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# SOME REMARKS ON A THEOREM OF H.-J.SCHMIDT

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# SOME REMARKS ON A THEOREM OF H.-J. SCHMIDT

S. Barov, G. Dimov, St. Nedev

**Abstract.** On the basis of a property of the hyperspace, H.-J. Schmidt [S] introduced implicitely a class of spaces called in the present paper *HS*-spaces. In [S, Theorem 11(3d)] it is stated that every Hausdorff *HS*-space is regular. Later on M.Paoli and E.Ripoli noted in [PR1] that the proof of the above assertion is incorrect, but the question of the correctness of the statement is open. We give a partial solution of this problem introducing a large class of spaces, which contains all Hausdorff spaces with countable character, where the theorem holds. Two reduction theorems are obtained as well.

**Keywords:** hyperspaces, upper semi-finite topology, *HS*-spaces, *F*-normal spaces, *K*-, $K'$-, $K''$- and $K^*$-spaces, continuous maps, closed maps, $T_2$-, $T_3$- and $T_4$-spaces.

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## 1. Introduction and Preliminary Results and Definitions

The following definition is motivated by the results of H.-J.Schmidt in [S].

1.1. Definition. A topological space $X$ is called a *HS*-space if, for every subspace $A$ of $X$, the map $\overline{i}_A : 2^{A,T} \longrightarrow 2^{X,T}$, defined by the formula $\overline{i}_A(B) = \mathrm{cl}_X B$, for every $B \in 2^A$, is a continuous map.

Here and below, for every topological space $(X,\mathcal{T})$, $2^X$ stands for the set of all non-empty closed subsets of $X$ and $\mathrm{cl}_X B$ - for the closure of the subset $B$ of $X$ in the space $X$. The set $2^X$ is endowed with the Tychonoff topology $\mathcal{T}_T$, which is known also as upper semi-finite topology [M], generated by the base $\mathcal{B} = \{ \langle U \rangle : U \in \mathcal{T}\}$, where $\langle U \rangle = \{ F \in 2^X : F \subset U\}$. The topological space $(2^X,\mathcal{T}_T)$ is denoted briefly by $2^{X,T}$.

The class of all *HS*-spaces (resp., all $T_i$-spaces, for $i$ = 1, 2, 3, 3.5, 4) will be denoted by $\mathcal{HS}$ (resp., by $\mathrm{T}_i$, $i$ = 1, 2, 3, 3.5, 4) and the class of all normal spaces - by $\mathcal{N}$ [1].

In [S] H.-J. Schmidt proved the following theorem.

1.2. Theorem.([S, Theorem 11(3d)]). $\mathcal{HS} \cap \mathrm{T}_2 \subseteq \mathrm{T}_3$.

M. Paoli and E. Ripoli noted in [PR1] that the proof of this theorem is incorrect, but the question of the correctness of the statement is open. In the present paper we give a partial solution of this question. More precisely: a) we give an internal (i.e. in terms of the space only) characterization of *HS*-spaces (see Theorem 2.10); b) we introduce a large class of spaces, called $\mathcal{K}^*$ (see Definition 2.17), containing all Hausdorff spaces with countable character (see 2.24 and 2.22), where 1.2 holds (see Theorem 2.19, where a stronger result is proved) and we show that 1.2 is true iff (= if and only if) the statement "$\mathrm{T}_2 \subset \mathcal{K}^*$" is true (see Theorem 2.20); using Theorem 2.19, we demonstrate that the limit of an inverse sequence of *HS*-spaces needs not be a *HS*-space (see Example 2.26); c) we show that the class $\mathcal{HS}$ is

1 In this paper we assume that $T_i$-spaces ($i$ = 3, 3.5, 4) are Hausdorff, while the regular and normal spaces are not assumed to be, in general, $T_1$-spaces.

invariant under closed mappings (see Theorem 2.11); d) we prove that 1.2 is true iff the statement " $\mathcal{HS} \cap T_2 = T_4$ " does (see Theorem 2.14). Moreover, some new classes of spaces, closely related to the problem discussed here, are introduced and briefly investigated. These are the class of $F$-normal spaces (see Definition 2.4 and Theorems 2.12, 2.28) and the classes of $K$-, $K'$- and $K''$-spaces (see Definition 2.23, Theorems 2.30, 2.31 and Examples 2.32, 2.33).

