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Compacta with Dense Ambiguous Loci of Metric Projections and Antiprojections

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Compacta with Dense Ambiguous Loci of Metric Projections and Antiprojectons*

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Abstract

In every strictly convexifiable Banach space X with $\dim X \geq 2$ there exists a dense G_δ set of compacta $\mathcal{A}$ in the Hausdorff set topology such that with respect to an arbitrary equivalent strictly convex norm in X both the metric projection and the metric antiprojecton genereted by any member of $\mathcal{A}$ are densely multi-valued.

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1 Introduction

Consider a Banach space X with norm $\|\cdot\|$. Designate by 2^X the family of all non-void sets in X. Let $M \in 2^X$. The multi-valued maping $P : X \longrightarrow M$ defined by

$$P(x, M) = \{y \in M : \|x - y\| = d(x, M)\}$$

with distance function $d(x, M) = \inf\{\|x - z\| : z \in M\}$ is called metric projection (the nearest point mapping) generated by M with respect to the norm $\|\cdot\|$, and the multi-valued mapping $Q : X \longrightarrow M$ defined by

$$Q(x, M) = \{y \in M : \|x - y\| = f(x, M)\},$$

with farthest distance function $f(x, M) = \sup\{\|x - z\| : z \in M\}$ is a metric antiprojection (the farthest point mapping) generated by M and $\|\cdot\|$.

Let $\mathcal{K}(X)$ and $\mathcal{F}(X)$ denote the families of non-empty compacta and non-empty finite subsets of X respectively.

A lot of papers have been devoted to the investigation of generic properties of metric projections and antiprojections. Far from being complete we mention works of [St], [Ko], [L2], [BF] concerning projections and [As], [L1] dealing with antiprojections. In all these works a certain "good" property of the multi-valued projection (resp. antiprojection) such as existence of a best approximation (resp. existence of a farthest element), uniqueness of the solution, or well-posedness is shown to be fulfilled for the points from a residual set, i.e. a dense and G_δ subset of the space. It is of a natural interest then to ask what happens with the sets of "bad" points, i.e. the sets whose elements fail to have unique solution for the metric projection (respectively the metric antiprojection). Following [Lu] we define ambiguous locus of a projection (resp. antiprojection) as the set of points of multi-valuedness of this projection (resp. antiprojecton). In [Za] Zamfirescu showed that in a finite dimensional Euclidean space most compacta, in the Hausdorff metric space of compacta, generate metric projections which are densely multi-valued. Recently in a series of papers [BM1], [BM2] and [BM3] De Blasi and Myjak extended this result of Zamfirescu and showed variuos analogous theorems for different Hausdorff spaces of sets in separable strictly convex Banach spaces. The aim of this work is to give a proof of the following

Theorem. In a strictly convexifiable Banach space X with $\dim X \geq 2$ there exists a dense G_δ set of compacta $\mathcal{A}$ in the Hausdorff set topology such that with respect to an arbitrary equivalent strictly convex norm in X both the metric projection and the metric antiprojection generated by any member of $\mathcal{A}$ are densely multi-valued. Moreover, the ambiguous loci of these projections and antiprojections are everywhere continual, i.e. loci's intersections with open sets in X contain continuum elements.

This result is an extention of a theorem of De Blasi and Myjak from [BM1] (see also [BM3]). It shows that the separability assumption can be dropped and that the set of compacta $\mathcal{A}$ is in some sense universal, i.e. it plays the same role with respect to any equivalent strictly convex norm in the space. The main construction is motivated by the construction of an example from [Zh].

