ИНСТИТУТ ПО МАТЕМАТИКА С ИЗЧИСЛИТЕЛЕН ЦЕНТЪР
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Chaotic attractor reconstruction and applications in astronomy

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Chaotic attractor reconstruction and applications in astronomy*

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Abstract

A short mathematical background of fractal dimensions, and some methods for computing spectrum of dimensions (capacity, information and correlation dimensions, etc.) are presented. Several aspects of attractors reconstructed from experimental data are also investigated.

A corresponding software system for IBM-PC is created and some of its possibilities are described. The system is used for calculating the spectrum of dimensions of fractal sets, discrete dynamical systems and attractors of differential equations. Also, the results of processing real astronomical data (electrophotometric data for the variable stars TT Ari and KR Aur) are obtained and discussed.

1 Introduction

There are different methods for finding the dimension of the attractors. The correlation integral method (introduced by Grassberger and Procaccia [7]) is one of the widespread methods for practical computations. This method can be applied in astronomy for objects, possessing any observed (and measured) variability (see Kolláth and Nuspl [11]). Such investigations aim at:

– determining low-dimensional attractor – global evolution of the system which can be described by non-linear differential equations showing deterministic chaos in their behaviour (cf. Auvergne and Baglin [2], Atmanspacher et al. [1], Harding et al. [8]);

– determining local scaling properties (white or shot noise) – a system consisting of a number of uncorrelated elements which appear at random and live only a short time (cf. Cannizzo and Goodings [5], Lehto et al. [13]);

– comparing similar variable astronomical objects (galaxies, variable stars, etc.) in order to do its classification (cf. Lehto et al. [13]).

The main problems in the determination of correlation dimension are high noise level of the observations and the small length of the data (see Lochner et al. [14], Norris and Matilski [17]). In this paper, we apply correlation integral method for a number of simulated light curves (including noise) and real data. Comparison analysis of the graphics obtained is the main tools in our investigations.

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2 Generalized attractor dimensions

Balatoni and Renyi [3] and then Grassberger and Procaccia [7] introduced the notion of *attractor dimensions*.

Let A be an attractor (limit attracting set) of a dynamical system. Cover A with volume elements (spheres, cubes, etc.) each with diameter ε. Let $N(\varepsilon)$ be minimum number of such volume elements needed to cover A. Generalized information of order q is defined by

$$I^{(q)}(\varepsilon) = \log_2 M^{(q)}(\varepsilon), \quad M^{(q)}(\varepsilon) = \left(\sum_{i=1}^{N(\varepsilon)} {p_i}^q \right)^{\frac{1}{q-1}}, \tag{1}$$

where p_i is the relative frequency with which a typical trajectory enters the i-th volume element of the covering. When $q \to 1$, $I^{(1)}(\varepsilon)$ is the well known Shennon information

$$I^{(1)}(\varepsilon) = S(\varepsilon) = - \sum_{i=1}^{N(\varepsilon)} p_i \log_2 p_i. \tag{2}$$

Generalized dimension of order q (spectrum of dimensions, Renyi dimensions) is introduced by

$$D^{(q)} = \lim_{\varepsilon \to 0} \frac{I^{(q)}(\varepsilon)}{\log_2 \varepsilon}. \tag{3}$$

Usually, dimension of order 0 is called *capacity*, *Hausdorff dimension* or *fractal dimension*, $D^{(1)}$ – *information dimension* and $D^{(2)}$ – *correlation dimension*.

$$D^{(0)} = \lim_{\varepsilon \to 0} \frac{- \log N(\varepsilon)}{\log \varepsilon}, \quad D^{(1)} = \lim_{\varepsilon \to 0} \frac{\sum_{i=1}^{N(\varepsilon)} p_i \log p_i}{\log \varepsilon}, \quad D^{(2)} = \lim_{\varepsilon \to 0} \frac{\log \sum_{i=1}^{N(\varepsilon)} {p_i}^2}{\log \varepsilon}. \tag{4}$$