Let us cite also the following corollary of Proposition 5 in [Se]:

1.3. Proposition. ([Se]). $\mathcal{N} \subset \mathcal{HS}$.

1.4. Notation. If $\mathcal{P}$ is a class of topological spaces, then we denote by $\urcorner\mathcal{P}$ the class defined in the following way: $X \in \urcorner\mathcal{P}$ iff $X \notin \mathcal{P}$.

For all notions and notations undefined here see [E].

Let us finally note, that many of the results of this paper were announced (without proofs) in [BDN].

## 2. The results

2.1. Notation. For any set $X$, we denote by $E(X)$ the set of all nonempty subsets of $X$.

2.2. Convention. Let $(X,\mathcal{T})$ be a topological space, $H \in 2^X$, $U \in \mathcal{T}$ and $H \subseteq U$. Then we will say that (H,U) is a pair in X.

2.3. Definition. A pair $(H,U)$ in $(X,\mathcal{T})$ is said to be $F$-embedded in $X$ if there exists a $V \in \mathcal{T}$ such that

i) $H \subseteq V$, and

ii) ( $\Phi \in 2^U$, $\Phi \subseteq V$ ) implies $\Phi \in 2^X$.

2.4. Definition. A topological space $X$ is said to be $F$-normal if every pair $(H,U)$ in $X$ is $F$-embedded in $X$. The class of all $F$-normal spaces will be denoted by $\mathcal{FN}$.

2.5. Definition. A topological space $X$ is said to be $LF$-normal if for every pair $(H,U)$ in $X$ and for every subspace $Y$ of $X$ such that $H \subseteq Y$, the pair $(H, U \cap Y)$ in $Y$ is $F$-embedded in $Y$. The class of all $LF$-normal spaces is denoted by $\mathcal{LFN}$.

2.6. Remarks. Obviously, $\mathcal{N} \subseteq \mathcal{LFN} \subseteq \mathcal{FN}$. The inclusions $\mathcal{N} \subset \mathcal{LFN}$ and $T_4 = \mathcal{N} \cap T_1 \subset \mathcal{LFN} \cap T_1$ are strong, since for any infinite set $X$ with the cofinite topology we have that $X \in (\mathcal{LFN} \cap T_1) \setminus T_2$ and, hence, $X \in (\mathcal{LFN} \cap T_1) \setminus T_4$ and $X \in \mathcal{LFN} \setminus \mathcal{N}$.

Now, we are going to show that the classes $\mathcal{HS}$ and $\mathcal{LFN}$ coincide (which, in particular, will imply 1.3).

2.7. Convention. Let $X$ be a topological space and $\Phi \in E(X)$. The statement " for every subspace $A$ of $X$ such that $\Phi \in 2^A$, the mapping $\bar{i}_A : 2^{A,T} \text{--->} 2^{X,T}$ is continuous at the point $\Phi$ of $2^A$ " will be shortened as " $i$ is continuous at $\Phi$ ".

2.8. Remark. $X \in \mathcal{HS}$ iff $i$ is continuous at any $\Phi \in E(X)$.

2.9. Lemma. Let $(X,\mathcal{T})$ be a topological space. Then $X \in \mathcal{HS}$ iff $i$ is continuous at any $F \in 2^X$.

Proof. ==>) This is trivial.

<==)Let $\Phi \in E(X)$. We will show that $i$ is continuous at $\Phi$, which will imply, by 2.8, that $X \in \mathcal{HS}$. Let $A$ be a subspace of $X$ such that $\Phi \in 2^A$. We have to show that the map $\bar{i}_A : 2^{A,T} \text{--->} 2^{X,T}$ is continuous at $\Phi$. Let $F = \mathrm{cl}_X \Phi$, $U \in \mathcal{T}$ and $F \subseteq U$. We put $C = A \cup F$. Then $F \subseteq C$ and $F \in 2^C$. Since the map $\bar{i}_C : 2^{C,T} \text{--->} 2^{X,T}$ is continuous at $F$, there exists an open in $C$ set $V_1$ such that $F \subseteq V_1$ and

$$\bar{i}_C(<V_1>) \subseteq <U>. \qquad (1)$$

Let $V = V_1 \cap A$. Then $V$ is open in $A$, $\Phi \subseteq V$ and $\bar{i}_A(<V>) \subseteq <U>$. Indeed, let $B \in 2^A$, $B \subseteq V$ and $B_1 = \mathrm{cl}_C B$. Then $B_1 \subseteq B \cup F \subseteq V \cup V_1 = V_1$ and hence, by (1), $\mathrm{cl}_X B = \mathrm{cl}_X B_1 \subseteq U$. Q.E.D.

