ИНСТИТУТ ПО МАТЕМАТИКА С ИЗЧИСЛИТЕЛЕН ЦЕНТЪР
INSTITUTE OF MATHEMATICS WITH COMPUTER CENTER

Peano Continua Generating Densely Multivalued Metric Projections

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БЪЛГАРСКА АКАДЕМИЯ НА НАУКИТЕ

1869

Preprint
July 1993
No 4

BULGARIAN ACADEMY OF SCIENCES

Department of Operations Research

Peano Continua Generating Densely Multivalued Metric Projections*

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Abstract

In every strictly convexifiable Banach space X with $\dim X \geq 2$ a Peano continuum $\tilde{K}$ exists such that with respect to an arbitrary strictly convex norm the metric projection generated by $\tilde{K}$ is multivalued on a dense subset of X. Moreover, there exists a starshaped Peano continuum $\tilde{S}$ in X with this property provided $\dim X \geq 3$.

AMS (1991) - Mathematics Subject Classification: 41A65, 54F15, 52A30, 54E52.

Key words and phrases: metric projection, Peano continuum, starshaped set, dense G_δ set.

*Research partially supported by the National Foundation for Scientific Research at the Bulgarian Ministry of Education, Science and Culture under contract No MM58/91.

1 Introduction

Suppose $(X, \|\cdot\|)$ is a strictly convex Banach space. A non-empty set $M \subset X$ generates a distance function $d(x, M) = \inf\{\|x - z\| : z \in M\}$ and a set-valued mapping $P(x, M) = \{y \in M : \|x - y\| = d(x, M)\}$ called metric projection or nearest point mapping. It is known that $P(\cdot, M)$ is single-valued at most points, i.e. on a dense subset of G_δ type, provided that M is a compact [St]. Zamfirescu [Za2] showed that $P(\cdot, M)$ might be multi-valued on a dense set, as well, and that this is the typical case in the Euclidean space $\mathbb{R}^n$ when $n \geq 2$: In the Hausdorff metric space of compacta $\mathcal{K}(\mathbb{R}^n)$ most compacta (in sense of Baire category) generate metric projections which are multi-valued on everywhere continual sets, and according to the proof of Stechkin's result they are single-valued on everywhere continual sets too. Recall that a set is everywhere continual in X if its intersection with any non-empty open subset of X contains continuum many elements.

Recently, De Blasi and Myjak extended Zamfirescu's result to the case of separable strictly convex Banach space X, $\dim X \geq 2$, and showed various analogous theorems for different Hausdorff complete metric spaces of non-empty sets, [BM1], [BM2] and [BKM] (jointly with Kenderov), namely

- $\mathcal{B}(X)$ the space of bounded and closed subsets of X,
- $\mathcal{K}(X)$ the space of compacta,
- $\mathcal{C}(X)$ the space of continua, i.e. connected compacta,
- $\mathcal{S}(X)$ the space of starshaped continua.

A subsequent extension of a result of that type concerning $\mathcal{K}(X)$ for arbitrary strictly convex Banach space X is obtained in [Zh2].

We shall prove here that in every strictly convexifiable Banach space X with $\dim X \geq 2$ a locally connected continuum $\tilde{K}$ exists such that with respect to any strictly convex norm defined on X the metric projection generated by $\tilde{K}$ is multivalued on an everywhere continual subset of X. It is shown that the constructed Peano continuum is a non-rectifiable curve of Hausdorff dimension 1. Moreover, if $\dim X \geq 3$ then there exists a starshaped locally connected continuum $\tilde{S}$ of Hausdorff dimension 2 with the same property while in a two-dimensional strictly convex space there is no starshaped Peano continuum with a densely multi-valued metric projection.

A paper of Gruber and Zamfirescu [GZ] is our motivation to consider fractal dimensions of continua. A theorem in [GZ] states that most elements of $\mathcal{S}(\mathbb{R}^n)$ have Hausdorff dimension 1.

It should be noted that according to [Ma] and [Bi] the Peano continua form a first Baire category set in $\mathcal{C}(X)$, and according to [Za1] the same is true in $\mathcal{S}(X)$, thus sets like $\tilde{K}$ and $\tilde{S}$ are not typical.

The paper consists of four sections including the present one. At the end of section 2 a general result about densely multi-valued metric projections is established. In section 3 the basic planar construction which modifies an example from [Zh1] is given. Section 4 contains the main result.