2 Preliminaries

Suppose $(X, \|\cdot\|)$ is a strictly convex Banach space with dimension $\dim X \geq 2$, and $Y \subseteq X$ is a closed subspace with $\text{codim} Y = 2$. Suppose $\|\cdot\|_N$ is an equivalent norm in Y, different and not necessarily strictly convex, to the induced norm $\|\cdot\|$ in X. Consider the Banach space $X' := \mathbb{R}^2 \times Y = \{x = (r, s, y) : r, s \in \mathbb{R}, y \in Y\}$ with norm $|\cdot|_N$ defined by

$$|x|_N^2 = r^2 + s^2 + \|y\|_N^2, \quad \text{for } x \in X'.$$

Since X and X' are isomorphic, it might be viewed that an equivalent norm $\|\cdot\|$ is defined in X' and from now on all the considerations will be made in X' which will be denoted as X for the sake of simplicity.

The equivalence of the two norms implies existence of positive constants γ_1 and γ_2 such that

$$\gamma_1 |x|_N \leq \|x\| \leq \gamma_2 |x|_N \quad \text{whenever} \quad x \in X \tag{1}$$

The usual Euclidean norm in $\mathbb{R}^2$ is denoted by $|\cdot|$. We make a stipulation to identify $\mathbb{R}^2$ with $\mathbb{R}^2 \times \{\theta\}$ with θ the origin of Y. The following simple observation proves the usefullness of the norm $|\cdot|_N$: The distances between elements in any two-dimensional plane which is parallel to $\mathbb{R}^2 \times \{\theta\}$ are measured by $|\cdot|$.

In order to avoid the ambiguity in notation of functions and multi-valued mappings we use the symbol of the corresponding norm. For instance, $d(x, M; |\cdot|_N)$ is the distance from the element x to the set $M \in 2^X$ with respect to the norm $|\cdot|_N$, while $f(\cdot, M; \|\cdot\|)$ is the farthest distance function generated by M with respect to the norm $\|\cdot\|$. Similarly, $B(x, \varepsilon; |\cdot|_N)$ is the open ball with center x and radius ε with respect to $|\cdot|_N$, and $B[x, \varepsilon; \|\cdot\|]$

stands for the closed ball with the same center and radius with respect to $\|\cdot\|$. The symbol of the norm will be dropped whenever the norm is explicitly indicated.

The Hausdorff distance for elements of 2^X with respect to $\|\cdot\|$ is defined by

$$H(M_1, M_2; \|\cdot\|) := \max\{\sup\{d(x, M_1; \|\cdot\|) : x \in M_2\},\\ \sup\{d(x, M_2; \|\cdot\|) : x \in M_1\}\}.$$

Observe that $\mathcal{K}(X)$ equipped with a Hausdorff distance is a complete metric space and $\mathcal{F}(X)$ is dense in it. Besides, the different Hausdorff distances generated by equivalent norms in X do generate one and the same set topology which is designated by $\mathcal{H}$. For balls in a Hausdorff metric space the symbol $\mathcal{O}$ is used. Let $K \in \mathcal{K}(X)$. It follows immediately from (1):

$$\mathcal{O}(K, \gamma_1\varepsilon; \|\cdot\|) \subset \mathcal{O}(K, \varepsilon; |\cdot|_N) \subset \mathcal{O}(K, \gamma_2\varepsilon; \|\cdot\|).$$

A subset $M \in 2^X$ is called *completely disconnected* whenever M contains at least two different elements and for every x and y from X, $x \neq y$, there is no continuous curve in M with end-points x and y, i.e. there is no continuous map $\phi : [0,1] \longrightarrow M$ such that $\phi(0) = x$ and $\phi(1) = y$.

A subset $M \in 2^X$ is called ε – *disconnected* (with respect to $|\cdot|_N$) whenever M contains at least two distant elements at greater than ε distance and for every x and y in M with $|x - y|_N > \varepsilon$ there is no continuous curve in M with end points x and y.

Obviously, $M \in 2^X$ is completely disconnected, if and only if M is ε-disconnected for every positive ε. Also, the notion of complete disconnectedness is purely topological.

For the elements of $\mathcal{F}(X)$ define a separator function (with respect to $|\cdot|_N$):

$$\text{sep}(F) = \min\{1, \{|x - y|_N : x, y \in F, x \neq y\}, \quad F \in \mathcal{F}(X).$$

The value of $\text{sep}(F)$ is 1 whenever F is a singleton.

