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A FEW REMARKS ON BOUNDED UNIVALENT FUNCTIONS

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Let $S_R(M)$, $M>1$, denote the class of functions $F(z)=z+\Sigma_{n=2}^{\infty} A_n z^n$ holomorphic and univalent in the disc $E=\{z:|z|<1\}$, satisfying the conditions: $|F(z)|<M$ for $z\in E$, $A_n=\bar{A}_n$ for $n=2, 3,\dots$ In the paper, estimations of several functionals defined on the class $S_R(M)$, $M>1$, are obtained. In particular, the property of the coefficient A_5, obtained here, seems to be rather interesting.

1. Let S denote the class of functions

(1) $$F(z)=z+\sum_{n=2}^{\infty} A_{nF} z^n$$

holomorphic and univalent in the disc $E=\{z:|z|<1\}$, S_R – its subclass composed of all functions with real coefficients. Let further $S(M)$ and $S_R(M)$, $M>1$, stand for subclasses of S and S_R, respectively, consisting of functions satisfying in the disc E the condition: $|F(z)|<M$.

Consider on the class $S_R(M)$, $M>1$, the functional

(2) $$H_N(F)=A_{NF}.$$

In [5] it was shown that if there exists a function $F^*\in S_R(M)$ realizing $\max_{F\in S_R(M)} H_N(F)$ and $\max_{F\in S_R(M)} H_{N+1}(F)$ at the same time, then $F^*\in\tilde{P}$, where $\tilde{P}$ is the family of functions $w=P(z;M;\tau)$, $P(0;M;\tau)=0$, $-1\leqq\tau\leqq 1$, satisfying in the disc E the equation

(3) $$\frac{w}{1-2\tau\dfrac{w}{M}+\left(\dfrac{w}{M}\right)^2}=\frac{z}{1-2\tau z+z^2}.$$

Of course, every function $P(z;M;\tau)$ belongs to $S_R(M)$ and maps the disc E onto the disc $|w|<M$ from which at most two appropriate segments of the real axis are removed.

One knows the estimation of the maximum of functional (2) when $N=4$ [21-23, 15, 8], for all values of $M>1$; for $M\geqq 11$, the only function realizing $\max_{F\in S_R(M)} H_4(F)$ is the Pick function $w=P(z;M;1)$ mapping the disc E onto the disc $|w|<M$ from which a segment of the real axis, issuing from the point $w=-1$,

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has been removed. For the remaining M, i.e. for $1<M<11$, the extremal functions with respect to $\max_{F\in S_R(M)} H_4(F)$ are not of class $\tilde{P}$.

Consequently, the question arises whether from the above information one can draw the conclusions that there exists a function realizing simultaneously the maxima of the functionals $H_4(F)$ and $H_5(F)$, $F\in S_R(M)$, for some M.

For the purpose, let us put

$$w=P(z;\ M;\ \tau)=z+\sum_{n=2}^{\infty} a_n(M,\ \tau)z^n,\ z\in E. \tag{4}$$

We shall prove

Theorem 1. *If $P\in\tilde{P}$, whereas the coefficient $a_5(M,\ \tau)$ is defined by relation* (4), *then*

$$\max_{-1\leqq\tau\leqq 1} a_5(M;\ \tau)=\begin{cases} a_5(M,0) & \text{for}\quad 4-\sqrt{6}<M<4+\sqrt{6},\\ a_5(M,1) & \text{for}\quad 1<M\leqq 4-\sqrt{6},\quad M\geqq 4+\sqrt{6}.\end{cases} \tag{5}$$

Proof. Let

$$\frac{z}{1-2\tau z+z^2}=z+\sum_{n=2}^{\infty} a_n(\tau)z^n,\quad z\in E,\quad \tau\in[-1,\ 1].$$

It is known that

$$a_n(\tau)=\begin{cases} (-1)^{n+1}n & \text{for}\quad \tau=-1,\\ \sin n\varphi/\sin\varphi & \text{for}\quad -1<\tau<1,\quad n=2,\ 3,\dots,\\ n & \text{for}\quad \tau=1,\end{cases} \tag{6}$$

where $e^{i\varphi}=\tau+i\sqrt{1-\tau^2}$, $\varphi\in[0,\ \pi]$.