2 Preliminaries

Let $(X, \|\cdot\|)$ be a strictly convex Banach space with dimension $\dim X \geq 2$. For $x \in X$ and $\varepsilon > 0$, the open (respectively closed) ball with center x and radius ε is denoted by $B(x, \varepsilon)$ (resp. $B[x, \varepsilon]$). A line segment with end-points $x, y \in X$ is denoted by $[x, y]$ and (x, y) means $[x, y] \setminus \{x, y\}$. The symbols $l(x, e)$ and $L(x, e_1, e_2)$ will stand for a line $\{z = x + te : t \in \mathbb{R}\}$, $\|e\| > 0$, and a plane $\{z = x + te_1 + se_2 : t, s \in \mathbb{R}\}$, $\|e_1\|, \|e_2\| > 0$ respectively. For a subset M of X we denote by $\mathrm{int} M$, $\mathrm{bd} M$, $\mathrm{co} M$ and $\mathrm{diam} M$ its interior, boundary, convex hull and diameter respectively. If M is starshaped then $\ker M$ is its kernel. The Hausdorff distance between bounded and non-empty sets is designated by H.

Connected compacta are called continua and metrizable locally connected continua are referred to as Peano continua [Ku]. As a consequence of the classical theorem of Hahn-Mazurkiewicz-Sierpinski [Ku] a continuum is a Peano continuum if and only if for every $\varepsilon > 0$ it is partitioned as a finite union of subcontinuua with smaller than ε diameters, and if and only if it is a continuous image of the interval $I = [0, 1]$. Hence the Peano continua from $\mathcal{C}(X)$ might be viewed as continuous curves taking values in X.

For $s > 0$, $\delta > 0$ and $M \subset X$, let

$$\mathcal{H}^s_\delta(M) = \inf\{\sum_{i=1}^{\infty} (\mathrm{diam} U_i)^s : \bigcup U_i \supset M, \mathrm{diam} U_i \leq \delta\}$$

To get the Hausdorff s-dimensional outer measure of M let $\delta \to 0$:

$$\mathcal{H}^s(M) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(M) = \sup_{\delta > 0} \mathcal{H}^s_\delta(M)$$

There is a unique value, $\dim M$, called Hausdorff dimension of M (c.f. [Fa]), such that

$$\mathcal{H}^s(M) = \infty \text{ if } 0 \leq s < \dim M, \quad \mathcal{H}^s(M) = 0 \text{ if } \dim M < s < \infty$$

provided M is a subset of a finite dimensional space. We denote the usual dimension of a linear space and the Hausdorff dimension of a set by one and the same symbol.

Definition 2.1 Let $y_0 \in M \subset X$. A line $l = l(y_0, e)$, $\|e\| = 1$, is called a (two-sided) pseudo-tangent to M at y_0 if there exists a sequence $(y_n) \subset M$, $y_n \neq y_0$ for all n, such that

(i) $\lim y_n = y_0$,

(ii) $\lim d(y_n, l)/\|y_n - y_0\| = 0$,

(iii) the sequence $(y_n - y_0/\|y_n - y_0\|)$ has two cluster points e and $-e$.

It has to be noted that the notion of a pseudo-tangent is invariant under equivalent renormings of X.

For a set $M \subset X$, denote by $\mathcal{N}_M^0$ (resp. $\mathcal{N}_M^1$) the family of all continuous images of I in M which are not points (resp. which are not line segments).

Definition 2.2 Let N be a proper subset of a compact M. N is said to be a $\mathcal{N}^0$-separator (resp. $\mathcal{N}^1$-separator) of M provided that N meets each element of $\mathcal{N}_M^0$ (resp. $\mathcal{N}_M^1$).

In order to prove a general result about the sets of multi-valuedness of metric projections we need two technical lemmas:

Lemma 2.3([Zh2]) Let $x_0, y_0 \in X$, $\|x_0 - y_0\| = d > 0$ and $l = l(y_0, e)$ with $\|e\| = 1$ be a line such that $\{y_0\} = l \cap B[x_0, d]$, and (y_n) be a sequence satisfying

(i) $\lim y_n = y_0$,

(ii) $\lim d(y_n, l)/\|y_n - y_0\| = 0$.

Then there exists a sequence (x_n) such that $\lim x_n = x_0$ and $\|y_0 - x_n\| = \|y_n - x_n\|$.