Lemma 1. Let K_1 and K_2 be two disjoint and non-void compacta in the Banach space $(X, \|\cdot\|)$ and $x \in X$ is an element such that $d(x, K_1) = d(x, K_2)$ (respectively $f(x, K_1) = f(x, K_2)$). Then arbitrary neighbourhood of x contains continualy many elements with the same property.

Proof: Projection part: Suppose $y_i \in P(x, K_i)$, $i = 1,2$. Denote $y_i(t) = (1-t)x + ty_i$, $i = 1,2$ for $t \in (0,1]$. Obviously, $y_i \in P(y_i(t), K_i)$, $i = 1,2$. Make use of the continuity principle applied to the function $\phi_t(u) = d(u, K_1) -$

$d(u, K_2)$ with $u \in [y_1(t), y_2(t)]$ and $\phi(y_1(t)) \le 0, \phi(y_2(t)) \ge 0$. There is $y(t) \in [y_1(t), y_2(t)]$ such that $d(y(t), K_1) = d(y(t), K_2)$. For different t the values of $y(t)$ are different and form a set with the power of continuum.

Antiprojection part: Suppose $y_i \in Q(x, K_i)$, $i = 1,2$. and define $y_i(t) = (1-t)x + ty_i, i = 1,2$ for $t < 0$. Then $y_i \in Q(y_i(t), K_i), i = 1,2$ and the continuity principle applied to the function $\psi_t(u) = f(u, K_1) - f(u, K_2)$ with $u \in [y_1(t), y_2(t)]$ and $\psi(y_1(t)) \ge 0, \psi(y_2(t)) \le 0$, completes the proof.

Lemma 2. Let $(X, \|\cdot\|)$ be a strictly convex Banach space, and $x_o, y_o \in X$ be such that $\|x_o - y_o\| = d > 0$. Suppose $L = \{y = y_o + te : t \in \mathbb{R}\}$ with $\|e\| = 1$ is a line such that $\{y_o\} = L \cap B[x_o, d]$, and there is a sequence (y_m) satisfying

(i) $\lim y_m = y_o$,

(ii) $\lim d(y_m, L)/\|y_m - y_o\| = 0$.

Then there exists a sequence (w_m), $\lim w_m = x_o$ such that $\|y_o - w_m\| = \|y_m - w_m\|$.

Proof. Denote $e_m = (y_m - y_o)/\|y_m - y_o\|$. According to (ii) the sequence (e_m) has at most two cluster points e and $-e$. With no change of indexation, assume $\lim(y_o + e_m) = y_o + e$. The case $\lim(y_o + e_m) = y_o - e$ is treated analogously.

Define the lines $L_m(y_o) = \{y = y_o + \lambda e_m : \lambda \in \mathbb{R}\}$ and $L_m(x_o) = \{x = x_o + \lambda e_m : \lambda \in \mathbb{R}\}$ and the sequences (u_m) and (v_m) such that $\{u_m\} = P(y_o, L_m(x_o); \|\cdot\|)$, $\{v_m\} = P(y_m, L_m(x_o); \|\cdot\|)$. Moreover, $\{y_o\} = P(u_m, L_m(y_o); \|\cdot\|), \{y_m\} = P(v_m, L_m(y_o); \|\cdot\|)$ and

$$\|y_o - y_m\| = \|u_m - v_m\|. \tag{2}$$

It follows that $\|y_o - u_m\| < \|y_m - u_m\|$ and $\|y_m - v_m\| < \|y_o - v_m\|$. Apply the continuity principle to the function $\phi_m(u) = \|y_o - w\| - \|y_m - w\|, w \in [u_m, v_m]$ in order to show the existence of $w_m \in (u_m, v_m)$ such that