Let us assume that a trajectory $\{x_i\}_{i=1}^{\infty}$ forms the attractor A. For a given $\varepsilon > 0$, generalized correlation function of order q is introduced by

$$C^{(q)}(\varepsilon) = \lim_{N \to \infty} \left(\frac{1}{N} \sum_{i=1}^{N} \left(\frac{1}{N} \sum_{j=1}^{N} \theta(\varepsilon - \|x_i - x_j\|) \right)^{q-1} \right)^{\frac{1}{q-1}}, \tag{5}$$

where x_i, x_j are trajectory points, θ is Heaviside function, i.e. $\theta(z) = 1$ for $z > 0$ and $\theta(z) = 0$ for $z \leq 0$. Then the correlation integral of order q is given by the following formula

$$C^{(q)} = \lim_{\varepsilon \to 0} \frac{\log C^{(q)}(\varepsilon)}{\log \varepsilon}. \tag{6}$$

3 Reconstruction of attractors

A remarkable result, first proved by Takens [21], allows a strange attractor to be reconstructed from a sampled time waveform of just one component of the state. For a system where one or more of the state variables cannot be measured directly, reconstruction may be the only way to observe the attractor. This is a useful technique in experimental settings. This is the usual case in astronomy.

Let an attractor A of an n-th-order system with flow φ_t be contained in an K-dimensional compact manifold $M \subset \mathbf{R}^{2K+1}$ as

$$F(x) = [\varphi_0^{(j)}(x), \varphi_\tau^{(j)}(x), \ldots, \varphi_{2N\tau}^{(j)}(x)]^T$$

where $\varphi_t^{(j)}(x)$ is the j-th component of $\varphi_t(x)$, j is arbitrary, and $\tau > 0$ is the sampling period, also arbitrary.

Generally, F is an embedding, that is, F diffeomorphically maps M onto compact K-dimensional manifold $M' \subset \mathbf{R}^{2K+1}$. This fact implies that given a sequence $\{x_n\}_{n=1}^N := \{\varphi_{n\tau}^{(j)}(x)\}_{n=1}^N$ that corresponds to a uniformly time-sampled component of a trajectory that lies on an attractor A, the sequence of points

$$\begin{array}{l}
[x_0 \; x_1 \; \cdots \; x_{2K}]^T \\
[x_1 \; x_2 \; \cdots \; y_{2K+1}]^T \\
\vdots \\
[x_i \; x_{i+1} \; \cdots \; x_{2K+i}]^T \\
\vdots \\
[x_{N-2K} \; x_{N-2K+1} \; \cdots \; y_N]^T
\end{array}$$

lies on a diffeomorphic copy of A.

Let $\{x_k\}_{k=0}^N$ be a sequence of measurements of an attractor, contained in K-dimensional manifold. Then we build sequences of d-dimensional points $X_i = (x_i, x_{i+1}, \ldots, x_{i+d-1})$, $i = 1, 2, \ldots, N-d$ and calculate the dimension in the space $\mathbf{R}^d$. According to Takens' theorem, the dimension will not depend on d when $d \geq 2K+1$. The correlation integral method is consists of first calculating the correlation function (5) in d-dimensional space and then determining the slope of the linear part of correlation function $C^{(q)}(\varepsilon)$ vs. ε.

4 Methods and program system

A program system is created for calculating the correlation integral $C^{(q)}$. It works on IBM-PC (386 and better) and supports EGA and VGA graphics display. The first (computational) part of the system calculates the generalized correlation function $C^{(q)}(\varepsilon)$ for sequences of values for ε and q, and for embedding dimensions $d = 1, 2, \ldots, d_{max}$. Our main efforts have been directed to an efficient realization of correlation integral method (cf. Parker and Chua [18]). Data up to 3000 points can be processed into a reasonable time interval.

Also, we implemented a version of the box counting method, (using udeas of Malinetzkii and Potapov [16]), which allow us to compute the dimensions $D^{(q)}$. Now this part of the program works efficiently only for small data sets.