Lemma 2.4 Let $x_0, y_0 \in X$, $\|x_0 - y_0\| = d > 0$ and $l_0 = l(y_0, e_0)$ with $\|e_0\| = 1$ and $l = l(y_0, e)$ with $\|e\| = 1$ be two lines such that $e_0 \neq e$ and both l_0 and l are supporting the ball $B[x_0, d]$. Let also (y_n) be a sequence satisfying conditions (i) and (ii) from the above lemma. Then there exists a sequence (x_n) such that $\lim x_n = x_0$ and $d(x_n, l_0) = \|x_n - y_n\|$.

Proof: Denote $e_n = (y_n - y_0)/\|y_n - y_0\|$. According to (ii) the sequence (e_n) has at most two cluster points e and $-e$. Without any change of indexes we consider the case $\lim(y_0 + e_n) = y_0 + e$. The other case is treated analogously.

Denote $L_n(y_0) = L(y_0, e_0, e_n)$, $L_n(x_0) = L(x_0, e_0, e_n)$ and let (u_n) and (v_n) be sequences such that $\{u_n\} = P(y_0, L_n(x_0))$ and $\{v_n\} = P(y_n, L_n(x_0))$.

Then $\{y_0\} = P(u_n, L_n(y_0)), \{y_n\} = P(v_n, L_n(y_0))$ and $\|y_0 - y_n\| = \|u_n - v_n\|$. Obviously, $\|y_0 - u_n\| < \|y_n - u_n\|$ and $\|y_n - v_n\| < \|y_0 - v_n\|$. Hence $d(u_n, l_0) < \|u_n - y_n\|$ and $\|v_n - y_n\| < d(v_n, l_0)$. Now applying the intermediate value theorem to the function $\phi_n(x) = d(x, l_0) - \|x - y_n\|$ defined on $[u_n, v_n]$ we obtain a point $x_n \in (u_n, v_n)$ such that $d(x_n, l_0) = \|x_n - y_n\|$.

It has to be proved that $\lim x_n = x_0$. Since (u_n) is a bounded sequence then (t_n) and (s_n) are bounded too. Let $u_0 = x_0 + t_0 e_0 + s_0 e$ be a cluster point of (u_n). If $u_0 \neq x_0$ then $\|y_0 - x_0\| < \|y_0 - u_0\|$ since $\{x_0\} = P(y_0, L(x_0, e_0, e))$. There is $\delta > 0$, $\delta < \|x_0 - u_0\|$, so that $\|y_0 - x_0\| < \|y_0 - u\|$ whenever $u \in B(u_0, \delta)$. However, the last inequality contradicts to the choice of u_n since for large n $\|y_0 - x_0\| < \|y_0 - u_n\| = d(y_0, L_n(x_0))$. Therefore $x_0 = \lim u_n = \lim x_n = \lim v_n$.

Proposition 2.5 Suppose M is a nowhere dense in X compact such that

(a) every point $y \in M$ lies on a pseudo-tangent to M at y,

(b) if $y \in (u, v) \subset M$ then there is a pseudo-tangent to M at y which does not contain u and v,

(c) there is a $\mathcal{N}^1$-separator N of M,

(d) the set $Q = \{x \in X : P(x, M) \cap N \neq \emptyset\}$ is nowhere dense in X.

Then the metric projection $P(\cdot, M)$ is multi-valued on a dense subset of X.

Proof: Let $x_0 \in X$ and $\varepsilon > 0$ be given. Since $M \cup Q$ is nowhere dense, we may assume with no loss of generality that $B(x_0, \varepsilon) \cap (M \cup Q) = \emptyset$. Assume now, that $P(\cdot, M)$ is single-valued on $B(x_0, \varepsilon)$. According to a result in [Si] the metric projection restricted on $B(x_0, \varepsilon)$ is a continuous mapping.

We shall prove that all points of $B(x_0, \varepsilon)$ are mapped on a line l_0 passing through y_0. If not so, there are $x_i \in B(x_0, \varepsilon)$ with $\{y_i\} = P(x_i, M)$ for i=1,2,3 such that co$\{y_1, y_2, y_3\}$ is a non-degenerated triangle. Hence the relative interior of co$\{x_1, x_2, x_3\}$ is non-empty and for a point x from it the image of $[x_1, x] \cup [x, x_2]$ via $P(\cdot, M)$ is contained in the line $l(y_1, y_2 - y_1)$. By similar reasonings we see that x belongs also to $l(y_2, y_3 - y_2)$ and $l(y_3, y_1 - y_3)$, thus proving that y_1, y_2 and y_3 are colinear.