$$\|y_o - w_m\| = \|y_m - w_m\|. \tag{3}$$

It is to be proved that $\lim u_m = x_o$. Indeed, $u_m = x_o + \lambda_m e_m$, $\lambda_m \in \mathbb{R}$. Since $\|u_m - y_o\| < \|x_o - y_o\|$, then $\|u_m - x_o\| = |\lambda_m| \le 2d$, i.e. the sequence of reals (λ_m) is bounded. Let λ_o be a cluster point. Then $u_o = x_o + \lambda_o e$ is a cluster point of (u_m). If we assume that $u_o \ne x_o$, then from the strict convexity of $\|\cdot\|$ and $P(x_o, L; \|\cdot\|) = \{y_o\}$ it follows that $\|x_o - y_o\| < \|u_o - y_o\|$. There exists then $\delta > 0$, $\delta < \|u_o - x_o\|$, such that $\|x_o - y_o\| < \|u - y_o\|$ whenever $u \in B(u_o, \delta; \|\cdot\|)$. The last inequality contradicts to the choice of

(u_m), since for infinitely many values of m $\|x_o - y_o\| < d(y_o, L_m(x_o); \|\cdot\|)$ and $u_m \neq x_o$. Hence $\lambda_o = 0$ and x_o is the only cluster point of (u_m). According to (2) and (i) it follows that $\lim v_m = \lim w_m = x_o$.

3 Proof of Main Result

It suffices to prove the statement of the theorem with the norm $\|\cdot\|$. The proof is partitioned on six steps.

The first step is a definition of a set $\mathcal{A}$: Designate by V_n the set of vertices of a regular 2^n-gon inscribed in the unit circumference in $\mathbb{R}^2$, i.e.

$$V_n = \{(\cos(i\pi/2^{n-1}), \sin(i\pi/2^{n-1}), \theta) : i = 0, 1, \ldots, 2^n - 1; \theta \in Y\}, n \geq 3.$$

It is a routine matter to verify that

$$\operatorname{sep}(V_n) = 2\sin(\pi/2^n). \tag{4}$$

Assign $\mathcal{U}_n := \bigcup\{\mathcal{O}(F + n^{-1}\operatorname{sep}(F)V_n, n^{-1}\operatorname{sep}(F)\sigma_n; |\cdot|_N) : F \in \mathcal{F}(X)\}$, with $\sigma_n = \sin^2(\pi/2^n)$, $n \geq 3$, and define

$$\mathcal{A} := \bigcap_{n=3}^{\infty} \mathcal{U}_n$$

.

Obviously $\mathcal{U}_n$ are open sets as unions of open balls. On the other hand all elements from the type $F + n^{-1}\operatorname{sep}(F)$ for $F \in \mathcal{F}(X)$ and $n \geq 3$ form a dense subset of $\mathcal{F}(X)$ in the $\mathcal{H}$-topology, and since $\mathcal{F}(X)$ is dense in $\mathcal{K}(X)$ then $\mathcal{U}_n$ are open and dense. Hence $\mathcal{A}$ is a dense G_δ subset of $(\mathcal{K}(X), \mathcal{H})$.

The second step of the proof is to show that any element A of $\mathcal{A}$ is a completely disconnected set. For that reason let $A \in \mathcal{A}$ and fix $n \geq 3$. There is a finite set F with k different elements, $k \geq 1$, such that

$$H(A, F + n^{-1}\operatorname{sep}(F)V_n; |\cdot|_N) < n^{-1}\operatorname{sep}(F)\sigma_n \tag{5}$$

Having in mind (4) we get

$$\operatorname{sep}(F + n^{-1}\operatorname{sep}(F)V_n) = 2n^{-1}\operatorname{sep}(F)\sqrt{\sigma_n} \tag{6}$$

It is seen from (5) and (6) that A might be viewed as union of $k2^n$ disjoint compacta A_i,

$$A = \bigcup_{i=1}^{k2^n} A_i \tag{7}$$

such that for $y_1 \in A_i$ and $y_2 \in A_j$

$$|y_1 - y_2|_N > 2n^{-1}\text{sep}(F)(\sqrt{\sigma_n} - \sigma_n), \quad \text{iff} \quad i \neq j \tag{8}$$

which implies that A is an ε_n-disconnected set, for $\varepsilon_n = 2n^{-1}(\sqrt{\sigma_n} - \sigma_n)$. Now let n go to infinity.