The second part represents the results (the calculated correlation functions) as different graphics. It privides means for an interactive work with the user when drawing basic graphics of the following kind:

– artificial 2-dimensional phase space x_k vs. $x_{(k+l)}$ for different l and for different subsequences of the data sequence x_k;

– the correlation integral of order q, $C^{(q)}$, as a logarithmic graphic $\log C^{(q)}(\varepsilon)$ vs. $\log \varepsilon$ for a single embedding dimension d or for a set of embedding dimensions;

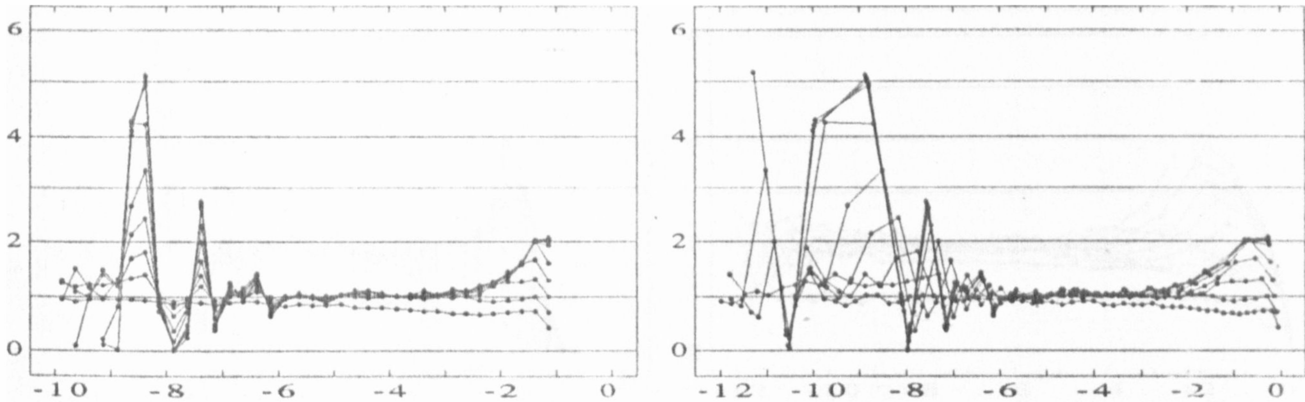

Figure 1: The correlation function of the attracting limit cycle for Van der Pol's equation ($N = 1000, d = 1 \ldots 10$) – slope vs. $\log \varepsilon$ and slope vs. $\log C^{(2)}(\varepsilon)$.

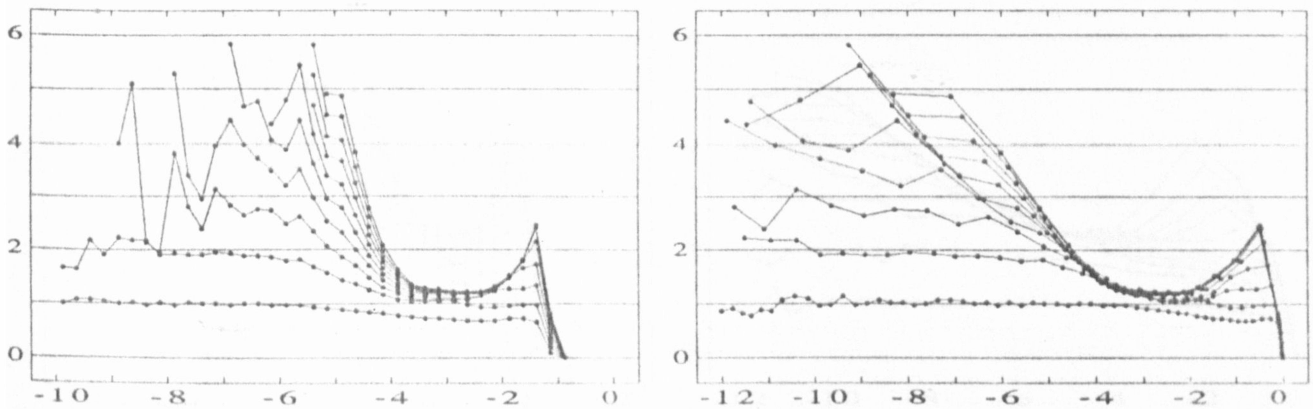

Figure 2: The correlation function of the attracting limit cycle for Van der Pol's equation plus 10 percent noise – slope vs. $\log \varepsilon$ and slope vs. $\log C^{(2)}(\varepsilon)$.

– the slope of the correlation function of order q vs. $\log \varepsilon$ and vs. $\log C^{(q)}(\varepsilon)$, again for a single embedding dimension d or for a set of embedding dimensions;

– the correlation integral of order q, $C^{(q)}$ vs. q and vs. embedding dimensions d.