2.10. Theorem. $\mathcal{HS} = \mathcal{LFN}$.

Proof. A) Let $(X,\mathcal{T}) \in \mathcal{HS}$. We will show that $X \in \mathcal{LFN}$. Let $(H,U)$ be a pair in $X$ and $Y$ be a subspace of $X$ such that $H \subset Y$. We have to show that the pair $(H, U \cap Y)$ in $Y$ is $F$-embedded in $Y$.

Put $A = U \cap Y$. Then $H \in 2^A$ and $\bar{i}_A(H) = H \in <U>$. Since the map $\bar{i}_A: 2^{A,T} \dashrightarrow 2^{X,T}$ is continuous, there exists an open set $V$ in $A$ such that $H \subseteq V$ and $\bar{i}_A(<V>) \subseteq <U>$. Obviously, $V$ is open also in $Y$. Let $\Phi \in 2^{U \cap Y} = 2^A$ and $\Phi \subseteq V$. Then $\Phi \in <V>$ and hence $\mathrm{cl}_X \Phi \subseteq U$. We obtain that $\Phi = \mathrm{cl}_A \Phi = Y \cap U \cap \mathrm{cl}_X \Phi = Y \cap \mathrm{cl}_X \Phi = \mathrm{cl}_Y \Phi$, i.e. $\Phi \in 2^Y$.

B) Let $(X,\mathcal{T}) \in \mathcal{LFN}$. We will show that $X \in \mathcal{HS}$. By 2.9, it is enough to prove that $i$ is continuous at each $F \in 2^X$.

Suppose there exists a $F_0 \in 2^X$ such that $i$ is not continuous at $F_0$. Then there exists a subspace $B$ of $X$ such that $F_0 \subseteq B$ and the map $\bar{i}_B: 2^{B,T} \dashrightarrow 2^{X,T}$ is not continuous at the point $F_0$ of $2^B$. Hence, there exists a $U_0 \in \mathcal{T}$ such that: a) $F_0 \subseteq U_0$ and b) for every open in $B$ set $V$, containing $F_0$, there exists a $\Phi_V \in 2^B$ such that $\Phi_V \subset V$ and $(\mathrm{cl}_X \Phi_V) \setminus U_0 \neq \emptyset$. Let us put $C = B \cap U_0$. Then the map $\bar{i}_C: 2^{C,T} \dashrightarrow 2^{X,T}$ is not continuous at the point $F_0$ of $2^C$ (since every set, open in $C$, is open in $B$ too). Let $Y = C \cup (X \setminus U_0)$. Then $F_0 \subset Y$. Since $X \in \mathcal{LFN}$, it follows that the pair $(F_0, Y \cap U_0)$ in $Y$ is $F$-embedded in $Y$. But $Y \cap U_0 = (C \cup (X \setminus U_0)) \cap U_0 = C \cap U_0 = B \cap U_0 = C$ and hence $C$ is open in $Y$ and the pair $(F_0, C)$ in $Y$ is $F$-embedded in $Y$. So, there exists an open in $Y$ set $V_0$ such that: i) $F_0 \subseteq V_0$, and ii) $(\Phi \subset V_0, \Phi \in 2^C)$ $\Longrightarrow \Phi \in 2^Y$. Then $V_1 = V_0 \cap C$ is open in $Y$ and in $C$. Hence $V_1$ is open in $B$ and $F_0 \subseteq V_1$. So, there exists a $\Phi_1 \in 2^B$ such that $\Phi_1 \subseteq V_1$ and

$$(\mathrm{cl}_X \Phi_1) \setminus U_0 \neq \emptyset. \qquad (2)$$

Since $\Phi_1 \subseteq C \subseteq B$, we obtain that $\Phi_1 \in 2^C$. Now, $\Phi_1 \subseteq V_1 \subseteq V_0$ and ii) imply that $\Phi_1 \in 2^Y$. Hence $M = (\mathrm{cl}_X \Phi_1) \cap (X \setminus U_0) \subset$ $\subset (\mathrm{cl}_X \Phi_1) \cap Y = \mathrm{cl}_Y \Phi_1 = \Phi_1 \subset C \subset U_0$, i.e. $M \subset (X \setminus U_0) \cap U_0 = \emptyset$, while (2) shows that $M \neq \emptyset$. Q.E.D.

2.11. Theorem. Let $f: X \xrightarrow{\text{onto}} Z$ be a closed map and $X \in \mathcal{HS}$. Then $Z \in \mathcal{HS}$.