For the third step a "tangent" property of the elements of $\mathcal{A}$ is needed.

Lemma 3. Suppose $A \in \mathcal{A}$ and $y_o \in A$. There exist a line $L = \{y = y_o + \lambda e : \lambda \in \mathbb{R}\}$, $\|e\| = 1$ and a sequence (y_m) in A such that

(i) $\lim y_m = y_o$,

(ii) $\lim d(y_m, L; \|\cdot\|)/\|y_m - y_o\| = 0$,

(iii) the sequence $(y_o + (y_m - y_o)/\|y_m - y_o\|)$ has two cluster points $y_o + e$ and $y_o - e$.

Proof. For every $n \geq 3$ there is $F \in \mathcal{F}(X)$ satisfying (5). Set for convenience $\tau_n = n^{-1}\text{sep}(F)$. There exist z and v, $z \in F$ and $v \in z + \tau_n V_n$, such that $|y_o - v|_N < \tau_n\sigma_n$. Denote by $v_1(n)$ and $v_2(n)$ the two neighbouring (and nearest with respect to $|\cdot|_N$) to v elements from $z + \tau_n V_n$. It follows from (6) that

$$|v - v_i(n)|_N = 2\tau_n\sqrt{\sigma_n}, \quad i = 1,2 \tag{9}$$

Designate by l the line in the plane $z + \mathbb{R}^2$, passing through v and parallel to the segment $[v_1(n), v_2(n)]$. After elementary calculations we get

$$d(v, [v_1(n), v_2(n)]; |\cdot|_N) = \tau_n(1 - \cos(\pi/2^{n-1})) =$$

$$d(v_i(n), l; |\cdot|_N), \quad i = 1,2. \tag{10}$$

Choose in accordance to (5) $y_{2n-1} \in A \cap B(v_1(n), \tau_n\sigma_n; |\cdot|_N)$ and $y_{2n} \in A \cap B(v_2(n), \tau_n\sigma_n; |\cdot|_N)$. Assign $e_n := (v_2(n) - v_1(n))/\|v_2(n) - v_1(n)\|$ and $L_n := \{y = y_o + \lambda e_n : \lambda \in \mathbb{R}\}$. Since all the members of the sequence (e_n) belong to $\mathbb{R}^2$ then it has a cluster point e, $\|e\| = 1$, which is a limit of a subsequence (e_{n_k}). Denote $L = \{y = y_o + \lambda e : \lambda \in \mathbb{R}\}$ and $\eta_n = \|e_n - e\|$.

The triangle inequality and (9) imply

$$2\tau_n(\sqrt{\sigma_n} - \sigma_n) \leq |y_{2n-j} - y_o|_N \leq 2\tau_n(\sqrt{\sigma_n} + \sigma_n), \quad j = 0,1. \tag{11}$$

Moreover

$$|y_{2n} - y_{2n-1}|_N \geq |v_2(n) - v_1(n)|_N - 2\tau_n\sigma_n =$$

$$2\tau_n\sqrt{\sigma_n}(2\cos(\pi/2^n) - \sqrt{\sigma_n}). \tag{12}$$

It follows from (1), (5), (10) and (11) that

$$\frac{d(y_{2n-j}, L_n; \|\cdot\|)}{\|y_{2n-j} - y_o\|} \le \frac{\gamma_2}{\gamma_1} \cdot \frac{d(y_{2n-j}, L_n; |\cdot|_N)}{|y_{2n-j} - y_o|_N} \le$$

$$\frac{\gamma_2}{\gamma_1} \cdot \frac{1 - \cos(\pi/2^{n-1}) + 2\sigma_n}{2(\sqrt{\sigma_n} - \sigma_n)} = \frac{\gamma_2}{\gamma_1} \cdot \frac{2\sqrt{\sigma_n}}{1 - \sqrt{\sigma_n}}, \; j = 0,1 \qquad (13)$$

