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Kähler Curvature Identities for Twistor Spaces

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Kähler curvature identities for twistor spaces[1]

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1.INTRODUCTION

In order to generalize some results of Kähler geometry, A.Gray [10] introduced and studied three classes of almost-Hermitian manifolds whose curvature tensor resembles that of Kähler manifolds. They are defined by the following curvature identities:

$\mathcal{AH}_1 : R(E,F,G,H) = R(E,F,JG,JH)$

$\mathcal{AH}_2 : R(E,F,G,H) = R(JE,JF,G,H) + R(JE,F,JG,H) + R(JE,F,G,JH)$

$\mathcal{AH}_3 : R(E,F,G,H) = R(JE,JF,JG,JH)$

(here J is the almost-complex structure).

These identities are very useful in the study of the action of the unitary group on the space of curvature tensors (cf. [16]) as well as for characterizing the Kähler manifolds in various classes of almost-Hermitian manifolds (cf. e.g. [9, 10, 15, 17, 18]). By a result of S.Goldberg [9] (cf. also [10]) every compact almost-Kähler manifold of class $\mathcal{AH}_1$ is Kählerian and it is an open question raised by A.Gray [10, Th.5.3] whether the same is true under the weaker condition $\mathcal{AH}_2$. We answer negatively to this question showing that the twistor space of a compact Einstein and self-dual 4-manifold with negative scalar curvature provides an example of a compact non-Kähler almost-Kähler manifold of class $\mathcal{AH}_2$.

Recall that the twistor space of an orientied Riemannian 4-manifold M is the (2-sphere) bundle $\mathcal{Z}$ on M whose fibre at any point $p \in M$ consists of all complex structures on T_pM compatible with the metric and the opposite orientation of M. The 6-manifold $\mathcal{Z}$ admits a 1-parameter family of Riemannian metrics h_t, $t > 0$, such that the natural projection $\pi : \mathcal{Z} \to M$ is a Riemannian submersion with totally geodesic fibres (cf.e.g. [7, 8, 19]). These metrics are compatible with the almost-complex structures J_1 and J_2 on $\mathcal{Z}$ introduced, respecrively, by Atiyah, Hitchin and Singer [1] and Eells-Salamon [6].

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The purpose of this note is to investigate the twistor spaces as a source of examples of almost-Hermitian manifolds of the classes $\mathcal{AH}_i$. Our main result is the following :

THEOREM. *Let M be a (connected) oriented Riemannian 4-manifold with scalar curvature s. Then:*

(i) $(\mathcal{Z}, h_t, J_n) \in \mathcal{AH}_3$ if and only if $(\mathcal{Z}, h_t, J_n) \in \mathcal{AH}_2$ if and only if M is Einstein and self-dual ($n = 1$ or 2).

(ii) $(\mathcal{Z}, h_t, J_1) \in \mathcal{AH}_1$ if and only if M is Einstein and self-dual with $s = 0$ or $s = 12/t$.

(iii) $(\mathcal{Z}, h_t, J_2) \in \mathcal{AH}_1$ if and only if M is Einstein and self-dual with $s = 0$.

The proof is based on an explicit formula for the sectional curvature of $(\mathcal{Z}, h_t)$ in terms of the curvature of M ([3]).

REMARKS. Let M be an Einstein self-dual manifold with scalar curvature s.

1. If $s < 0$ and $t = -12/s$, then $(\mathcal{Z}, h_t, J_2)$ is an almost-Kähler manifold ([14]) of class $\mathcal{AH}_2$. This manifold is not Kählerian since the almost-complex structure J_2 is never integrable ([6]). So $(\mathcal{Z}, h_t, J_2)$ gives a negative answer to the Gray question.

Note that the only known examples of compact Einstein and self-dual manifolds with negative scalar curvature are compact quotiens of the unit ball in $\mathbb{C}^2$ with the metric of constant negative curvature or the Bergman metric (for a description of the twistor space of the unit ball in $\mathbb{C}^2$ cf. [19]).

2. Let $s = 0$. Then $(\mathcal{Z}, h_t, J_2)$, $t > 0$, is a quasi-Kähler manifold ([14]) of class $\mathcal{AH}_1$ which is not Kählerian. So the Goldberg result cannot be extended to quasi-Kähler manifolds. In the case when $M = \mathbb{R}^4$, the twistor space is $\mathcal{Z} = \mathbb{R}^4 \times S^2$ and we recover an example found by A.Gray [10].