Proof. By 2.10, it is enough to show that $Z \in \mathcal{LFN}$. Let $(\Phi, U)$ be a pair in $Z$ and let $Y$ be a subspace of $Z$ such that $\Phi \subseteq Y$. We have to prove that the pair $(\Phi, U \cap Y)$ in $Y$ is $F$-embedded in $Y$.

Since $X \in \mathcal{HS}$ and hence, by 2.10, $X \in \mathcal{LFN}$, the pair $(f^{-1}\Phi, f^{-1}(U \cap Y)) = (f^{-1}\Phi, f^{-1}U \cap f^{-1}Y)$ is $F$-embedded in $f^{-1}Y$. Then there exists an open in $f^{-1}Y$ set $V$ such that: i) $f^{-1}\Phi \subseteq V$, and ii) $B \subseteq V$ and $B \in 2^{f^{-1}(U \cap Y)}$ imply that $B \in 2^{f^{-1}Y}$. Obviously, the same holds for the set $V' = V \cap f^{-1}U$. The map $\varphi = f_Y: f^{-1}Y \longrightarrow Y$ is closed since $f$ is closed (see [E, 2.1.4]). Hence, there exists an open in $Y$ set $W$ such that $\Phi \subseteq W$ and $\varphi^{-1}(W) \subseteq V'$ (see [E, 1.4.12]). Then $f^{-1}\Phi \subseteq f^{-1}(W) = \varphi^{-1}(W) \subseteq V' \subseteq f^{-1}(U \cap Y)$ and, consequently, $\Phi \subseteq W \subseteq U \cap Y$. Let now $B' \subseteq W$ and $B' \in 2^{U \cap Y}$. Then $B =$ $= \varphi^{-1}(B') = f^{-1}(B') \subseteq V'$ and $B \in 2^{f^{-1}(U \cap Y)}$. Hence $B \in 2^{f^{-1}Y}$. Since $\varphi$ is a quotient map, this shows that $B' \in 2^Y$. So, $Z \in \mathcal{HS}$. Q.E.D.

2.12. Theorem. Let $f: X \xrightarrow{\text{onto}} Z$ be a closed map and $X \in \mathcal{FN}$. Then $Z \in \mathcal{FN}$.

Proof. Put $Y = Z$ in the proof of 2.11. Q.E.D.

The next definition will be used in the proof of Theorem 2.14 and further in the text.

2.13. Definition. A pair $(H,U)$ in a topological space $(X,\mathcal{T})$ is called nonseparable pair if $(\mathrm{cl}_X V) \setminus U \neq \emptyset$ for every $V \in \mathcal{T}$ such that $H \subseteq V$.

2.14. Theorem. The following assertions are equivalent:

a) $\mathcal{HS} \cap T_2 \subseteq T_3$;

b) $\mathcal{HS} \cap T_2 \subseteq T_{3.5}$;

c) $\mathcal{HS} \cap T_2 \subseteq T_4$;

d) $\mathcal{HS} \cap T_2 = T_4$.

Proof. Obviously, d)==>c)==>b)==>a) and c)==>d) (see 1.3 or 2.10 and 2.6). Hence, in order to prove the theorem, we have to show that a)==>c).

Let $X \in \mathcal{HS} \cap T_2$. Then, by a), $X \in T_3$. Suppose $X \notin T_4$. This means there exists a nonseparable pair $(F,U)$ in $X$. Let $Y = X/F$, i.e. the quotient space $Y$ is obtained by identifying the points of the closed subset $F$ of $X$, and let $\varphi: X \dashrightarrow X/F = Y$ be the natural map. Then $\varphi$ is a closed map and, hence, by 2.11, $Y \in \mathcal{HS}$. Obviously, we have that $Y \in T_2$ (since $X \in T_3$). Now, the condition a) implies that $Y \in T_3$. But the pair $(\varphi(F), \varphi(U))$ is, obviously, a nonseparable pair in $Y$, which shows that $Y \notin T_3$ - a contradiction. Hence $X \in T_4$. Q.E.D.

2.15. Remark. The inclusion $\mathcal{HS} \cap T_1 \subseteq T_2$ doesn't hold: any infinite space $X$ with the cofinite topology testifies to this (see 2.6).

2.16. Definition. Let $(X,\mathcal{T})$ be a topological space and $(H,U)$ be a pair in $X$. The pair $(H,U)$ is said to be $N$-embedded in $X$ if for every $V \in \mathcal{T}$ with $H \subset V \subset U$, there exists a subset $B_V$ of $V$ such that $\emptyset \neq (\mathrm{cl}_X B_V) \setminus V \subseteq X \setminus U$.