Consider now the alternative:

(I) Either $P(x, M) = \{y_0\}$ for all $x \in B(x_0, \varepsilon)$,

(II) or there is $x \in B(x_0, \varepsilon)$ with $\{y\} = P(x, M)$ such that for some $u, v \in M$ $y \in (u, v)$.

In case (I) let l be a pseudo-tangent to M at y_0. Obviously, l is a supporting line for the ball $B(x_0, d)$, $d = d(x_0, M) > 0$, i.e. $l \cap B(x_0, d) = \{y_0\}$. Applying Lemma 2.3 we obtain a sequence (x_n), $\lim x_n = x_0$ such that $\|y_0 - x_n\| = \|y_n - x_n\|$. But now, for large n x_n belongs to $B(x_0, \varepsilon)$ and this violates the single-valuedness assumption.

The case (II) is treated analogously after applying Lemma 2.4. The proof is completed.

Corollary 2.6 Suppose M is a nowhere dense compact with pseudo-tangents at all points and there is a N^0-separator N of M such that the set $\{x \in X : P(x, M) \cap N \neq \emptyset\}$ is nowhere dense too. Then the metric projection $P(\cdot, M)$ is multi-valued on a dense subset of X.

The case of totally disconnected compact M is considered in [Zh2].

3 Planar Construction

In the Euclidean plane $(\mathbb{R}^2, |\cdot|)$ a Cartesian coordinate system Oxy is used. The closed unit disk(ball) is denoted by B.

Let $\Delta_n = [P_{n1}, P_{n2}, ..., P_{n2^n}]$ be a regular 2^n-gon inscribed in B with vertices $P_{ni} = (\cos(\pi(i-1)/2^{n-1}), \sin(\pi(i-1)/2^{n-1}))$ for $i = 1, 2, ..., 2^n$, $n \geq 1$. For $q_n = \sin(\pi/2^n)/(1 + \sin(\pi/2^n))$ let $D_{ni} = (1 - q_n)P_{ni} + q_n B$, i.e. D_{ni} are the images of B via homothets with centers P_{ni} and scaling factor equal to q_n. Define $M_n = \bigcup_{i=1}^{2^n} D_{ni}$, $\mathcal{M}_n = \{D_{ni} : i = 1, 2, ..., 2^n\}, n = 1, 2, ...$ The scalars q_n are chosen such that any two adjoining disks from $\mathcal{M}_n$ have only one boundary point in common and thus M_n are continua.

Now, a Peano continuum $K \subset \mathbb{R}^2$ will be constructed as a countable intersection of a nested family of continua K_n such that for each n the set K_n is a finite union of closed disks with equal radii r_n.

Assign $K_0 = B$, $\mathcal{K}_0 = \{B\}$. We are going to define K_n and $\mathcal{K}_n$ under the inductive assumption that K_j and $\mathcal{K}_j$ have already been defined, so that the disks of $\mathcal{K}_j$ have radii $r_j = q_1 q_2 ... q_j$ for j =1,2,...,$n-1$. (See Figure visualizing the sets K_0, K_1, K_2 and K_3.)

For $D \in \mathcal{K}_{n-1}$ let T_D be the translation carrying O to the center of D. Set

$$K_n = \bigcup\{T_D(r_{n-1}M_n) : D \in \mathcal{K}_{n-1}\}$$

and

$$\mathcal{K}_n = \{T_D(r_{n-1}E) : D \in \mathcal{K}_{n-1}, E \in \mathcal{M}_n\}.$$

The elements of $\mathcal{K}_n$ are closed disks with radii equal to $r_n = q_1 q_2 ... q_n$. Put finally $K = \bigcap_{n=o}^{\infty} K_n$.

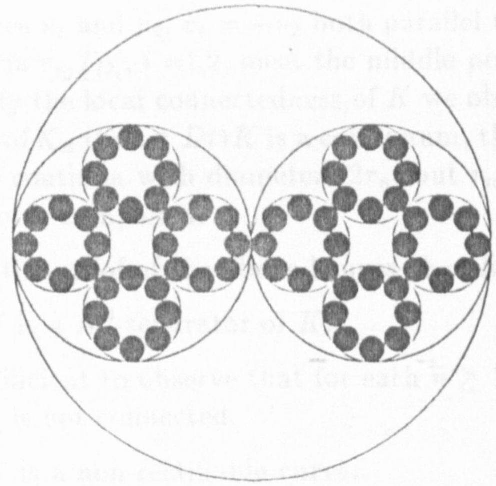

Figure. The union of filled disks is K_3.