By a result of Vaisman [18] any compact Hermitian surface of class $\mathcal{AH}_1$ is Kählerian. This is not true in higher dimensions since $(\mathcal{Z}, h_t, J_1)$ is a Hermitian manifold ([1]) of class $\mathcal{AH}_1$ which is not Kählerian ([8]).

Note that by a result of Hitchin [12] the only compact Einstein self-dual manifolds with $s = 0$ are the flat 4-tori, the $K3$-surfaces with the Calabi-Yau metric and the quotients of $K3$-surfaces by $\mathbb{Z}_2$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

3. If $s > 0$ and $t = 12/s$, then $(\mathcal{Z}, h_t, J_1)$ not only belongs to the class $\mathcal{AH}_1$ but it is actually a Kähler manifold ([8]). In fact in this case $M = S^4$ or $M = \mathbb{CP}^2$ ([8], [13]) and $\mathcal{Z} = \mathbb{CP}^3$ or $\mathcal{Z} = SU(3)/S(U(1) \times U(1) \times U(1))$ with their standard Kähler structures.

2. PRELIMINARIES

Let M be a (connected) oriented Riemannian 4-manifold with metric g. Then g induces a metric on the bundle $\Lambda^2 TM$ of 2-vectors by the formula

$$g(X_1 \wedge X_2, X_3 \wedge X_4) = 1/2 \det(g(X_i, X_j))$$

The Riemannian connection of M determines a connection on the vector bundle $\Lambda^2 TM$ (both denoted by ∇) and the respective curvatures are related by

$$R(X \wedge Y)(Z \wedge T) = R(X,Y)Z \wedge T + X \wedge R(Y,Z)T$$

for $X, Y, Z, T \in \chi(M)$; $\chi(M)$ stands for the Lie algebra of smooth vector fields on M. (For the curvature tensor R of M we adopt the following definition $R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$). The curvature operator $\mathcal{R}$ is the self-adjoint endomorphism of $\Lambda^2 TM$ defined by

$$g(\mathcal{R}(X \wedge Y), Z \wedge T) = g(R(X,Y)Z, T)$$

for all $X, Y, Z, T \in \chi(M)$. The Hodge star operator defines an endomorphism $*$ of $\Lambda^2 TM$ with $*^2$=Id. Hence

$$\Lambda^2 TM = \Lambda^2_+ TM \oplus \Lambda^2_- TM$$

where $\Lambda^2_\pm TM$ are the subbundles of $\Lambda^2 TM$ corresponding to the (± 1)-eigenvectors of $*$. Let (E_1, E_2, E_3, E_4) be a local oriented orthonormal frame of TM. Set

$$\begin{aligned} s_1 &= E_1 \wedge E_2 - E_3 \wedge E_4 & \bar{s}_1 &= E_1 \wedge E_2 + E_3 \wedge E_4 \\ s_2 &= E_1 \wedge E_3 - E_4 \wedge E_2 & \bar{s}_2 &= E_1 \wedge E_3 + E_4 \wedge E_2 \\ s_3 &= E_1 \wedge E_4 - E_2 \wedge E_3 & \bar{s}_3 &= E_1 \wedge E_4 + E_2 \wedge E_3 \end{aligned} \tag{2.1}$$

Then (s_1, s_2, s_3) (resp. $(\bar{s}_1, \bar{s}_2, \bar{s}_3)$ is a local oriented orthonormal frame of $\Lambda^2_- TM$ (resp. $\Lambda^2_+ TM$). The block-decomposition of $\mathcal{R}$ with respect to the above spliting of $\Lambda^2 TM$ is

$$\mathcal{R} = \begin{bmatrix} s/6.\mathrm{Id} + \mathcal{W}_+ & \mathcal{B} \\ {}^t\mathcal{B} & s/6.\mathrm{Id} + \mathcal{W}_- \end{bmatrix}$$

where s is the scalar curvature of M; $s/6.\mathrm{Id} + \mathcal{B}$ and $\mathcal{W} = \mathcal{W}_+ + \mathcal{W}_-$ represent the Ricci tensor and the Weyl conformal tensor, respectively. The manifold M is said to be self-dual (anti-self-dual) if $\mathcal{W}_- = 0$ ($\mathcal{W}_+ = 0$). It is Einstein exactly when $\mathcal{B} = 0$.