Here, we include only calculations of correlation function of order 2 $\log C^{(2)}(\varepsilon)$, as a first step of our investigations. The slope in these cases can be clearly seen when drawing graphics slope vs. ε and slope vs. $C^{(q)}(\varepsilon)$.

5 Results

5.1 Simulated light curves

Some simulated light curves produced from deterministic functions are used as test examples for the correlation integral method and for our software system. The dimensions of a few attracting sets of the discrete dynamical systems Logistic and Henon maps (for various values of the parameters) have been calculated. Also, the dimension of the Serpinski triangle fractal set has been obtained using a random algorithm for building this set (see Barnsley [4]). Data from some trajectories of differential equations (represented attractind sets) have been processed.

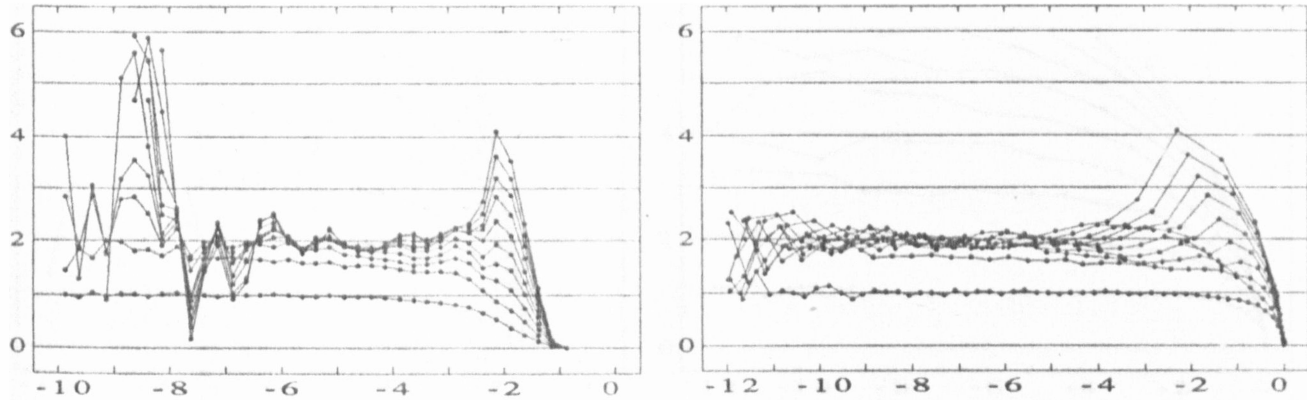

Figure 3: The correlation function of the Lorenz light curve ($N = 1200, d = 1 \ldots 10$) – slope vs. $\log \varepsilon$ and slope vs. $\log C^{(2)}(\varepsilon)$.

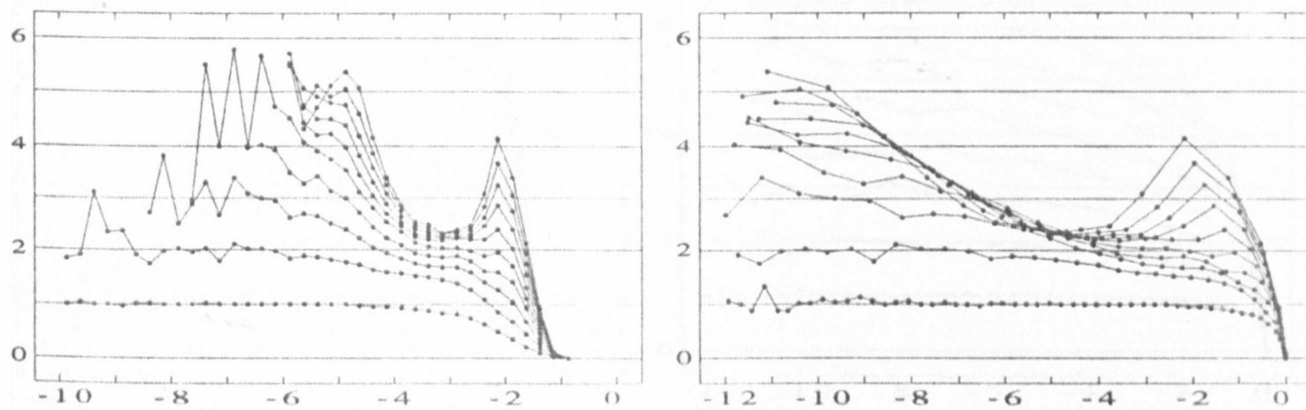

Figure 4: The correlation function of the Lorenz light curve with 10 percent noise – slope vs. $\log \varepsilon$ and slope vs. $\log C^{(2)}(\varepsilon)$.