Let now $y \in X, \quad y \neq y_o$. In order to evaluate the distance from y to the line L by making use of the distances from y to L_n, pick $w_n \in P(y, L_n; \|\cdot\|)$ and consider the inequalities:

$$d(y, L; \|\cdot\|) \le \|y - w_n\| + d(w_n, L; \|\cdot\|) \le \|y - w_n\| + \|w_n - y_o\| . \|e_n - e\| \le$$
$$\le (1 + \eta_n) d(y, L_n; \|\cdot\|) + \eta_n \|y - y_o\| \qquad (14)$$

Redefine a sequence (y_m) with abuse of notation:

$$y_m = \begin{cases} y_{2n_k - 1} & \text{for } m = 2k - 1 \\ y_{2n_k} & \text{for } m = 2k \end{cases}$$

herein k is the integer part of $(m + 1)/2$.

The statement (i) follows from (1) and (11):

$$\|y_m - y_o\| \le \gamma_2 |y_m - y_o|_N \le 2\gamma_2 \tau_{n_k} (\sqrt{\sigma_{n_k}} + \sigma_{n_k}) < 2\gamma_2 n_k^{-1}$$

In order to get (ii) compare (13) with (14)

$$d(y_m, L; \|\cdot\|)/\|y_m - y_o\| \le (1 + \eta_{n_k}) d(y_m, L; \|\cdot\|)/\|y_m - y_o\| + \eta_{n_k} \le$$
$$2\gamma_2 \gamma_1^{-1} (1 + \eta_{n_k}) \sqrt{\sigma_{n_k}} / (1 - \sqrt{\sigma_{n_k}}) + \eta_{n_k}$$

The expression from the right side of the last inequality tends to 0 when m increases unboundedly.

It remains to be proved (iii). Set $u_m = y_o + (y_m - y_o)/\|y_m - y_o\|$. The statement (ii) is equivalent with $\lim d(u_m, L; \|\cdot\|) = 0$. The local compactness of L entails existence of a cluster point $u = y_o + \lambda e$ for (u_m). It follows from the continuity of $\|\cdot\|$ that $|\lambda| = 1$, i.e. either $u = y_o + e$ or $u = y_o - e$. Now, it is to be shown that (u_m) is not convergent whence both $y_o + e$ and $y_o - e$ are cluster points. To reach this goal it suffices to prove that (u'_m), with $u'_m = y_o + (y_m - y_o)/\nu_m$, and $\nu_m = |y_m - y_o|_N$, is not a convergent seguence:

$$|u'_{2k} - u'_{2k-1}|_N = |\nu_{2k}^{-1}(y_{2k} - y_{2k-1}) + (\nu_{2k}^{-1} - \nu_{2k-1}^{-1})(y_{2k-1} - y_o)|_N \ge$$

after applying (11) and (12)

$$\nu_{2k}^{-1}(|y_{2k}-y_{2k-1}|_N - |\nu_{2k}-\nu_{2k-1}|) \geq \nu_{2k}^{-1}(|y_{2k}-y_{2k-1}|_N - 4\tau_{n_k}\sigma_{n_k}) \geq$$

$$\geq (2\cos(\pi/2^{n_k}) - 3\sqrt{\sigma_{n_k}})/(1+\sqrt{\sigma_{n_k}})$$

The last expression tends to 2 with k going to infinity. The proof of Lemma 3 is completed.

