-\lambda_k)(\lambda_k-\lambda_i)=0.$$

Hence it follows $\lambda_i=\lambda_j=\lambda_k$. Indeed we have to consider two cases:

Case 1. $\lambda_i=\lambda_j$. In (3.7) we made a cyclic change of $i$, $j$, $k$ and use $\lambda_i=\lambda_j$:

$$(\lambda_i-\lambda_k)\{3g((\nabla_{e_j}J)e_k,\ Je_i)+g((\nabla_{e_k}J)e_i,\ Je_j)\}=0,$$

(3.9) $\qquad\qquad (\lambda_i-\lambda_k)\{g((\nabla_{e_k}J)e_i,\ Je_j)+3g((\nabla_{e_i}J)e_j,\ Je_k)\}=0.$

If $g((\nabla_{e_k}J)e_i,\ Je_j)=0$ the last equation implies $\lambda_i=\lambda_k$, i. e. $\lambda_i=\lambda_j=\lambda_k$. So we assume $g((\nabla_{e_k}J)e_i,\ Je_j)\neq0$. In (3.5) we change $i$ and $k$ and we use $\lambda_i=\lambda_j$ and (3.8)

$$\{17\lambda_i-\lambda_k-16(2m-1)\nu\}g((\nabla_{e_k}J)e_j,\ Je_i)+3(\lambda_k-\lambda_i)g((\nabla_{e_j}J)e_k,\ Je_i)=0.$$

Hence, using (3.9), we obtain $\lambda_i=(2m-1)\nu$. On the other hand, (3.5) and (3.8) result

$$3\lambda_i+\lambda_k-4(2m-1)\nu=0$$

and so we find $\lambda_k=(2m-1)\nu$, i. e. $\lambda_i=\lambda_j=\lambda_k$.

C a s e 2. $\lambda_j = \lambda_k$. From (3.7) we obtain

$$(\lambda_i - \lambda_j)\{g((\nabla_{e_i}J)e_j, Je_k) - 3g((\nabla_{e_j}J)e_i, Je_k)\} = 0.$$

If $g((\nabla_{e_j}J)e_i, Je_k) = 0$ this implies $\lambda_i = \lambda_j$, so $\lambda_i = \lambda_j = \lambda_k$. But $g((\nabla_{e_j}J)e_i, Je_k) \neq 0$ is the Case 1.

So we have $\lambda_i = \lambda_j = \lambda_k$ and using (3.5) and (3.8), we find $\lambda_i = (2m-1)\nu$. If $m = 3$ $M$ is Einsteinian in $p$. Let $m > 3$. For $s \neq i, j, k$ we have

$$(\nabla_{e_i}R)(e_s, Je_s, e_j, Je_k) + (\nabla_{e_s}R)(Je_s, e_i, e_j, Je_k) + (\nabla_{Je_s}R)(e_i, e_s, e_j, Je_k) = 0.$$

Because of (2.1) this implies

$$(\nabla_{e_i}S)(e_j, e_k) + \{\lambda_j + \lambda_s - 2(2m-1)\nu\}g((\nabla_{e_i}J)e_j, Je_k) = 0.$$

Hence, using $\lambda_j = \lambda_k = (2m-1)\nu$ and (3.8), we derive $\lambda_s = (2m-1)\nu$. Consequently $M$ is Einsteinian in $p$.

Now we assume that

$$g((\nabla_x J)y, z) = 0$$

whenever $x, y, z$ are choosen among the basic vectors $e_i, Je_i; i = 1, \ldots, m$ and $x \neq y$, $z, Jy, Jz$. In (2.3) we put $x = Je_i, y = v = e_j, z = -Ju = e_k$ for $i \neq j \neq k \neq i$. Using (2.1), we obtain

$$(\nabla_{e_i}S)(e_i, e_j) + \{\lambda_j + \lambda_k - 2(2m-1)\nu\}g(Je_i, (\nabla_{e_j}J)e_j) = 0.$$

From this equality and (3.2) it follows that if $g(Je_i, (\nabla_{e_j}J)e_j) \neq 0$ for some $i, j$, then $\lambda_s = \lambda_k$ for $s, k \neq j$. Consequently if $(\nabla_{e_s}J)e_s \neq 0$ for any $s \neq j$ then $M$ is Einsteinian in $p$.

Let us assume that $M$ is not Einsteinian in $p$. Then $M$ is not Einsteinian in a neighbourhood $U$ of $p$. We shall prove that $M$ is an $AK_3$-manifold in $U$. Let $q \in U$. If $M$ is a Kähler manifold in $q$, $M$ is almost Kähler in $q$. Let $M$ is not Kähler in $q$. Let $f_i, Jf_i, i = 1, \ldots, m$, be an orthonormal basis of $T_q(M)$, such that $Sf_i = \mu_i f_i, i = 1, \ldots, m$. Since $M$ is non Kähler and non Einsteinian in $q$ we may assume that $(\nabla_{f_1}J)f_1 \neq 0$, $\mu_2 = \ldots = \mu_m = \mu$ and

(3.10)
$$(\nabla_x J)y = 0, \quad g((\nabla_{f_1}J)x, y) = 0$$

whenever $x, y$ are chosen among $f_i, Jf_i$ for $i > 1$. Analogously to (3.2)