Van der Pol's equation

$$\dot{x} = y,$$
$$\dot{y} = (1 - x^2)y - x$$

is a classical example of an attracting limit cycle. This attractor is 1-dimensional, of course (see Figure 1). The effect of adding 10 percent white noise is seen on Figure 2. Our experimental data consist of x-measurements, obtained by numerically calculated limit cycle using Runge-Kutta scheme of order 2.

Lorenz attractor (see Lorenz [15]) is produced by one of the simplest sets of differential equations demonstrating chaotic behaviour:

$$\dot{x} = -\sigma x + \sigma y$$
$$\dot{y} = -xz + rx - y$$
$$\dot{z} = xy - bz,$$

where $\sigma = 10, r = 28, b = 8/3$. The dimension of the attractor is approximately 2.05 (Grassberger and Procaccia [7]).

An Euler numerical scheme is applied for chaotic trajectory calculation with step 0.02. More detailed investigation of this attractor is given by Lehto et al.[13]. Our results are given on Figures 3 and 4.

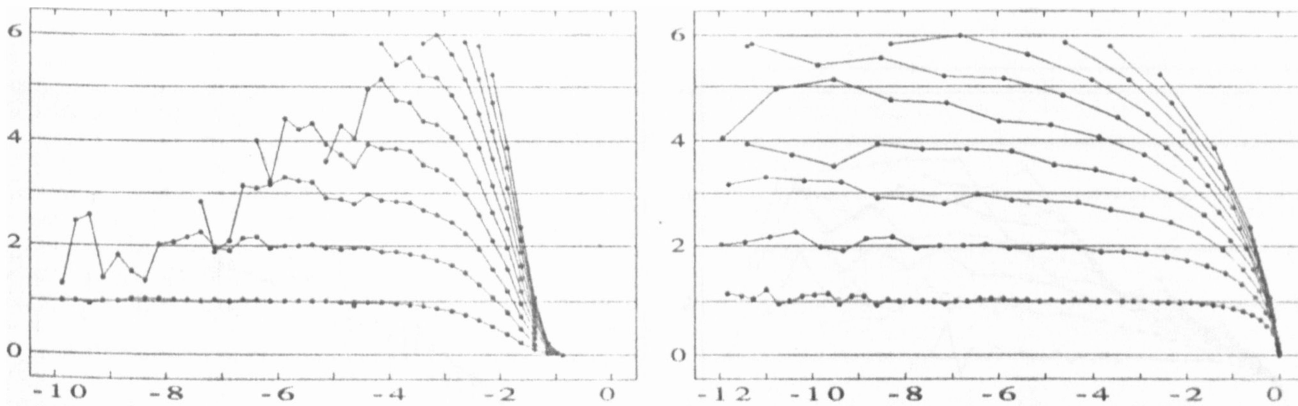

Figure 5: The correlation function for Gaussian distributed noise ($\sigma = 1, N = 1200, d = 1 \ldots 10$) – slope vs. $\log \varepsilon$ and slope vs. $\log C^{(2)}(\varepsilon)$.