So, from equation (3) we have

$$\sum_{n=1}^{\infty} a_n(M,\ \tau)z^n=\left[1-\frac{2\tau}{M}\sum_{n=1}^{\infty} a_n(M,\ \tau)z^n+\frac{1}{M^2}\left(\sum_{n=1}^{\infty} a_n(M,\ \tau)z^n\right)^2\right]\sum_{k=1}^{\infty} a_k(\tau)z^k,$$

$$a_1(M,\ \tau)=1,\quad a_1(\tau)=1,$$

whence

$$\sum_{n=1}^{\infty} a_n(M,\ \tau)z^n=\sum_{k=1}^{\infty} a_k(\tau)z^k-\frac{2\tau}{M}\sum_{n,k=1}^{\infty} a_n(M,\ \tau)a_k(\tau)z^{n+k} \tag{7}$$

$$+\frac{1}{M^2}\sum_{n,l,k=1}^{\infty} a_n(M,\ \tau)a_l(M,\ \tau)a_k(\tau)z^{n+l+k}.$$

In turn, comparing the corresponding coefficients in (7), we shall obtain the following recurrence formula for the coefficients of the function $w=P(z;\, M;\, \tau)$

(8)
$$a_s(M,\ \tau)=a_s(\tau)-\frac{2\tau}{M}\sum_{\substack{1\leqq n,k\leqq s-1\\ n+k=s}} a_n(M,\ \tau)a_k(\tau)$$
$$+\frac{1}{M^2}\sum_{\substack{1\leqq n,l,k\leqq s-2\\ n+l+k=s}} a_n(M,\ \tau)a_l(M,\ \tau)a_k(\tau),\quad s=2,\ 3\ldots$$

Making use of form (6) of the coefficients $a_k(\tau)$, from (8) we have successively

(9)
$$a_2(M,\ \tau)=2\left(1-\frac{1}{M}\right)\tau,$$
$$a_3(M,\ \tau)=\left(1-\frac{1}{M}\right)\left[4\left(1-\frac{1}{M}\right)\tau^2-\left(1+\frac{1}{M}\right)\right],$$
$$a_4(M,\ \tau)=2\left(1-\frac{1}{M}\right)\tau\left[4\left(1-\frac{1}{M}\right)^2\tau^2-\left(2-\frac{3}{M^2}\right)\right]$$
$$a_5(M,\ \tau)=$$
$$\left(1-\frac{1}{M}\right)\left[16\left(1-\frac{1}{M}\right)^3\tau^4-12\left(1-\frac{1}{M}\right)\left(1-\frac{2}{M^2}\right)\tau^2+\left(1+\frac{1}{M}\right)\left(1-\frac{2}{M^2}\right)\right],$$

where $M>1,\ -1\leqq\tau\leqq 1$.

Property (5) is obtained from the fourth formula in (9) by examining the function $a_5(M,\ \tau)$ as a function of $\tau\in[-1,\ 1]$, with a fixed $M>1$.

At the same time, note that $a_5(M,\ 1)$ is the fifth coefficient of the Pick function $w=P(z;\, M;\ 1)$ we mentioned before [6].

Since $M_0=4+\sqrt{6}<11$, therefore from the information given earlier one can draw only the following evident

Corollary 1. *For $M<11$, in the class $S_R(M)$ there is no function realizing $\max_{F\in S_R(M)} A_{4F}$ and $\max_{F\in S_R(M)} A_{5F}$ at the same time.*

Moreover, note that [24]:

Corollary 2. *If the Pick function $w=P(z;\ M;\ 1)$ is a function realizing $\max_{F\in S_R(M)} A_{5F}$ and $\max_{F\in S_R(M)} A_{6F}$ at the same time, then $M\geqq 4+\sqrt{6}$.*

2. In [9] Lewandowski and Wajler introduced and studied the class T_M, $M>1$, of functions of form (1) holomorphic in the disc E, given by the structural formula

(10)
$$F(z)=\int_{-1}^{1} P(z;\ M;\ \tau)\,d\mu(\tau),\quad z\in E,$$

where μ is any function non-decreasing on the interval $[-1, 1]$, $\mu(1)-\mu(-1)=1$, while P is a function defined by formula (3). It turns out that T_M is a subclass of bounded typically-real functions.