The twistor space of M is the 2-sphere bundle $\pi : \mathcal{Z} \to M$ consisting of all unit vectors of Λ^2_-TM. The Riemannian connection ∇ of M gives rise to a splitting $T\mathcal{Z} = \mathcal{H} \oplus \mathcal{V}$ of the tangent bundle of $\mathcal{Z}$ into horizontal and vertical components. Further we consider the vertical space $\mathcal{V}_\sigma$ at $\sigma \in \mathcal{Z}$ as the orthogonal complement of σ in $\Lambda^2_- T_pM$, $p = \pi(\sigma)$.

Each point $\sigma \in \mathcal{Z}$ defines a complex structure K_σ on T_pM, $p = \pi(\sigma)$, by

$$g(K_\sigma X, Y) = 2g(\sigma, X \wedge Y), \quad X, Y \in T_pM. \tag{2.2}$$

This structure K_σ is compatible with the metric g and the opposite orientation of M at p. The 2-vector 2σ is dual to the fundamental 2-form of K_σ.

Denote by $\times$ the usual vector product in the oriented 3-dimensional vector space $\Lambda^2_- T_pM$, $p \in$M. Then it is checked easily that

$$g(R(a)b, c) = -g(\mathcal{R}(b \times c), a) \tag{2.3}$$

for $a \in \Lambda^2 T_pM$, $b, c \in \Lambda^2_- T_pM$.

Following [1] and [6],define two almost- complex structures J_1 and J_2 on $\mathcal{Z}$ by

$$J_n V = (-1)^n \sigma \times V \text{ for } V \in \mathcal{V}_\sigma$$

$$J_n X^h_\sigma = (K_\sigma X)^h_\sigma \text{ for } X \in T_pM, p = \pi(\sigma).$$

It is well-known ([1]) that J_1 is integrable (i.e. comes from a complex structure) iff M is self-dual. Unlike J_1, the almost-complex structure J_2 is never integrable ([6]).

As in [8], define a Riemannian metric h_t on $\mathcal{Z}$ by

$$h_t = \pi^* g + t g^v$$

where $t > 0$, g is the metric of M and g^v is the restriction of the metric of $\Lambda^2 TM$ on the vertical distribution $\mathcal{V}$. The metric h_t is compatible with the almost-complex structures J_1 and J_2.

3. PROOF OF THE THEOREM

It is easy to see that $\mathcal{AH}_1 \subset \mathcal{AH}_2 \subset \mathcal{AH}_3$. First we prove that if $(\mathcal{Z},, h_t, J_n) \in \mathcal{AH}_3$, then M is Einstein and self-dual .The natural projection $\pi : (\mathcal{Z}, h_t) \to (M, g)$ is a Riemannian submersion with totally geodesic fibres. Applying the O'Neill formulas (cf. e.g. [2]) one can obtain coordinate-free formulas for different curvatures of the twistor space $(\mathcal{Z}, h_t)$ in terms of the curvature of the base manifold M. Denote by R and R_t the Riemannian curvature tensors of (M, g) and its twistor space $(\mathcal{Z}, h_t)$, respectively. If $E, F \in T_\sigma \mathcal{Z}$ and $X = \pi_* E$, $Y = \pi_* F$, $A = \mathcal{V}E$, $B = \mathcal{V}F$ where $\mathcal{V}$ means "vertical component", then ([3])

$$\begin{aligned} R_t(E,F,E,F) = {} & R(X,Y,X,Y) - tg((\nabla_X \mathcal{R})(X \wedge Y), \sigma \times B) \\ & + tg((\nabla_Y \mathcal{R})(X \wedge Y), \sigma \times A) - 3tg(\mathcal{R}(\sigma), X \wedge Y) g(\sigma \times A, B) \\ & - t^2 g(R(\sigma \times A)X, R(\sigma \times B)Y) + t^2/4 \|R(\sigma \times B)X + R(\sigma \times A)Y\|^2 \\ & - 3t/4 \|R(X \wedge Y)\sigma\|^2 + t(\|A\|^2 \|B\|^2 - g(A,B)^2) \end{aligned} \tag{3.1}$$

