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ПЛИСКА

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THE LOEWNER EQUATION FOR CONFORMAL MAPPINGS OF STRIPS

VICTOR V. GORJAINOV

The well-known Loewner method for studying univalence is based on a special differential equation generating univalent functions. In this paper similarly a differential equation generating conformal mappings of a strip into itself is constructed.

1. Introduction. An approach to the Bieberbach conjecture due to Loewner [1], is based on characterization of the class $\mathfrak{L}$ of functions φ holomorphic and univalent in the unit disk which satisfy the conditions $\varphi(0)=0$, $\varphi'(0)>0$ and $|\varphi(z)|<1$ for $|z|<1$. The class $\mathfrak{L}$ is a semigroup under composition of functions. Just as in the theory of Lie transformation groups one can define "infinitesimal transformations" and one-parameter semigroups, or a little more generally, paths emanating from the identity whose tangent at any point is an infinitesimal transformation. Loewner showed that the totality of functions lying on such paths is a dense subset in $\mathfrak{L}$. The paths are described in terms of differential equation.

Let $\mathfrak{T}$ be the class of functions f holomorphic and univalent in the strip $\Pi=\{z:|\mathrm{Im}\, z|<\pi\}$ that satisfy the conditions $f(\Pi)\subseteq\Pi$, $f(\mathrm{R})=\mathrm{R}$ and $f'(x)>0$ for $x\in\mathrm{R}$. The class $\mathfrak{T}$ is also a semigroup under composition of functions. Just as in Loewner's paper [1] we derive a differential equation that generates a conformal mappings of the strip Π into itself. The existence and uniqueness theorem for this equation is established as well.

2. Infinitesimal transformations. Let $\{f_t\}_{t\geqq 0}$ be one-parameter family of functions f_t of $\mathfrak{T}$ such that $f_0(z)\equiv z$ and $w=f_t(z)$ is differentiable on t locally uniformly in Π. Then the derivative

$$\frac{\partial}{\partial t} f_t(z)|_{t=0}=v(z)$$

is called an infinitesimal transformation of the semigroup $\mathfrak{T}$.

L e m m a 1. *Any infinitesimal transformation $v(z)$ of $\mathfrak{T}$ is a holomorphic function in the strip Π and satisfies the condition* $(\mathrm{Im}\, z)(\mathrm{Im}\, v(z))\leqq 0$ *for* $z\in\Pi$.

P r o o f. Let $v(z)$ be the infinitesimal transformation that corresponds to the one-parameter family $\{f_t\}_{t\geqq 0}$. Since

$$\frac{f_t(z)-z}{t}\rightarrow v(z)$$

locally uniformly in Π as $t\to 0$, $v(z)$ is holomorphic in Π. The equality $f_t(\mathsf{R})=\mathsf{R}$ implies $\operatorname{Im} v(x)=0$ for $x\in\mathsf{R}$.

Furthermore, it follows from Schwarz lemma that the mapping $z\to f_t(z)$ does not increase the hyperbolic metric in Π with length element

$$ds=\frac{|dz|}{\cos(\operatorname{Im} z/2)}.$$

The shortest (under hyperbolic metric) are from z to the real axis belongs to the line which is orthogonal to the real axis. Hence $|\operatorname{Im} f_1(z)|\leqq|\operatorname{Im} z|$ and $(\operatorname{Im} z)(\operatorname{Im} v(z))\leqq 0$. This completes the proof.

3. Neighborhoods of the identity. Suppose that $f\in\mathfrak{T}$. By Schwarz lemma we have

$$\frac{|f'(z)|}{\cos(\operatorname{Im} f(z)/2)}\leqq\frac{1}{\cos(\operatorname{Im} z/2)},$$

where the equality is attained only for the translation along the real axis. As a consequence, the inequalities $f'(x)\leqq 1$ is satisfied for $x\in\mathsf{R}$.