From (4), (5), (9) and (10) we obtain (cf. the general theorem in [9])

Corollary 3. *If* $F\in T_M$, $M>1$, *then*

$$A_{5F}\leqq\begin{cases}a_5(M,0) & \text{for}\quad 4-\sqrt{6}<M<4+\sqrt{6},\\ a_5(M,1) & \text{for}\quad 1<M\leqq 4-\sqrt{6},\quad M\geqq 4+\sqrt{6},\end{cases}$$

and

$$A_{4F}\leqq\begin{cases}a_4(M,0) & \text{for}\quad \dfrac{4-\sqrt{2}}{2}<M<\dfrac{4+\sqrt{2}}{2},\\ a_4(M,1) & \text{for}\quad 1<M\leqq\dfrac{4-\sqrt{2}}{2},\quad M\geqq\dfrac{4+\sqrt{2}}{2}.\end{cases}$$

So, it can be seen that in the classs T_M the Pick function $w=P(z;M;1)$ is extremal with respect to functional (2) when $N=4$ and $M\in(1, (4-\sqrt{2})/2]$ or $M\geqq(4+\sqrt{2})/2$ and when $N=5$ and $M\in(1, 4-\sqrt{6}]>$or $M\geqq 4+\sqrt{6}$. What is more, for $N=3$, 4, 5 and the corresponding M, $\max_{F\in T_M} H_N(F)$ is non-positive.

3. As we know, the problem of estimating the coefficients in the classes $S(M)$ and $S_R(M)$ for all values of M is a difficult task. When M is sufficiently close to 1, the estimation of the coefficients in the above classes was obtained by Siewierski [17-19] and Schiffer and Tammi [16]. It is also known [6] that if M is sufficiently large, the only function extremal with respect to the maximum of the functional A_{NF}, $F\in S_R(M)$, N-even, is the Pick function $w=P(z;M;1)$, $z\in E$. Whereas, for all values of M, only the estimations of the coefficients A_{2F}, A_{3F}, A_{4F} are known in the class $S_R(M)$ [11, 14, 20, 3, 15, 21, 8]. And so, for instance, one knows that if $F\in S_R(M)$, then

$$A_{3F}\leqq 1+2\lambda^2-\frac{4}{M^\lambda}+\frac{1}{M^2}\quad\text{for}\quad M\geqq e, \tag{11}$$

where λ is the greater root of the equation $\lambda\log\lambda=-1/M$. What is more, the extremal function $w=F^{**}(z)$ satisfies in this case the equation (cf. [2]):

$$\frac{1}{M}(1-w^{-1})A(w)-2\lambda\log\frac{w[A(w)-w+1-2M\lambda]}{A(w)-1+(1-2M\lambda)w}=z-z^{-1}+2\lambda\log z, \tag{12}$$

where $A(w)=[w^2+2(2M\lambda-1)w+1]^{1/2}$ stands for that branch which, for $w=0$, takes the value 1.

Consequently, it can be seen that the Pick function $w=P(z;M;1)$ which is the only extremal function for $\max_{F\in S_R(M)} A_{2F}$ when $M>1$ and $\max_{F\in S_R(M)} A_{4F}$ when $M\geqq 11$, is not extremal for $\max_{F\in S_R(M)} A_{3F}$ when $M>1$; in particular, for $M\geqq e$ (cf. (11)). Till now, the estimations of the coefficients A_{5F} for each $M>1$ are not known in the class $S_R(M)$. It seems interesting to ask whether in this case the Pick function can be extremal for some values of M. Calculating the fifth coefficient of the Pick function $w=P(z;M;1)=z+\Sigma_{n=2}^{\infty}P_n(M)z^n$, $z\in E$, we shall get (cf. [6]):

$$P_5(M)=5-\frac{40}{M}+\frac{105}{M^2}-\frac{112}{M^3}+\frac{42}{M^4}. \tag{13}$$