Let $\sigma \in \mathcal{Z}$, $p = \pi(\sigma)$ and $X, Y \in T_p$M. Since $\mathcal{Z} \in \mathcal{AH}_3$, it follows from (3.1) that

$$\begin{aligned} & R(X,Y,X,Y) - R(K_\sigma X, K_\sigma Y, K_\sigma X, K_\sigma Y) = \\ & = 3t/4(\|R(X \wedge Y)\sigma\|^2 - \|R(K_\sigma X \wedge K_\sigma Y)\sigma\|^2) \end{aligned} \tag{3.2}$$

where K_σ is the complex structure on T_pM determined by σ via (2.2). Fix $\tau \in \mathcal{Z}_p$, $\tau \perp \sigma$ and $E \in T_pM$, $\|E\| = 1$. Since $K_\sigma \circ K_\tau = -K_{\sigma \times \tau}$, $(E_1, E_2, E_3, E_4) = (E, K_\sigma E, K_\tau E, K_{\sigma \times \tau} E)$ is an oriented orthonormal basis of T_pM such that $\sigma = s_1$, $\tau = s_2$ and $\sigma \times \tau = s_3$ where s_1, s_2, s_3 are defined by (2.1). Since $R(X \wedge Y)\sigma$ is a vertical vector at σ, one has by (2.3)

$$\|R(X \wedge Y)\sigma\|^2 = g(\mathcal{R}(\tau), X \wedge Y)^2 + g(\mathcal{R}(\sigma \times \tau), X \wedge Y)^2 \tag{3.3}$$

Denote

$$V_i = X \wedge E_i - K_\sigma X \wedge K_\sigma E_i; \quad \bar{V}_i = X \wedge E_i + K_\sigma X \wedge K_\sigma E_i, \quad i = 1,\dots,4.$$

Then (3.2) and (3.3) give

$$(3.4) \qquad \begin{array}{c} \frac{4}{3t} g(\mathcal{R}(V_i), \bar{V}_i) = g(\mathcal{R}(\tau), V_i) g(\mathcal{R}(\tau), \bar{V}_i) + \\ g(\mathcal{R}(\sigma \times \tau), V_i) g(\mathcal{R}(\sigma \times \tau), \bar{V}_i), \end{array} \quad i = 1,\dots,4$$

If $X = \sum_{i=1}^{4} \lambda_i E_i$, then

$$(3.5) \qquad \begin{array}{ll} V_1 = -\lambda_3 s_2 - \lambda_4 s_3 & \bar{V}_1 = -\lambda_2(\bar{s}_1 + s_1) - \lambda_3 \bar{s}_2 - \lambda_4 \bar{s}_3 \\ V_2 = \lambda_3 s_3 - \lambda_4 s_2 & \bar{V}_2 = \lambda_1(\bar{s}_1 + s_1) - \lambda_3 \bar{s}_3 + \lambda_4 \bar{s}_2 \\ V_3 = \lambda_1 s_2 - \lambda_2 s_3 & \bar{V}_3 = -\lambda_4(\bar{s}_1 - s_1) + \lambda_1 \bar{s}_2 + \lambda_2 \bar{s}_3 \\ V_4 = \lambda_1 s_3 + \lambda_2 s_2 & \bar{V}_4 = \lambda_3(\bar{s}_1 - s_1) - \lambda_2 \bar{s}_2 + \lambda_1 \bar{s}_3 \end{array}$$

Substituting (3.5) into (3.4) and then varying $(\lambda_1, \dots, \lambda_4)$ one sees that the identity (3.4) implies

$$(3.6) \quad \frac{4}{3t} g(\mathcal{R}(\tau), \bar{s}_k) = g(\mathcal{R}(\tau), \tau) g(\mathcal{R}(\tau), \bar{s}_k) + g(\mathcal{R}(\sigma \times \tau), \tau) g(\mathcal{R}(\sigma \times \tau), \bar{s}_k), \; k = 1,2,3.$$

It follows from the curvature identity defining the class $\mathcal{AH}_3$ that the Ricci tensor of $(\mathcal{Z}, h_t)$ is J_n-Hermitian, $n = 1$ or 2. Then, by [4, formula (3.1)] one has

$$(12 - ts(p) + 6tg(\mathcal{W}_-(\sigma), \sigma))\mathcal{B}(\sigma) = 0$$

where s is the scalar curvature of M. This implies that either $\mathcal{B}_p \equiv 0$ or $12 - ts(p) + 6tg(\mathcal{W}_-(\sigma), \sigma) = 0$ for all $\sigma \in \mathcal{Z}_p$. In the second case, $ts(p) = 12$ since Trace $\mathcal{W}_- = 0$. Therefore $(\mathcal{W}_-)_p = 0$. Suppose that $\mathcal{B}_p \neq 0$. Then (3.6) becomes