2.17. Definition. A space $X$ is said to be a $K^*$-space if either $X \in \mathcal{N}$ or there exist a nonseparable pair $(H, U)$ in $X$ and a subspace $Y$ of $X$ such that $H \subseteq Y$ and the pair $(H, U \cap Y)$ in $Y$ is $N$-embedded in $Y$.

The class of all $K^*$-spaces is denoted by $\mathcal{K}^*$.

2.18. Proposition. Let $(H, U)$ be a pair in $(X, \mathcal{T})$. Then $(H, U)$ is $N$-embedded in $X$ iff $(H, U)$ is not $F$-embedded in $X$.

Proof. ==>) Let $(H, U)$ be $N$-embedded in $X$ and suppose that $(H, U)$ is also $F$-embedded in $X$. Then there exists a $V \in \mathcal{T}$ such that $H \subseteq V$ and $(\Phi \in 2^U, \Phi \subset V) \Longrightarrow (\Phi \in 2^X)$. Since $(H, U)$ is $N$-embedded in $X$, there exists a subset $B$ of $V \cap U$ such that

$$\emptyset \neq (\mathrm{cl}_X B) \setminus (V \cap U) \subset X \setminus U.$$

Then $(\mathrm{cl}_U B) \setminus (V \cap U) = (\mathrm{cl}_X B) \cap U \cap (X \setminus (V \cap U)) = ((\mathrm{cl}_X B) \setminus (V \cap U)) \cap U \subset (X \setminus U) \cap U = \emptyset$. Hence, $\Phi = \mathrm{cl}_U B \subset V \cap U \subset V$ and $\Phi \in 2^U$. This implies that $\Phi \in 2^X$. Thus $(\mathrm{cl}_X B) \setminus (V \cap U) = \Phi \setminus (V \cap U) = \emptyset$, which is a contradiction. Hence, $(H, U)$ is not $F$-embedded in $X$.

<==) Let $(H, U)$ be not $F$-embedded in $X$. We will show that $(H, U)$ is $N$-embedded in $X$. Indeed, let $V \in \mathcal{T}$ and $H \subset V \subset U$. Then there exists a $\Phi \in 2^U$ such that $\Phi \subset V$ and $\Phi \notin 2^X$. Further, $(\mathrm{cl}_X \Phi) \cap V = \mathrm{cl}_V \Phi = (\mathrm{cl}_U \Phi) \cap V = \Phi \cap V = \Phi$, and $(\mathrm{cl}_X \Phi) \setminus \Phi \neq \emptyset$. Thus $(\mathrm{cl}_X \Phi) \setminus V = (\mathrm{cl}_X \Phi) \setminus ((\mathrm{cl}_X \Phi) \cap V) = (\mathrm{cl}_X \Phi) \setminus \Phi \neq \emptyset$ and $(\mathrm{cl}_X \Phi) \setminus V = (\mathrm{cl}_X \Phi) \setminus \Phi = (\mathrm{cl}_X \Phi) \setminus \mathrm{cl}_U \Phi = (\mathrm{cl}_X \Phi) \setminus ((\mathrm{cl}_X \Phi) \cap U) = (\mathrm{cl}_X \Phi) \setminus U \subset X \setminus U$. Hence, the pair $(H, U)$ is $N$-embedded in $X$. Q.E.D.

2.19. Theorem. a) $\mathcal{HS} \cap \mathcal{K}^* = \mathcal{N}$ and hence $\mathcal{HS} \cap T_2 \cap \mathcal{K}^* = T_4$;

b) $\mathcal{HS} = \mathcal{N} \cup (\neg \mathcal{K}^*)$ and, equivalently, $\mathcal{K}^* = \mathcal{N} \cup \neg \mathcal{HS}$;

c) If $\mathcal{P}$ is a class of spaces such that $\mathcal{HS} \cap \mathcal{P} \subseteq \mathcal{N}$, then $\mathcal{P} \subseteq \mathcal{K}^*$.

Proof. a) Obviously, $\mathcal{N} \subset \mathcal{HS} \cap \mathcal{K}^*$. Hence, we have to show that $\mathcal{HS} \cap \mathcal{K}^* \subseteq \mathcal{N}$.