Consider a function extremal with respect to $\max_{F\in S_R(M)} A_{3F}$, i.e. the function $w=F^{**}(z)=z+\Sigma_{n=2}^{\infty}A_n(M)z^n$, $z\in E$, satisfying equation (12). After toilsome calculations we obtain

$$\begin{gathered} A_4(M)=2\lambda\left(1+\lambda^2-\frac{6}{M}\lambda+\frac{1}{M^2}\right), \\ A_5(M)=1+2\lambda^2+2\lambda^4+\frac{3}{M^2}+\frac{2}{M^4}-\frac{12}{M}\lambda+\frac{34}{M^2}\lambda^2-\frac{16}{M}\lambda^3-\frac{52}{3M^3}\lambda, \end{gathered} \tag{14}$$

where, as previously, λ is the greater root of the equation $\lambda\log\lambda=-1/M$.

The following theorem holds:

Theorem 2. *There exists an $M_0>1$ such that, for each $M>M_0$,*

$$A_5(M)>P_5(M). \tag{15}$$

Proof. Indeed, from formulae (13) and (14) we have

$$A_5(M)-P_5(M)=-4+2\lambda^2+2\lambda^4-\frac{4}{M}(4\lambda^3+3\lambda-10)+\frac{34}{M^2}(\lambda^2-3)$$

$$-\frac{4}{M^3}\left(\frac{13}{3}\lambda-28\right)+\frac{40}{M^4}.$$

Substituting $1/M=-\lambda\log\lambda$, let us consider the function

$$Q(\lambda)=A_5(M)-P_5(M)=-4+2\lambda^2+2\lambda^4+4\lambda(4\lambda^3+3\lambda-10)\log\lambda$$

$$+34\lambda^2(\lambda^2-3)\log^2\lambda+4\lambda^3\left(\frac{13}{3}\lambda-28\right)\log^3\lambda+40\lambda^4\log^4\lambda.$$

Calculating the values of the successive derivatives of the function $Q(\lambda)$, we have $Q'(1)=0$, $Q''(1)=0$, $Q'''(1)=-40$.

Since $e\leqq M<+\infty$, therefore $1/e\leqq\lambda<1$. So, the last equality of those given above implies that there exists some $\lambda_0<1$ such that, for $\lambda_0<\lambda<1$, the function $Q''(\lambda)$ is decreasing. Moreover, $Q''(1)=0$. Consequently, $Q''(\lambda)$ is positive in this neighbourhood, and hence, $Q'(\lambda)$ is increasing for $\lambda_0<\lambda<1$. Since $Q'(1)=0$, therefore $Q'(\lambda)$ is negative for $\lambda_0<\lambda<1$ and, finally, $Q(\lambda)$ is a decreasing function in this interval and, as can be seen, $Q(1)=0$. So, $Q(\lambda)>0$ for $\lambda_0<\lambda<1$.

In consequence, there exists an $M_0>1$ such that, for all $M>M_0$, inequality (15) is satisfied.

This immediately implies

Corollary 4. *There exists an $M_0>1$ such that, for each $M>M_0$, the Pick function $w=P(z;\ M;\ 1)$ is not extremal for* $\max_{F\in S_R(M)} A_{5F}$.

The above results seem to make probable the hypothesis that, for $N=7, 9,\ldots$, the Pick function will not be extremal with respect to the maximum of A_{NF}, $F\in S_R(M)$, when M is sufficiently large.

Corollary 5. *Since the function defined by equation* (12), *realizing* $\max_{F\in S_R(M)} A_{3F}$, *belongs to the class $S(M)$, $M>1$, therefore the Pick function cannot be extremal with respect to the maximum of* $\operatorname{Re} A_{5F}$, *$F\in S(M)$, either, for M sufficiently large.*

The estimation of $\max_{F\in S_R(M)} A_{NF}$, N — even, for M sufficiently large, quoted earlier [6] and Theorem 2 directly imply

Corollary 6. *There is no function realizing simultaneously* $\max_{F\in S_R(M)} A_{5F}$ *and* $\max_{F\in S_R(M)} A_{NF}$, *$N$ — even, if M is sufficiently large.*

4. In [7], when examining the functional $H(F)$, $F\in S_R$, depending on a finite number of the coefficients A_{nF} and satisfying some general assumptions, it is proved that, for M sufficiently large, only the Pick functions $w=P(z;\ M;\ \varepsilon)$, $z\in E$, $\varepsilon=\pm 1$, can realize the maximum of (F) in the class $S_R(M)$.