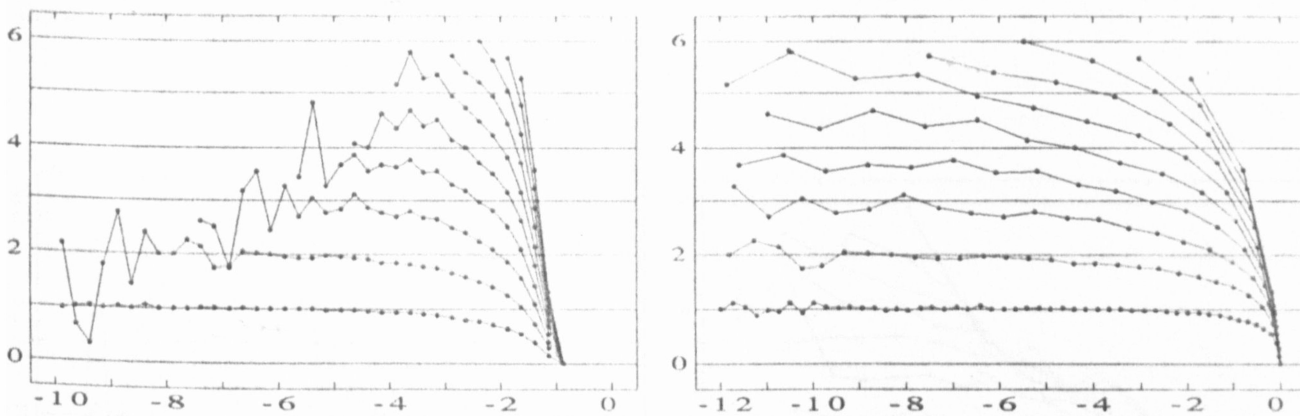

Figure 6: The correlation function for uniformly distributed white noise ($N = 1200, d = 1 \ldots 10$) – slope vs. $\log \varepsilon$ and slope vs. $\log C^{(2)}(\varepsilon)$.

5.2 White noise.

The dimension of the ideal white noise (random quantity, uniformly or normally distributed) in $\mathbf{R}^d$ is equal to the embedding dimension d. In practice, as Figures 5 and 6 shows, the correlation function has a slope which increases with the higher embedding dimension.

5.3 TT Arietis and KR Aurigae.

Processing data of these cataclysmic stars were obtained by Kraicheva et al. [12] for TT Arietis and Popov, Antov [20] for KR Aurigae.

The observations were made in U-color of the standard UBV system using single channel photon-counting photoelectric photometer, attached to 60 cm Cassergain telescope in the National Astronomical Observatory Rozhen. Photometric data reduction has been made by the program system APR (Kirov et al. [9]). Two sources of TT Ari and one of KR Aur are processed and prsented here.

A definite assertion for existence (or non existence) of a low-dimensional attractor can not be given on the basis of the presented graphical results (Figures 7, 8 and 9). Probably, the further more detailed investigations, using the full possibilities of the software system can be lead up to a definite answer. Also, Fourier technques to search for and remove any strong periodic components will be used and some predictive procedures can help us to establish the presence of deterministic chaos.

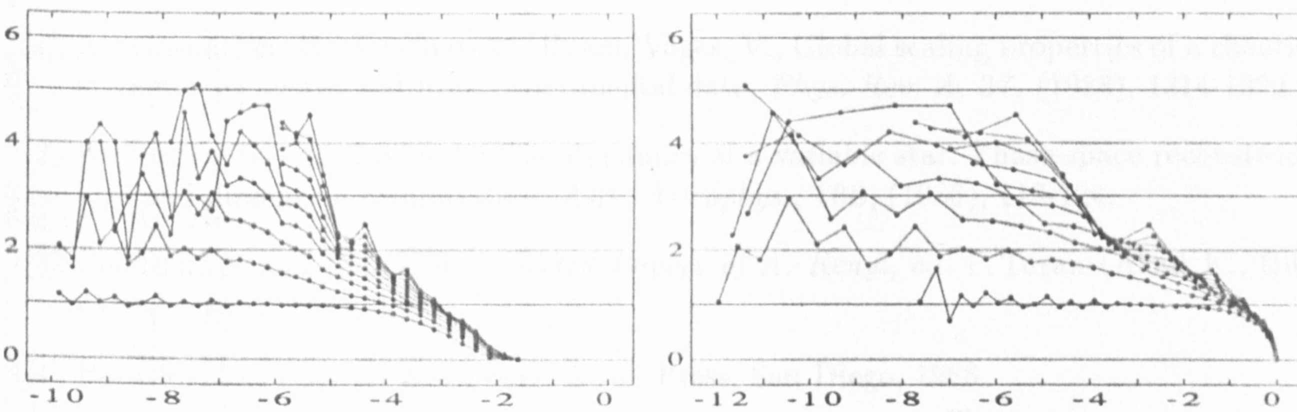

Figure 7: The correlation function for first TT Ari light curve ($N = 1050, d = 1 \ldots 10$) – slope vs. $\log \varepsilon$ and slope vs. $\log C^{(2)}(\varepsilon)$.