Lemma 2. *For any $\alpha\in(0,1)$ there exist a constant M_α, depending only on α, such that the inequality*

$$|f(z)-z-f(0)|\leqq(1-f'(0))M_\alpha$$

holds for $z\in\Pi_\alpha$ and $f\in\mathfrak{T}$, where

$$\Pi_\alpha=\{z:|\operatorname{Im} z|<\alpha\pi,\quad |\operatorname{Re} z|<1/(1-\alpha)\}.$$

Proof. Let $f\in\mathfrak{T}$ and $f(0)=0$. Then the function $\varphi(\zeta)=g\circ f\circ g^{-1}(\zeta)$ satisfies the hypotheses of Schwarz lemma, where

$$g(z)=\frac{e^{z/2}-1}{e^{z/2}+1}$$

is conformal mapping of the strip Π onto the unit disk. Since $\varphi'(0)=f'(0)$, the function

$$\psi(\zeta)=\frac{\varphi(\zeta)/\zeta-f'(0)}{1-f'(0)\varphi(\zeta)/\zeta}$$

also satisfies the hypotheses of Schwarz lemma. The inequality $|\psi(\zeta)|\leqq|\zeta|$, which may be rewritten as

$$\left|\frac{\dfrac{g\circ f(z)}{g(z)}-f'(0)}{1-f'(0)\dfrac{g\circ f(z)}{g(z)}}\right|\leqq|g(z)|,$$

implies:

$$\left|\frac{g\circ f(z)}{g(z)}-f'(0)\right|\leqq\frac{|g(z)|(1-(f'(0))^2)}{1-|g(z)|\,f'(0)}.$$

Using the inequality $|g(z)|<1$, we obtain

$$|g\circ f(z)-g(z)|\leqq(1-f'(0))\frac{3}{1-|g(z)|}.$$

The value $|g(z')-g(z'')|/|z'-z''|$ is separated from zero locally uniformly in Π. Therefore there exists a constant M_α, depending only on α, such that the inequality

$$|f(z)-z|\leqq(1-f'(0))M_\alpha$$

holds for $z\in\Pi_\alpha$, $\alpha\in(0, 1)$. If $f(0)\neq 0$ then we apply the inequality from above to $f(z)-f(0)$. This completes the proof of the lemma.

4. One-parameter semigroups. The semigroup $\mathfrak{T}$ endowed with topology induced by locally uniform convergence in Π is a topological semigroup. A one-parameter semigroup in $\mathfrak{T}$ is a continuous homomorphism $\varphi:\mathsf{R}^+\to\mathfrak{T}$, where $\mathsf{R}^+=\{t\in\mathsf{R}:t\geqq 0\}$. It is convenient to denote the image of t under φ by φ_t. Thus $\{\varphi_t\}$ is a family of functions in $\mathfrak{T}$. The fact that $t\to\varphi_t$ is a homomorphism follows from the identities

(i) $\varphi_0(z)\equiv z$,

(ii) $\varphi_{t+s}=\varphi_t\circ\varphi_s$.

The group of translations along the real axis is the maximal subgroup in $\mathfrak{T}$. Denote by $\mathfrak{T}_0$ the subsemigroup in $\mathfrak{T}$ of functions f, normalized by the condition $f(0)=0$.

Theorem 1. *Let $t\to\varphi_t$ be a one-parameter semigroup in $\mathfrak{T}_0$. Then $w=f_t(z)$ is absolutely continuous for $t\geqq 0$ and satisfies the differential equation*

$$\frac{dw}{dt}=-h(w),$$

where $h(w)$ is a holomorphic function satisfying the conditions $h(0)=0$ and $(\operatorname{Im} w)(\operatorname{Im} h(w))\geqq 0$ for $w\in\Pi$.

Proof. Let $t\to\varphi_t$ be a one-parameter semigroup in $\mathfrak{T}_0$. Then the function

$$\beta(t)=\frac{\partial}{\partial z}\varphi_t(z)|_{z=0}$$

is continuous and satisfies the conditions $\beta(0)=1$, $\beta(t+s)=\beta(t)\beta(s)$ for all $t, s\in\mathsf{R}^+$. Therefore $\beta(t)=e^{-at}$, where $a\geqq 0$. If $a=0$, then $\beta(t)\equiv 1$ and $\varphi_t(z)\equiv z$ by Schwarz lemma.