The set K is a compact as an intersection of a nested sequence of compacta. It is easily seen that $\mathrm{int} K = \emptyset$ and

$$\lim_{n\to\infty} H(K_n, K) = 0. \tag{3.1}$$

With K we associate sets V_n, N_n for $n \geq 1$ and V and N.

$$V_n = \{T_D(r_{n-1}P_{ni}) : D \in \mathcal{K}_{n-1}, \quad i = 1, 2, ..., 2^n\}, \quad V = \bigcup_{n=1}^{\infty} V_n$$

$$N_n = \{x \in K_n : \exists D_1, D_2 \in \mathcal{K}_n, \quad D_1 \neq D_2, \{x\} = D_1 \cap D_2\}, \quad N = \bigcup_{n=1}^{\infty} N_n.$$

The properties of K are collected in the subsequent lemmas.

Lemma 3.1 K is a Peano continuum.

Proof: It should be established at first that K is a connected set. Having in mind (3.1) it suffices to prove that K_n are continua. It is seen that K_n is connected if and only if $N_{n-1} \subset K_n$ since M_n is connected. Let $x \in N_{n-1}$ and $m < n$ be the least integer for which there are disks $D_1, D_2 \in \mathcal{K}_m$ with centers c_1 and c_2 such that $D_1 \neq D_2$, $\{x\} = D_1 \cap D_2$ and both D_1 and D_2 are contained in a disk from $\mathcal{K}_{m-1}$. Then $[c_1, c_2]$ is parallel to a side of Δ_m. There is a vertice v of Δ_{m+1} which is perpendicular to that side, hence there are two vertices v_1 and v_2, $v_1 = -v_2$ both parallel to $[c_1, c_2]$. Then the images of $\{v_1, v_2\}$ via $r_m T_{D_i}$, $i = 1,2$, meet the middle point $x = (c_1 + c_2)/2$.

In order to verify the local connectedness of K we observe that for every n and every disk D of $\mathcal{K}_n$ the set $D \cap K$ is a continuum, thus K is partitioned as a finite union of continua with diameters $2r_n$, but r_n tends to 0 when n increases. The proof is completed.

It is seen from the proof of the above lemma that $N \subset V \subset K$.

Lemma 3.2 N is a N^0-separator of K.

Proof: It is sufficient to observe that for each $n \geq 1$ N_n separates K_n, i.e. the set $K_n \setminus N_n$ is not connected.

Lemma 3.3 K is a non-rectifiable curve.

Proof: Let K be parametrized by a continuous mapping $\varphi : I \longrightarrow \mathbb{R}^2$, i.e. $\varphi(I) = K$. For a fixed integer n, $n \geq 1$, let $T = \{t_1 < t_2 < ... < t_m\}$ be a partition of I such that $\varphi(T) = V_{n+1}$ and for each $i = 1,...,m-1$ $\varphi(t_i) \neq \varphi(t_{i+1})$. Up to a subpartition of T we may assume also that any two adjoining points t_i and t_{i+1} are mapped by φ in one disk from $\mathcal{K}_n$. This is so, because N_n separates K_n and $\mathcal{K}_n$ is a finite set. If $\varphi(t_i)$ and $\varphi(t_{i+1})$ belong to different disks for some i, then there is a finite set of reals between t_i and t_{i+1} such that the images of any two neighboring numbers are in one disk. Therefore,

$$\sum_{i=1}^{m-1} |\varphi(t_i) - \varphi(t_{i+1})| \geq \sigma_n$$

where σ_n is a sum of the perimeters of all 2^{n+1}-gons inscribed in disks of $\mathcal{K}_n$. However,

$$\lim_{n\to\infty} \sigma_n = 2\prod_{n=1}^{\infty} \frac{2^n \sin(\pi/2^n)}{1+\sin(\pi/2^n)}$$

and the infinite product is not convergent since the n-th multiplier does not approach 1.

Lemma 3.4 The Hausdorff dimension of K is 1.