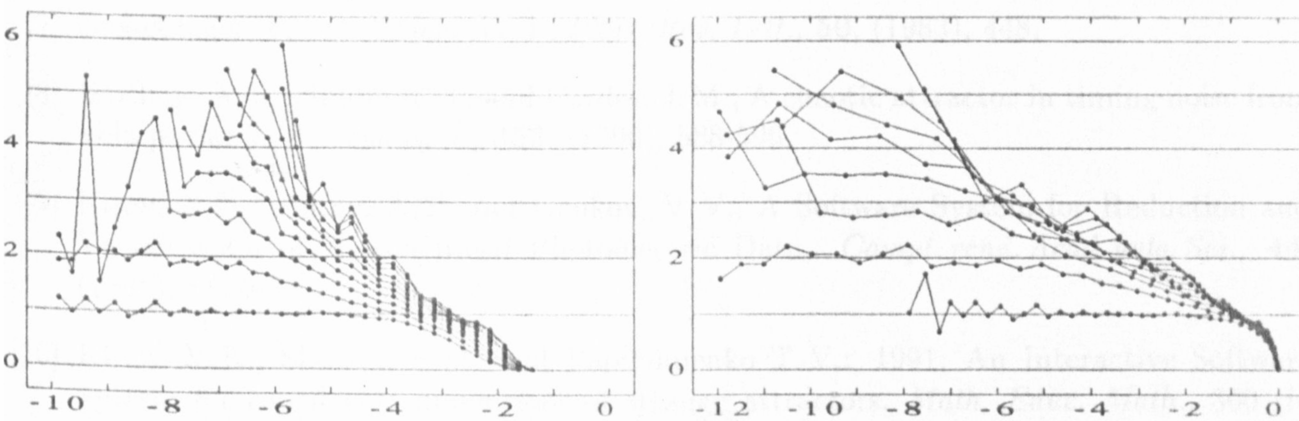

Figure 8: The correlation function for second TT Ari light curve ($N = 950, d = 1 \ldots 10$) – slope vs. $\log \varepsilon$ and slope vs. $\log C^{(2)}(\varepsilon)$.

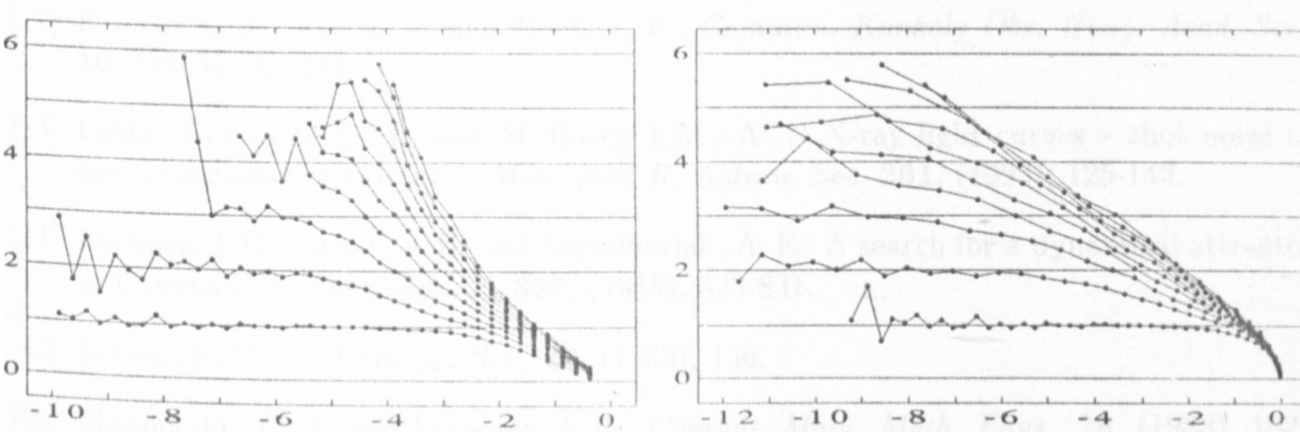

Figure 9: The correlation function for KR Aur light curve ($N = 1200, d = 1 \ldots 10$) – slope vs. $\log \varepsilon$ and slope vs. $\log C^{(2)}(\varepsilon)$.

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