Now assume that $a>0$. Since the functions $g\circ\varphi_t\circ g^{-1}$, $t\geqq 0$, satisfy the hypotheses of Schwarz lemma, where g is the same as in the proof of Lemma 2, then for each $\alpha\in(0, 1)$ there exists a number $\gamma\in(0, 1)$, depending only on α, such that $\varphi_t(\Pi_\alpha)\subseteq\Pi_\gamma$ for every φ_t, $t\geqq 0$. Therefore from Lemma 2 we get

$$|\varphi_t(z)-\varphi_s(z)|=|\varphi_{t-s}\circ\varphi_s(z)-\varphi_s(z)|\leqq(1-e^{-a(t-s)})M_\gamma$$

for all s, t, $0\leqq s\leqq t$, and $z\in\Pi_\alpha$. Hence $w=\varphi_t(z)$ is absolutely continuous for $t\geqq 0$. Furthermore,

$$\lim_{t\to s}\frac{\varphi_t(z)-\varphi_s(z)}{t-s}=\lim_{t\to s}\frac{\varphi_{t-s}\circ\varphi_s(z)-\varphi_s(z)}{t-s}=v\circ\varphi_s(z),$$

where $v(z)$ is an infinitesimal transformation of $\mathfrak{T}_0$. Now the theorem follows from the Lemma 1.

5. Loewner equation. We replace the equation from Theorem 1 by a little more general one which leads to the equation

$$\frac{dw}{dt}=-H(w, t), \tag{1}$$

similar to Loewner equation.

Our main result is the following theorem.

Theorem 2. *Let $H(z, t)$ be a function which is:*

(i) *holomorphic in the strip Π for any fixed t in $[0, T]$;*

(ii) *measurable with respect to t;*

(iii) $(\operatorname{Im} z)(\operatorname{Im} H(z, t))\geqq 0$ *for all $z\in\Pi$, $t\in[0, T]$;*

(iv) *$H(0, t)$, $H'(0, t)$ are summable.*

Then if $z\in\Pi$ and $s\in[0, T)$ the differential equation (1) has a unique absolutely continuous solution $w=w(t, z, s, H)$ with initial condition $w|_{t=s}=z$. Furthermore, $w^H_{t,s}: z\to w(t, z, s, H)$ is a conformal mapping of Π into itself. If the function $H(z, t)$ satisfies the additional condition

$$\sup_{x\in\mathbb{R}}\left|\int_0^T H(x, t)dt\right|<\infty.$$

then the real axis is invariant under all mappings $w^H_{t,s}$, $0\leqq s\leqq t\leqq T$, that is $w^H_{t,s}(\mathbb{R})=\mathbb{R}$.

Note. We write $H'=\partial H/\partial z$.

Proof. Since $(\operatorname{Im} z)(\operatorname{Im} H(z, t))\geqq 0$ for $z\in\Pi$, $t\in[0, T]$, the function

$$(H\left(\left(2\ln\frac{1+\zeta}{1-\zeta}, t\right)-H(0, t)\right)/H'(0, t)$$

is typically real [2] for each $t\in[0, T]$. By [2] there exist constants M_α, N_α, depending only on α, such that

$$|H(z, t)| \leqq M_\alpha H'(0, t) + |H(0, t)|, \quad |H'(z, t)| \leqq N_\alpha H'(0, t)$$

for all $z \in \Pi_\alpha$, $\alpha \in (0, 1)$. Since the rectangle Π_α is a convex domain, we have also

$$|H(z', t) - H(z'', t)| \leqq N_\alpha H'(0, t)|z' - z''|$$

for all $z', z'' \in \Pi_\alpha$.

Using these inequalities and the general theory of differential equations, we obtain the local existence of a unique absolutely continuous solution of (1). We need the following lemma for to prove the global existence.

Lemma 3. *The mapping $w_{t,s}^H : z \to w(t, z, s, H)$ defined as a transition along solutions of differential equality* (1) *does not increase the hyperbolic metric in* Π.

Proof. Evidently it is enough to prove the inequality

$$\frac{d}{dt}\left(\frac{|dw|}{\cos(\operatorname{Im} w/2)}\right) \leqq 0, \tag{2}$$

where the dependence of w from H is determined by (1). Since

$$\frac{d}{dt}|dw| = -\operatorname{Re} H'(w, t)|dw|, \quad \frac{d \operatorname{Im} W}{dt} = -\operatorname{Im} H(w, t),$$

the inequality (2) can be rewritten in the form

$$\frac{\cos\frac{\operatorname{Im} w}{2} \operatorname{Re} H'(w, t) + \frac{1}{2}\sin\frac{\operatorname{Im} w}{2} \operatorname{Im} H(w, t)}{\cos(\operatorname{Im} w/2)^2}|dw| \geqq 0.$$