$$(8 - ts(p)) g(\mathcal{B}(\tau), \bar{s}_k) = 0, \quad k = 1,2,3.$$

Hence $g(\mathcal{B}(\tau), \bar{s}_k) = 0$, $k = 1,2,3$, since $ts(p) = 12$. It follows that $\mathcal{B}_p = 0$, a contradiction. Thus $\mathcal{B} \equiv 0$ and the arguments in [4] show that $\mathcal{W}_- = 0$. In fact, consider $\mathcal{W}_-$ as a self-adjoint endomorphism of $\Lambda^2_- T_p M$, $p \in M$, and denote by μ_1, μ_2, μ_3 its eingenvalues. Since $\mathcal{R}(\sigma) = (s/6)\sigma + \mathcal{W}_-(\sigma)$ for $\sigma \in \Lambda^2_- T_p M$ and $\|\mathcal{R}(\cdot)\| =$ const on every fibre of $\mathcal{Z}$ ([4, formula (3.2)]) we have $|\mu_1 + s/6| = |\mu_2 + s/6| = |\mu_3 + s/6|$. Moreover, $\mu_1 + \mu_2 + \mu_3 =$ trace $\mathcal{W}_- = 0$. Hence either $\mu_1 = \mu_2 = \mu_3 = 0$ or $\{\mu_1, \mu_2, \mu_3\} = \{s/3, s/3, -2s/3\}$. It follows that either $\|\mathcal{W}_-\| \equiv 0$ or $\|\mathcal{W}_-\|^2 \equiv 2s^2/3$. So we have to consider only the case when $\|\mathcal{W}_-\|^2 \equiv 2s^2/3$. Since M is Einstein, $\delta\mathcal{W}_- = 0$(cf. e.g. [2, §16.5]) and Proposition 5.(iii) of [5] gives $\nabla\mathcal{W}_- = 0$. For every oriented Riemannian 4-manifold with $\delta\mathcal{W}_- = 0$, one has [2, §16.73]

$$\Delta\|\mathcal{W}_-\|^2 = -s\|\mathcal{W}_-\|^2 + 18 \det \mathcal{W}_- - 2\|\nabla\mathcal{W}_-\|^2$$

which implies in our case $s = 0$. Hence $\mathcal{W}_- = 0$.

Now let M be Einstein and self-dual. Then $\mathcal{R}|\Lambda^2_- TM = s/6$. Id. Note also that $K_\sigma X \wedge K_\sigma Y - X \wedge Y \in \Lambda^2_- T_pM$ for each $X, Y \in T_pM, p = \pi(\sigma)$. Using (3.1) and the well-known expression of the Riemannian curvature tensor by means of sectional curvatures (cf. e.g.[11]) a direct computation shows that the twistor space $(\mathcal{Z}, h_t, J_n)$ is of class $\mathcal{AH}_2$.

Thus the statement (i) is proved.

To prove (ii) and (iii) assume first that $(\mathcal{Z}, h_t, J_n) \in \mathcal{AH}_1$. Since $\mathcal{AH}_1 \subset \mathcal{AH}_3$ it follows from (i) that the base manifold M is Einstein and self-dual. Using (3.1) one sees that the Kähler curvature identity $\mathcal{AH}_1$ holds for the horizontal vectors of $\mathcal{Z}$ iff

$$g(X \wedge Y, \mathcal{R}(Z \wedge T - K_\sigma Z \wedge K_\sigma T)) - 8t(s/24)^2 g(X \wedge Y, Z \wedge T - K_\sigma Z \wedge K_\sigma T) = 0$$

for every $\sigma \in \mathcal{Z}$ and $X, Y, Z, T \in T_pM$, $p = \pi(\sigma)$. Since $Z \wedge T - K_\sigma Z \wedge K_\sigma T \in \Lambda^2_- T_pM$ and $\mathcal{R}|\Lambda^2_- T_pM = s/6$ Id, the above identity implies that either $s = 0$ or $st = 12$.

Now suppose M is Einstein and self-dual. Then a direct computation involving (3.1) shows that: if $s = 0$, both almost-complex structures J_1 and J_2 satisfy the Kähler curvature identity $\mathcal{AH}_1$; if $st = 12$, this identity is satisfied only by J_1.

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