Proof: Since K is not a point then $\dim K \geq 1$. On the other hand, making use of monotonicity of $\mathcal{H}^s_\delta(K)$ with respect to δ, we write:

$$\mathcal{H}^s(K) = \lim_{n\to\infty} \mathcal{H}^s_{2r_n}(K) \leq \lim_{n\to\infty} \sum\{(\operatorname{diam} D)^s : D \in \mathcal{K}_n\} =$$

$$\lim_{n\to\infty} 2^{\frac{n(n+1)}{2}}(2r_n)^s = 2^s \lim_{n\to\infty} \prod_{i=1}^{n} 2^i q_i^s = 2^s \lim_{n\to\infty} \prod_{i=1}^{n} a_i(s) b_i(s)$$

where $a_i(s) = \sin(\pi/2^i)/(1+\sin(\pi/2^i))$ and $b_i(s) = \sin^{s-1}(\pi/2^i)$. Since $\lim a_i(s) = \pi$, and for $s > 1$ $\lim b_i(s) = 0$, then $\mathcal{H}^s(K) = 0$ for $s > 1$. Therefore $\dim K = 1$.

Lemma 3.5 Every point $y_0 \in K$ lies on a pseudo-tangent.

Proof: Let $D \in \mathcal{K}_{n-1}$, $D' \in \mathcal{K}_n$, $D'' \in \mathcal{K}_{n+1}$ and $y_0 \in D'' \subset D' \subset D$. Let also $v, v_1, v_2 \in V_n \cap D$, $v \in D''$ and v_1 and v_2 be v's two neighboring vertices. It will be used later that v_1 and v_2 depend on n, i.e. $v_1 = v_1(n)$ and $v_2 = v_2(n)$. Denote by c the center of D'' and by $l_n = l(y_0, e_n)$, $|e_n| = 1$, the straight line through y_0 which is parallel to $[v_1, v_2]$. An elementary calculation shows that

$$|v_i - y_0| \geq r_{n-1}\sin(\pi/2^{n-1}) - r_{n+1}, \quad i = 1,2$$

On the other hand for large n $[v_1, v_2]$ does not intersect D'', i.e. $r_{n-1}(1-\cos(\pi/2^{n-1})) > 2r_{n+1}$, and $d(v_i, l_n) \leq r_{n-1}(1-\cos(\pi/2^{n-1}))$. Hence

$$\frac{d(v_i, l_n)}{|v_i - y_0|} \leq \frac{1-\cos(\pi/2^{n-1})}{\sin(\pi/2^{n-1}) - q_n q_{n+1}} = \frac{\sin(\pi/2^n)}{\cos(\pi/2^n) - a_n q_{n+1}} \tag{3.2}$$

with $a_n = 2^{-1}(1+\sin(\pi/2^n))^{-1}$. The expressions in (3.2) tend to 0.

Let e be a cluster element of the sequence (e_n). Without any change of indexation we assume that $\lim e_n = e$. Denote $\eta_n = |e_n - e|$ and $l = \{y = y_0 + te : t \in \mathbb{R}\}$. Let $y \neq y_0$ and $u_n \in P(y, l_n)$. The following estimation for distances holds:

$$d(y,l) \leq |y - u_n| + d(u_n, l) \leq |y - u_n| + |u_n - y_0| . |e_n - e| \leq$$

$$(1+\eta_n) d(y, l_n) + \eta_n |y - y_0| \tag{3.3}$$

Define a sequence (y_n):

$$y_n = \begin{cases} v_1(m) & \text{for} \quad n = 2m-1 \\ v_2(m) & \text{for} \quad n = 2m \end{cases}$$

Then by (3.3)

$$\frac{d(y_n, l)}{|y_n - y_0|} \leq (1+\eta_m)\frac{d(y_n, l_n)}{|y_n - y_0|} + \eta_m,$$

here m is the integer part of $(n+1)/2$.

Having in mind (3.2) we see that the expressions tend to 0 when n increases. It is a routine matter to verify that $(y_n - y_0)/|y_n - y_0|$ has two cluster points. Therefore l is a pseudo-tangent.

Lemma 3.6 Every line through every element y_0 of N is a pseudo-tangent to M.

Proof: Let $n \geq 1$ be an integer and $\{y_0\} = D_1 \cap D_2$ for $D_1, D_2 \in \mathcal{K}_n$. Denote by l_0 the line which separates D_1 and D_2. Let l_{ni} be the lines through y_0 which make elementary angles equal to $\pi i/2^n$, $i = 1, 2, ..., 2^n - 1$, with l_0. These lines contain points from V_m for every $m \geq n$. Obviously, l_{ni} are pseudo-tangents. Thus there is a dense set of pseudo-tangents. In order to prove that any line through y_0 is a pseudo-tangent we make use of (3.3).