Having in mind the integral representation of typically real functions it is enough for proving (2) to establish that the expression

$$Q(h, z) = \operatorname{Re} h'(z) + \frac{1}{2}\operatorname{tg}\frac{\operatorname{Im} z}{2} \operatorname{Im} h(z)$$

is nonnegative for all $z \in \Pi$ and all functions

$$h(z) = (e^z - 1)/(e^z + \beta), \quad \beta > 0.$$

It is easy to see that

$$q(\beta, e^z) Q(h, z) \geqq 0$$

for $z \in \Pi$, where

$$q(\beta,\ \xi)=(|\xi|^2+\beta^2)\operatorname{Re}\xi+2\beta|\xi|^2+\frac{\operatorname{Im}\xi}{2}|\xi+\beta|^2\operatorname{tg}\frac{\arg\xi}{2}$$

and β is the same as in the definition of $h(z)$. Now we need to show that $q(\beta,\ \xi)\geqq 0$ for all ξ and $\beta>0$.

It is evident that $q(\beta,\ \bar{\xi})=q(\beta,\ \xi)$ and $q(\beta,\ \xi)>0$ when $\operatorname{Re}\xi>0$. Therefore it will be enough to show that $q(\xi,\ \beta)\geqq 0$ when $\xi=x+iy$ and $y>0$. The restriction for ξ implies the following equality

$$\operatorname{tg}\frac{\arg\xi}{2}=\sqrt{1+\frac{x^2}{y^2}}-\frac{x}{y}.$$

Hence

$$q(\beta,\ \xi)=x(|\xi|^2+\beta^2)+2\beta|\xi|^2+\frac{y}{2}|\xi+\beta|^2\left(\sqrt{1+\frac{x^2}{y^2}}-\frac{x}{y}\right)$$

$$=x(|\xi|^2+\beta^2)+2\beta|\xi|^2+\frac{1}{2}(|\xi|-x)(|\xi|^2+2\beta x+\beta^2)$$

$$=\frac{1}{2}(|\xi|+x)(|\xi|+\beta)^2+\beta(|\xi|^2-x^2)\geqq 0.$$

This completes the proof of the lemma.

We turn now to the question of global existence of the solution. It is clear that $w=w(t,\ x,\ s,\ H)$ is real for $x\in\mathsf{R}$. Furthermore, it is not difficult to verify that the following inequality

$$|w(t,\ x,\ s,\ H)|\leqq|x|+\int_s^t|H(0,\ \tau)|d\tau \tag{3}$$

is holds. We can see from (3) that there exists a solution $w=w(t,\ x,\ s,\ H)$ for $s\leqq t\leqq T$ and for every $x\in\mathsf{R}$. By Lemma 3 we conclude that $w=w(t,\ z,\ s,\ H)$ exists for $s\leqq t\leqq T$ and for all $z\in\Pi$.

Now to complete the proof we need to show that the additional condition on $H(z,\ t)$ implies the relation $w_{t,s}^H(\mathsf{R})=\mathsf{R}$ for every $t\in[s,\ T]$. We can see from the hypothesis (iii) of the theorem that $H(x,\ t)$ increases when $x\in\mathsf{R}$. Therefore there exist finite or infinite limits

$$\eta^{\pm}(t)=\lim_{x\to\pm\infty}H(x,\ t).$$

Furthermore, $x(H(x,\ t)-H(0,t))\geqq 0$ for all $x\in\mathsf{R}$ and $t\in[0,\ T]$. Now we obtain from P. Fatou's and B. Levi's theorems about sequences of positive measurable functions that

$$\lim_{x\to\pm\infty} \int_s^t H(x, \tau)d\tau = \int_s^t \eta^{\pm}(\tau)d\tau$$

for all $s, t \in [0, T]$. Using (3), we see also that

$$x - w(t, x, s, H) = \int_s^t H(w(\tau, x, s, H), \tau)\, d\tau \leqq \int_s^t \eta^+(\tau)dt.$$

Similarly

$$x - w(t, x, s, H) \geqq \int_s^t \eta^-(\tau)d\tau.$$

These inequalities show that the distortion of mapping $w_{t,s}^H$ is bounded on R. Therefore the equality $w_{t,s}^H(\mathsf{R}) = \mathsf{R}$ is true. The proof is complete.

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Academy of Sciences of the Ukrainian SSR
Institute of Applied Mathematics and Mechanics
340048 Donetsk USSR

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