Consider the following functionals defined on the class S_R:

$$A_{2F}(A_{3F}-A_{1F}) \quad (A_{1F}=1), \tag{16}$$

$$A_{2F}(A_{4F}-A_{2F}), \tag{17}$$

$$A_{2F}(A_{5F}-A_{3F}). \tag{18}$$

In the case of the first two functionals, it is not difficult to verify that the assumptions of Theorem 2 in [7] are satisfied; the third functional was considered in the paper we cite.

So, in the case of functionals (16) and (18), the only function realizing their maxima in the class $S_R(M)$ for M sufficiently large is the Pick function $w=P(z;\ M;\ 1)$, $z\in E$. Whereas for functional (17), the maximum in the class $S_R(M)$ for large M is realized by the functions $w=P(z;\ M;\ 1)$ and $w=P(z;\ M;\ -1)$, $z\in E$.

These results are premisses for raising the following hypothesis:

There exists an $\hat{M}>1$ such that, for all $M>\hat{M}$, the maximum of the functional $A_{2F}(A_{n+1,F}-A_{n-1,F})$ in the class $S_R(M)$ is realized by the Pick function $w=P(z;\ M;\ 1)$ (n — even) or the two functions $w=P(z;\ M;\ 1)$ and $w=P(z;\ M;\ -1)$ (n — odd).

5. Let $F\in S$. Denote by $S^{(2)}$ the class of functions

$$h(z)=\sqrt{F(z^2)}=z+c_3z^3+c_5z^5+\ldots, \quad z\in E. \tag{19}$$

Between functions of the classes S and $S^{(2)}$ there holds a one-to-one correspondence.

Littlewood-Paley proved the following well-known theorem [10]:

If $h\in S^{(2)}$, then $|c_n|<A$, $n=3, 5,\dots$, where A is an absolute constant. These authors also raised the hypothesis that the constant $A=1$. It is easy to notice that this hypothesis is true for $n=3$, whereas for $n=5$, it was demolished [1].

It becomes natural to ask whether the analogous hypothesis is true in the class $S^{(2)}(M)$, i.e. in the class of functions of form (19), when $F\in S(M)$. For example, from the estimations obtained by Siewierski [17-19] and Schiffer and Tammi [16] it follows easily that, for every n, there exists an $M_n>1$ such that, for all $M<M_n$ and each function $h\in S^{(2)}(M)$, we have $|c_n|\leqq 1$.

From Pick's estimation [11] we know that if $F\in S(M)$, then $|A_{2F}|\leqq 2(1-M^{-1})$, $M>1$, thus, by (19),

$$|c_3|=\frac{1}{2}|A_{2F}|\leqq 1-M^{-1}<1 \quad \text{for each} \quad M>1.$$

Consequently, the corresponding hypothesis of Litllewood-Paley for the class $S^{(2)}(M)$ is confirmed for $n=3$ with any M.

Since, by (19),

$$c_5=\frac{1}{2}\left(A_{3F}-\frac{1}{4}A_{2F}^2\right),$$

therefore, making use of the estimation of the Goluzin functional in the class $S(M)$ [2], we have

$$|c_5|\leqq\begin{cases}\frac{1}{2}(1-M^{-2}) & \text{when } 1<M\leqq e^{4/3},\\ M^{-2}(1-\beta)^2+\frac{1}{2}(1-M^{-2}) & \text{when } M>e^{4/3},\end{cases}$$

where $1<\beta<M$ is a root of the equation

$$\beta\,\mathrm{Log}\frac{\beta}{M}+\frac{1}{3}\beta+1=0. \tag{20}$$

Of course, $(1-M^{-2})/2<1$ for each $M>1$, in particular, for $1<M\leqq e^{4/3}$.

It is not difficult to check that the expession $M^{-2}(1-\beta)^2+(1-M^{-2})/2$, $\beta=\beta(M)$, is an increasing function of the variable M, and from the examination of equation (20) it follows that M, being a function of β, increases from $e^{4/3}$ to $+\infty$. So, we obtain the following:

Corollary 7. *There exists exactly one $M_5\in(e^{4/3}, +\infty)$ such that, for each $M\leqq M_5$, $|c_5|\leqq 1$ and, for $M>M_5$, this estimation in the class $S^{(2)}(M)$ is false.*

The effective determination of the sequence $M_5, M_7, \ldots,$ seems to be an interesting task (cf. the relationships between the hypothesis of Littllewood-Paley and that of Bieberbach).