4 General Case

Theorem 4.1 Suppose X is a strictly convexifiable Banach space.

(I) If $\dim X \geq 2$ then there exists a Peano continuum $\tilde{K}$ of Hausdorff dimension 1 in X such that for any strictly convex norm in X the metric projection generated by $\tilde{K}$ is multi-valued on a dense set which is everywhere continual in X,

(II) If $\dim X \geq 3$ then there exists a starshaped Peano continuum $\tilde{S}$ of Hausdorff dimension 2 in X such that for any strictly convex norm in X the metric projection generated by $\tilde{S}$ is multi-valued on an everywhere continual subset of X.

Proof: Let Y be a two-dimensional subspace of X and $f : \mathbb{R}^2 \longrightarrow Y$ be a linear isomorphism. Denote by $\tilde{K}$ the image of the constructed in the preceding section Peano continuum K, i.e. $\tilde{K} = f(K)$. Denote also $\tilde{N} = f(N)$, $\tilde{V} = f(V)$ etc. Suppose $\|\cdot\|$ is a strictly convex norm in X. There are positive constants c_1 and c_2 such that for any $x, y \in \mathbb{R}^2$, $c_1|x-y| \leq \|f(x) - f(y)\| \leq c_2|x-y|$.

All the properties of K are shared by $\tilde{K}$. In order to verify that $\dim \tilde{K} = 1$ we make use of the next

Lemma 4.2([Fa] Lemma 1.8) Let $f : M_1 \longrightarrow M_2$ be a surjective mapping such that $\|f(x) - f(y)\| \leq c|x-y|$ $x, y \in M_1$ for a constant c. Then $\mathcal{H}^s(M_2) \leq c^s \mathcal{H}^s(M_1)$.

Trivial checks establish the other properties of $\tilde{K}$ analogous to those of K listed in lemmas 3.1-3.6. From now on the parts (I) and (II) are considered separately.

Part (I): In order to apply Corollary 2.6, we have to show only that $\text{int}Q = \emptyset$, where $Q = \{x \in X : P(x, \tilde{K}) \cap \tilde{N} \neq \emptyset\}$. For this purpose let $x \in Q$, $y \in P(x, \tilde{K}) \cap \tilde{N}$ and $C(x) = Y \cap B[x, d(x, \tilde{K})]$. Since any line through y is a pseudo-tangent, then $C(x)$ has an empty relative interior in Y. Thus

$$d(x, \tilde{K}) = d(x, Y) \text{ whenever } x \in Q.$$

If Q is dense in some open and non-empty set U then the above equality holds for all points in U, but this is impossible since $\tilde{K}$ is a nowhere dense compact and $\|\cdot\|$ is strictly convex.

Denote by W the set of points of multi-valuedness of $P(\cdot, \tilde{K})$. According to Corollary 2.6 W is dense in X. In order to show that W is everywhere continual in X we employ the following

Lemma 4.3([Zh2]) Let M_1 and M_2 be two disjoint and non-void compacta and $x \in X$ be a point such that $d(x, M_1) = d(x, M_2)$ Then arbitrary neighbourhood of x contains continually many elements with the same property.

Suppose $x \in W \backslash Q$ and $y_1, y_2 \in P(x, \tilde{K})$, $\|y_1 - y_2\| = d > 0$. There is a sufficiently large integer n so that y_1 and y_2 belong to two different sets $\tilde{D}_1$ and $\tilde{D}_2$ from $\tilde{\mathcal{K}}_n$. Since $G = \text{bd}\tilde{D}_1 \cap \tilde{N}_n$ is a finite set (in fact it contains no more than three points) and $G \cap P(x, \tilde{K}) = \emptyset$ then there is $\delta > 0$ and an open neighbourhood U of x such that $M =: \tilde{K} \backslash \bigcup_{v \in G} B(v, \delta)$ does not contain y_i, i =1,2, and both $P(\cdot, \tilde{K})$ and $P(\cdot, M)$ coincide on U. It remains to apply Lemma 4.2 in order to complete the proof of (I).