Let $X \in \mathcal{HS} \cap \mathcal{K}^*$. Suppose that $X \notin \mathcal{N}$. Since $X \in \mathcal{K}^*$, there exist a nonseparable pair $(H,U)$ in $X$ and a subspace $Y$ of $X$ such that $H \subseteq Y$ and the pair $(H, U \cap Y)$ in $Y$ is $N$-embedded in $Y$. Since $X \in \mathcal{HS}$ and $\mathcal{HS} = \mathcal{LFN}$ (see 2.10), the pair $(H, U \cap Y)$ in $Y$ is $F$-embedded in $Y$. But this contradicts 2.18. So, $X \in \mathcal{N}$.

b) In a) we have shown, in fact, that $(\mathcal{K}^* \setminus \mathcal{N}) \cap \mathcal{HS} = \emptyset$. This, obviously, implies that $\mathcal{HS} \subseteq \mathcal{N} \cup (\neg\mathcal{K}^*)$. Let us prove now that $\mathcal{N} \cup (\neg\mathcal{K}^*) \subseteq \mathcal{HS}$. Since $\mathcal{N} \subset \mathcal{HS}$ (see 2.6 and 2.10 or 1.3), we have only to show that $\neg\mathcal{K}^* \subseteq \mathcal{HS}$.

Let $X \in \neg\mathcal{K}^*$ and suppose that $X \notin \mathcal{HS}$, i.e. that $X \notin \mathcal{LFN}$ (by 2.10). Then there exist a pair $(H,U)$ in $X$ and a subspace $Y$ of $X$ such that $H \subseteq Y$ and the pair $(H, U \cap Y)$ in $Y$ is not $F$-embedded in $Y$. The pair $(H,U)$ in $X$ is a nonseparable pair (otherwise the pair $(H, U \cap Y)$ in $Y$ should be $F$-embedded in $Y$) and, by 2.18, the pair $(H, U \cap Y)$ in $Y$ is $N$-embedded in $Y$. Hence, $X \in \mathcal{K}^*$ – a contradiction.

c) Let $\mathcal{P}$ be a class of spaces such that $\mathcal{HS} \cap \mathcal{P} \subseteq \mathcal{N}$. Then, by b), $\mathcal{P} \cap (\neg\mathcal{K}^*) = \mathcal{P} \cap (\mathcal{HS} \setminus \mathcal{N}) \subseteq \mathcal{N} \setminus \mathcal{N} = \emptyset$, i.e. $\mathcal{P} \subseteq \mathcal{K}^*$. Q.E.D.

2.20. Theorem. The following assertions are equivalent:

a) $\mathcal{HS} \cap T_2 \subseteq T_3$;

b) $T_2 \subseteq \mathcal{K}^*$.

Proof. a)==>b). By 2.14, the assertion a) implies that $\mathcal{HS} \cap T_2 = T_4$. Hence, by 2.19c), we obtain that $T_2 \subseteq \mathcal{K}^*$.

b)==>a) If $T_2 \subseteq \mathcal{K}^*$, then, by 2.19a), $\mathcal{HS} \cap T_2 = \mathcal{HS} \cap T_2 \cap \mathcal{K}^* =$

$= T_4 \subset T_3$ . Q.E.D.

2.21.Definition.([DIT],[O]). A topological space $X$ is called a $gF$-space if for every subset $A$ of $X$ and for every $x \in (\mathrm{cl}_X A) \setminus A$, there exists a subset $B$ of $A$ such that $\{x\} = (\mathrm{cl}_X B) \setminus B$. The class of all $gF$-spaces will be denoted by $g\mathcal{F}$.

2.22.Remark.([DIT],[O]). Obviously, every Frechet-Urysohn $T_2$-space (and hence every $T_2$-space with countable character) is a $gF$-space.

2.23.Definitions. A topological space $(X,\mathcal{T})$ is called a

i) $K$-space if for every $U \in \mathcal{T}$ and for every $x \in (\mathrm{cl}_X U) \setminus U$ there exists a subset $B$ of $U$ such that $\{x\} = (\mathrm{cl}_X B) \setminus B$;

ii) $K'$-space if every nonseparable pair $(H,U)$ in $X$ is $N$-embedded in $X$;

iii) $K''$-space if either $X \in \mathcal{N}$ or there exists a pair $(H,U)$ in $X$ which is $N$-embedded in $X$.

The class of all $K$-spaces (resp., $K'$-spaces; $K''$-spaces) will be denoted by $\mathcal{K}$ (resp., $\mathcal{K}'$; $\mathcal{K}''$).