Continue the proof of the theorem. The fourth step is to show that for given $A \in \mathcal{A}$ there is a dense subset of X whose elements have at least two best approximations for the projection mapping $P(\cdot, A; \|\cdot\|)$. Assume the contrary. There exist $x_o \in X$ and $\varepsilon > 0$ such that $P(\cdot, A; \|\cdot\|)$ is single-valued in the ball $B(x_o, \varepsilon; \|\cdot\|)$. Therefore all elements of this ball are projected onto a single point $y_o \in A$. Indeed, if $x_1, x_2 \in B(x_o, \varepsilon; \|\cdot\|), x_1 \neq x_2$, and $\{y_i\} = P(x_i, A; \|\cdot\|), i = 1,2$, then the upper semi-contunuity of $P(\cdot, A; \|\cdot\|)$ [Si] and its single-valuedness imply the existence of a continuous curve $\kappa := \{y \in A : \{y\} = P(tx_2 + (1-t)x_1, A; \|\cdot\|), 0 \leq t \leq 1\}$ $\kappa(0) = y_1, \kappa(1) = y_2$ connecting y_1 and y_2 whence $y_1 = y_2$ as A is completely disconnected.

Assume now, with no loss of generality, that $\|x_o - y_o\| = d > 0$ since $\text{int}A = \oslash$ and let L be the line through y_o satisfying (i), (ii) and (iii) from Lemma 3 for a sequence (y_m). In order to apply Lemma 2 we need only prove that L is a supporting line for $B[x_o, d; \|\cdot\|]$. Suppose this is not true, i.e. there exists $x \in L \cap B(x_o, d; \|\cdot\|)$. Then for some $\delta > 0$ $\text{co}(B(x, \delta; \|\cdot\|) \cup \{y_o\}) \subset B(x_o, d; \|\cdot\|) \cup \{y_o\}$ (here co stands for the convex hull of a set). It follows from (iii) that $B(x_o, d; \|\cdot\|)$ contains elements from A and y_o is not a best approximation which is a contradiction. Hence $\{y_o\} = L \cap B[x_o, d; \|\cdot\|]$.

To conclude the fourth step of the proof apply Lemma 2. There is a sequence (w_m) such that $\lim w_m = x_o$ and (3) holds: $\|y_o - w_m\| = \|y_m - w_m\|$. Therefore, for large m w_m is in the ε-neighbourhood of x_o and $P(\cdot, A; \|\cdot\|)$ is multi-valued at w_m since $y_o, y_m \in A$ and $y_o \neq y_m$. Thus, the assumption for single-valuedness of P in $B(x_o, \varepsilon; \|\cdot\|)$ leads to a contradiction.

The next fifth step is to show that the ambiguous locus of a projection generated by arbitrary $A \in \mathcal{A}$ is everywhere uncountable (in fact everywhere continual). Let $x_o \in X$ and $y_1, y_2 \in P(x_o, A; \|\cdot\|)$, $\|y_1 - y_2\| = r > 0$. Choose n sufficiently large so that $n^{-1}(\sqrt{\sigma_n} - \sigma_n) < r/2\gamma_2$. There is $F \in \mathcal{F}(X)$ such that (5), (7) and (8) hold. Since $2n^{-1}\text{sep}(F)(\sqrt{\sigma_n} - \sigma_n) < r\gamma_2^{-1} \leq |y_1 - y_2|_N$ then by (8) there are i and j, $i \neq j$, such that $y_1 \in A_i$ and $y_2 \in A_j$. The set

A might be represented as union of two disjoint compacta K_1 and K_2. It remains to apply Lemma 1.

The final part of the proof concerns metric antiprojections. We proceed in analogous way. The assumption that there is $A \in \mathcal{A}$ such that $Q(\cdot, A; \|\cdot\|)$ is single-valued in some open set leads to the conclusion that Q maps all the elements from this open set onto a singleton, because an antiprojection generated by a compact is upper semi-continuous [Bl], and A is comletely disconnected. Further, apply consequitively Lemmata 3, 2 and 1 (the antiprojection part). The proof of the theorem is completed.

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