Part (II): Since $\dim X > 2$ there is $e \in X \backslash Y$. Denote by Z the linear span of $\{e\} \cup Y$ and consider the set

$$\tilde{S} = \{t(e + y) : y \in \tilde{K}, t \in [-1, 1]\}$$

It is obviously a starshaped Peano continuum of Hausdorff dimension 2 with $\ker \tilde{S} = \theta$, where θ is the origin of X. The set $\hat{N} = \{t(e + y) : y \in \tilde{N}, t \in [-1, 1]\}$ is a $\mathcal{N}^1$-separator of $\tilde{S}$. In applying Proposition 2.5 we need verify condition (d) only, i.e. $\hat{Q} = \{x \in X : P(x, \tilde{S}) \cap \hat{N} \neq \emptyset\}$ is nowhere dense in X. If not so, then there is an open set $U \subset X$ such that $U \cap \tilde{S} = \emptyset$ and $\hat{Q}$ is dense in U. Denote $E = (\{\theta\} \cup (e + \tilde{K}) \cup (-e + \tilde{K}))$ and consider the alternative:

(i) Either $P(x, \tilde{S}) \subset E$ whenever $x \in U$,

(ii) or there is an open and non-empty set $U_1 \subset U$ such that $P(x, \tilde{S}) \subset \tilde{S} \backslash E$ whenever $x \in U_1$.

The case (i) has been treated already. In the latter case observe that for every $y \in \tilde{S} \backslash E$ any line through y which is contained in Z is a pseudo-tangent to $\tilde{S}$ at y, hence

$$d(x, \tilde{S}) = d(x, Z) \quad \text{whenever} \quad x \in U_1. \tag{4.1}$$

However, $\tilde{S}$ is nowhere dense in Z and (4.1) cannot be true.

For the remaining part of the proof our reasonings are very similar to those concerning part (I) when proving that the set of points of multi-valuedness of the metric projection is everywhere continual. We need only consider two separate cases again: $P(\cdot, \tilde{S})$ maps an open set U on E, and $P(\cdot, \tilde{S})$ maps an open set U on $\tilde{S} \backslash E$. The proof is completed.

The next result shows that the minimal dimension at which (II) holds is 3.

Proposition 4.4 Suppose S is a starshaped Peano continuum in a two-dimensional strictly convex space X. If the metric projection $P(\cdot, S)$ is multi-valued on a dense subset of some open set $U \subset X$, then S has non-empty interior.

Proof: Let $P(\cdot, S)$ be multi-valued at some $x_0 \in U$. Since the norm is strictly convex, then $\ker S \cap P(x_0, S) = \emptyset$. For $y_0 \in P(x_0, S)$ denote $x_t = (1-t)x_0 + ty_0$ and choose $t > 0$ sufficiently small so that $x_t \in U$. It follows from a well-known lemma (c.f. [St]) that $\{y_0\} = P(x_t, S)$. Let (x_n) be a converging to x_t sequence such that the metric projection is multi-valued at each point x_n. For every integer n there is $y_n \in P(x_n, S)$ such that $y_n \neq y_0$. The continuity of P at x_t implies $\lim y_n = y_0$.

Let now $z_0 \in \ker S$ and $\varepsilon = \|y_0 - z_0\|/2 > 0$. Since S is locally connected at y_0, then for a sufficiently large number n a continuous path in $S \cap B(y_0, \varepsilon)$ connect y_0 with y_n. Therefore, $\text{co}\{z_0, y_0, y_n\} \cap (X \backslash B(y_0, \varepsilon)$ is a subset of S, and obviously it has non-empty interior.

Remarks. It has to be mentioned that theorem 4.1 provides new information not only for non-separable Banach spaces since most continua in $\mathcal{C}(X)$ and $\mathcal{S}(X)$ are not Peano continua. According to a result of Mazurkiewicz [Ma], later extended by Bing [Bi], the (hereditarily) indecomposable continua form a dense G_δ subset of $\mathcal{C}(X)$. A continuum is indecomposable if it cannot be represented as union of two proper subcontinua. It is seen that an indecomposable continuum is not locally connected at any of its points. This is no longer true for continua in $\mathcal{S}(X)$ since the starshaped sets are locally connected at the points of their kernels. However, an analogous result of Zamfirescu [Za1] still holds and it shows that the set of Peano continua in $\mathcal{S}(X)$ is of first Baire category (although formulated for $S(\mathbb{R}^n)$ the proof of Zamfirescu combined with [BKM] works in arbitrary Banach space). Hence the existence of sets like $\tilde{K}$ and $\tilde{S}$ even in $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively is not guaranteed by any of before quoted theorems.

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