2.24.Remark. Obviously, $g\mathcal{F} \subseteq \mathcal{K} \subseteq \mathcal{K}' \subseteq \mathcal{K}'' \subseteq \mathcal{K}^*$.

2.25.Remark. The theorem from [PR2, n.1] asserts that if $X$ is a Hausdorff countably compact space with countable character, then $X \in \mathcal{HS}$. We will show that this assertion is not true. (The fact that the proof of Theorem 1 from n.1 of [PR2] is incorrect was mentioned in MR # 88a:54020.) Indeed, J. Vaughan constructed in [V] a Hausdorff countably compact space $X_0$ with countable character which is not normal. By Remarks 2.22 and 2.24 we get that $X_0 \in \mathcal{K}^*$. So, if $X_0$ were a $HS$-space, then, by Theorem 2.19a), $X_0$ should be a normal space - a contradiction. Thus, $X_0$ is not a

$HS$-space.

2.26.Remark. Theorem 2.19a) together with Remarks 2.22 and 2.24 show that any space $X \in T_2 \setminus T_4$ with countable character is not a $HS$-space( i.e. $X$ contains a subspace $A$ for which the map $\bar{i}_A: 2^{A,T} \text{--->} 2^{X,T}$ is not continuous). Since the square $X$ of the Sorgenfrey line $L$ is such a space, we obtain that $L^2 = X \notin \mathcal{HS}$, which is the content of Theorem f) in n.3 of [PR1].

2.27.Example. A limit of an inverse sequence of $T_4$-spaces, which is not a $HS$-space.

Let $\mathbb{N}$, $\mathbb{Q}$ and $\mathbb{P}$ be the subspaces of the real line $\mathbb{R}$ (endowed with its natural topology) consisting of all natural, all rational and all irrational numbers respectively and let $X = \mathbb{R}_{\mathbb{Q}}$ be the Michael line (see [E, 5.1.32]). Then, as shown by E. Michael, the space $Y = X \times \mathbb{P}$ is not normal (see [E, 5.1.32]). Since $\mathbb{P}$ is homeomorphic to $\mathbb{N}^{\mathbb{N}}$ and $Y$ has a countable character, the standard representation of the infinite Cartesian product $X \times \mathbb{N}^{\mathbb{N}}$ as the limit of an inverse sequence (see [E, 2.5.3]) and our Theorem 2.19a) (together with 2.22 and 2.24) show that the space $Y$ is the desired example. Q.E.D.

2.28.Theorem. a) $\mathcal{FN} \cap \mathcal{K}'' = \mathcal{N}$;

b) $\mathcal{FN} = \mathcal{N} \cup (\rceil \mathcal{K}'')$;

c) If $\mathcal{P}$ is a class of spaces such that $\mathcal{FN} \cap \mathcal{P} \subseteq \mathcal{N}$, then $\mathcal{P} \subseteq \mathcal{K}''$.

Proof. The proof of this theorem can be obtained from the proof of Theorem 2.19 putting there $Y = X$. Q.E.D.

Now, we are going to present examples testifying that the first three inclusions in 2.24 are strong; as a first step in this direction we prove an auxiliary theorem (which generalizes the construction of one of the examples), using the following definition.

2.29.Definition.([A]). A topological space $X$ is called a funnel-shaped space if for every point $x$ of $X$ there exists a well ordered by inclusion local base $\mathcal{B}(x)$ at $x$.

2.30.Theorem. Let $(X,\mathcal{T})$ be a topological space which can be mapped by a continuous one-to-one map $f$ onto a Hausdorff funnel-shaped space $(Y,\mathcal{O})$ such that $\chi(y,Y)=\tau$, for every $y \in Y$, where $\tau$ is an infinite regular cardinal number. Then the space $X$ can be homeomorphically embedded as a closed nowhere dense subset of a Hausdorff $K$-space $Z$.

Proof. Obviously, there is no loss of generality in assuming that the set $Y$ coincides with the set $X$ and that $f(x)=x$ for every $x \in X$. Then $\mathcal{O} \subseteq \mathcal{T}$. Denoting by $\lambda$ the initial ordinal number of cardinality $\tau$, let $W$ be the set of all ordinal numbers less than or equal to $\lambda$.

Since the space $(Y,\mathcal{O})$ is funnel-shaped and the character of $Y$ at any point $y \in Y$ is equal to $\tau$, we can fix, for every $y \in Y$, a well-ordered by inclusion local base $\mathcal{B}(y) = \{ V_{\alpha,y} : \alpha < \lambda \}$ for $Y$ at $y$.