(3.2')
$$(\nabla_{f_j}S)(f_i, f_j) + \{\mu_i + \mu_j - 2(2m-1)\nu\}g(Jf_i, (\nabla_{f_j}J)f_j) = 0$$

holds good and according to (3.10) this implies

(3.11)
$$(\nabla_{f_j}S)(f_i, f_j) = (\nabla_{Jf_j}S)(f_i, Jf_j) = 0 \text{ for } j > 1, j \neq i.$$

In (2.3) we put $x = f_i, y = -Jv = f_j, z = -Ju = f_1$ for $i \neq j \neq 1 \neq i$ and using (2.1), (3.10) and (3.11) we obtain

(3.12)
$$(\nabla_{f_i}S)(f_j, f_j) + (\nabla_{f_j}S)(f_1, f_1) - (\nabla_{f_1}S)(f_i, f_1)$$
$$+ 2\{\mu - (2m-1)\nu\}g(Jf_i, (\nabla_{f_1}J)f_1) = 0.$$

Now let $k \neq i$. From

$$(\nabla_{f_i}R)(f_k, Jf_k, Jf_k, f_k) + (\nabla_{f_k}R)(Jf_k, f_i, Jf_k, f_k) + (\nabla_{Jf_k}R)(f_i, f_k, Jf_k, f_k) = 0$$

it follows

$$2(\nabla_{f_i}S)(f_k, f_k) - (\nabla_{f_k}S)(f_i, f_k) + \{\mu_i + \mu_k - 2(2m-1)\nu\}g(Jf_i, \ (\nabla_{f_k}J)f_k)$$

$$-(\nabla_{Jf_k}S)(f_i, Jf_k) + \{\mu_i + \mu_k - 2(2m-1)\nu\} \ g(Jf_i, (\nabla_{Jf_k}J)Jf_k) = 0.$$

Hence using (3.2′), we derive

(3.13) $$(\nabla_{f_i}S)(f_k, f_k) = (\nabla_{f_k}S)(f_i, f_k) + (\nabla_{Jf_k}S)(f_i, Jf_k).$$

Now (3.11) and (3.13) imply

$$(\nabla_{f_i}S)(f_j, f_j) = 0 \quad \text{for} \quad i, j > 1, \ i \ne j.$$

Then (3.12) takes the form

$$(\nabla_{f_i}S)(f_1, f_1) - (\nabla_{f_1}S)(f_i, f_1) + 2\{\mu - (2m-1)\nu\}g(Jf_i, (\nabla_{f_1}J)f_1) = 0$$

and using (3.13), we obtain

$$(\nabla_{Jf_1}S)(f_i, Jf_1) + 2\{\mu - (2m-1)\nu\}g(Jf_i, (\nabla_{f_1}J)f_1) = 0$$

which implies

(3.14) $(\nabla_{f_1}S)(f_i, f_1) + (\nabla_{Jf_1}S)(f_i, Jf_1) + 2\{\mu - (2m-1)\nu\}g(Jf_i, (\nabla_{f_1}J)f_1 + (\nabla_{Jf_1}J)Jf_1) = 0.$

Since $M$ is non Einsteinian in $q$ the first equation of (3.2) and (3.14) result

(3.15) $$(\nabla_{f_1}J)f_1 + (\nabla_{Jf_1}J)Jf_1 = 0.$$

From (3.10) and (3.15) it follows easily that $M$ is an almost Kähler manifold in $q$. Consequently it is an almost Kähler manifold in $U$ and hence an $AK_2$-manifold in $U$. If $M$ is a Kähler manifold in $U$ it is of constant holomorphic sectional curvature [2] and hence Einsteinian in $U$ which contradicts our assumption. Let $M$ is non Kähler in $q$ (we shall use the above notations for the basis of $T_q(M)$) and let

$$(\nabla_{f_1}J)f_i = \alpha_i f_1 + \beta_i Jf_1 \quad \text{for} \quad i > 1.$$

In (2.2) we put $x = u = f_i, \ y = z = f_1$:

$$\nu - \frac{1}{6}(\mu + \mu_1) + \frac{2m-1}{3}\nu = -\frac{1}{2}(\alpha_i^2 + \beta_i^2)$$

for $i > 1$ which implies

(3.16) $$\alpha_i^2 + \beta_i^2 = \alpha_j^2 + \beta_j^2 \quad \text{for} \quad i, j > 1.$$

Now we put in (2.2) $(x = f_i, \ y = z = f_1, \ u = f_j), \ (x = f_i, \ y = z = f_j, \ u = Jf_j)$ respectively and we obtain

$$\alpha_i \alpha_j + \beta_i \beta_j = 0,$$

(3.17)

$$\alpha_i \beta_j - \alpha_j \beta_i = 0,$$

respectively. But (3.16) and (3.17) imply $\alpha_i = \beta_i = 0$ for $i > 1$ which is a contradiction. This proves Lemma.

Now we prove the theorem. Since $M$ is Einsteinian (2.1) takes the form

$$R = \nu\pi_1 + \lambda\pi_2$$

with a constant $\lambda$. Consequently $M$ is a real space form or a complex space form [7].

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