6. The estimations of coefficients in the class $S_R(M)$, mentioned in Sections 1 and 3, lead in a natural way to the consideration of $\max_{F\in S_R(M)} A_{kF}$, where n is some even number, k — odd.

Pietrasik [12] showed that there exists a $M_{n,k}>1$ such that in the class $S_R(M)$, $M>M_{n,k}$, the estimation

$$A_{nF}\,A_{kF}\leqq P_n(M)P_k(M)$$

takes place.

The same author also obtained an estimation of $\max_{F\in S_R(M)} A_{2F}A_{3F}$, $M>1$, and, in particular, proved [13] that in the class $S_R(M)$ the estimations

$$A_{2F}A_{3F}\leqq\begin{cases} -2\left(1-\frac{1}{M}\right)^2\left(3-\frac{5}{M}\right) & \text{if } 1<M<\frac{13}{11},\\ 2\left(1-\frac{1}{M}\right)^2\left(3-\frac{5}{M}\right) & \text{if } M\geqq\frac{13}{3}\end{cases}$$

hold; in the first case, the extremal function is the function $w=P(z;M;-1)$, while in the other — the function $w=P(z;M;1)$, $z\in E$.

In consequence, the following hypothesis was raised [4]: for each $n=3,4,\ldots,$ there exists an $M_n^*>1$ such that, for $M\in(1, M_n^*)$, $\max_{F\in S_R(M)} A_{2F}A_{nF}$ is attained for the Pick functions $w=P(z;M;1)$ or $w=P(z;M;-1)$.

The result below is a confirmation of this hypothesis for $n=4$.

Theorem 4. *For any function $F\in S_R(M)$, where*

$$1<M<\frac{10-\sqrt{46}}{3}\quad or\quad M>\frac{10+\sqrt{46}}{3}, \tag{21}$$

the estimation

$$A_{2F}A_{4F}\leqq 4\left(1-\frac{1}{M}\right)^2\left(2-\frac{8}{M}+\frac{7}{M^2}\right) \tag{22}$$

takes place; equality holds for the Pick function $w=P(z;M;1)$, $z\in E$.

Proof. Jokinen showed in [8] that in the class $S_R(M)$ the inequality

$$A_{4F}\leqq -\frac{7}{12}A_{2F}^3+\frac{1}{2}\left(4-\frac{9}{M}\right)A_{2F}^2+\frac{2}{3}\left(1-\frac{1}{M^3}\right) \tag{23}$$

holds if

$$-\frac{2}{3}\frac{1}{M}\leqq A_{2F}\leqq 2\left(1-\frac{1}{M}\right).$$

On the other hand, if $F \in S_R(M)$ is a function extremal with respect to the functional $A_{2F} A_{4F}$, then $G(z) = -F(-z)$ is an extremal function, too. Consequently, without loss of generality one may assume that $A_{2F} > 0$. From inequality (23) we have:

$$A_{4F} A_{2F} \leqq -\frac{7}{12} A_{2F}^4 + \frac{1}{2}\left(4 - \frac{9}{M}\right) A_{2F}^3 + \frac{2}{3}\left(1 - \frac{1}{M^3}\right) A_{2F}, \quad A_{2F} > 0.$$

Examining the right-hand side of the above inequality as a function of $A_{2F} \in [0, 2(1-M^{-1})]$, we obtain the estimation (22) for M satisfying conditions (21). As can easily be verified, the equality in (22) holds for the function $w = P(z; M; 1)$.

Since $(10+\sqrt{46})/3 < 11$, therefore, finally, it is worth observing that the part of Theorem 3 which concerns the interval $((10+\sqrt{46})/3, +\infty)$ gives an estimation of the functional $A_{2F} A_{4F}$, $F \in S_R(M)$, by the Pick function in a wider domain of the parameter M than it formally follows from the results of Pick, Tammi, Jokinen.

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