Let $Z$ be the Cartesian product of the sets $Y$ and $W$. We will define a topology $\mathcal{T}'$ on the set $Z$ in the following way:

a) if $(y,\alpha) \in Z$ and $\alpha \neq \lambda$ then $(y,\alpha)$ is an isolated point of $Z$;

b) for every point $z=(y,\lambda)$ of $Z$, the local base $\mathcal{B}'(z)$ for $Z$ at $z$ is the family $\{ (V_{\alpha,y} \times (\alpha,\lambda)) \cup (U \times \{\lambda\}) : \alpha < \lambda,\ y \in U \in \mathcal{T}$ and $U \subseteq V_{\alpha,y} \}$, where $(\alpha,\lambda) = \{ \beta \in W : \alpha < \beta < \lambda \}$. Then, obviously, $(Z,\mathcal{T}')$ is a Hausdorff space and the map

$i:(X,\mathcal{T}) \dashrightarrow (Z,\mathcal{T}')$, $x \dashrightarrow (x,\lambda)$, is a homeomorphic embedding and $i(X)$ is a closed nowhere dense subset of $(Z,\mathcal{T}')$. We will show that $(Z,\mathcal{T}')$ is a $K$-space.

Let $O \in \mathcal{T}'$ and $z \in (\mathrm{cl}_Z O) \setminus O$. Then, obviously, $z \in Y \times \{\lambda\}$, i.e. $z = (y,\lambda)$ for some $y \in Y$. Let us put $O_1 = O \setminus (Y \times \{\lambda\})$. We will show that $z \in \mathrm{cl}_Z O_1$. Indeed, supposing that $z \notin \mathrm{cl}_Z O_1$, we will obtain a neighbourhood $V$ of $z$ in $Z$ such that $V \cap O_1 = \emptyset$ and $V \cap O \cap (Y \times \{\lambda\}) \neq \emptyset$. Let $V' = V \cap O$. Then $V' \in \mathcal{T}'$, $V' \subset O$, $V' \neq \emptyset$ and $V' \cap O_1 = \emptyset$. Hence $V' \subset Y \times \{\lambda\} = i(X)$ and this is a contradiction since $\mathrm{Int}_Z i(X) = \emptyset$. So, $z \in \mathrm{cl}_Z O_1$.

Now, for every $\alpha < \lambda$, we put $V_\alpha = V_{\alpha,y} \times (\alpha,\lambda]$ and define a point $b_\alpha = (y_\alpha,\xi_\alpha) \in O_1$ in the following way:

1) if $\alpha = 1$, then $b_1$ is any point from $V_1 \cap O_1$;

2) let $\beta < \lambda$ and assume that $b_\alpha$ has already been defined for every $\alpha < \beta$ in such a way that $b_\alpha \in V_{\varphi(\alpha)} \cap O_1$, where $\varphi:[1,\beta) \dashrightarrow W$ is some increasing function and $b_{\alpha'} \neq b_{\alpha''}$, for $\alpha' \neq \alpha''$. We shall define the point $b_\beta$. Let us prove, first, that the set $F_\beta = \{ b_\alpha : \alpha < \beta \}$ is closed in $Z$. Indeed, since $\tau$ is a regular cardinal number, there exists a $\gamma_\beta \in W \setminus \{\lambda\}$ such that $\gamma_\beta > \xi_\alpha$ for every $\alpha < \beta$. Then, for every $z' = (y',\lambda) \in Y \times \{\lambda\}$, we have that $O_{z'} = V_{\gamma_\beta,y'} \times (\gamma_\beta,\lambda]$ is a neighbourhood of $z'$ in $Z$ and $O_{z'} \cap F_\beta = \emptyset$. Hence, $F_\beta$ is a closed subset of $Z$ and $F_\beta \subset O_1$. Since $b_\alpha = (y_\alpha,\xi_\alpha) \in V_{\varphi(\alpha)}$, we have that $\xi_\alpha > \varphi(\alpha)$ and, consequently, $\gamma_\beta > \varphi(\alpha)$ for every $\alpha < \beta$. Putting $\varphi(\beta) = \gamma_\beta$ and choosing a point $b_\beta$ from the set $V_{\varphi(\beta)} \cap O_1$, we complete the construction of the points $\{ b_\alpha : \alpha < \lambda \}$.

If we put now $B = \{ b_\alpha : \alpha < \lambda \}$ then, obviously, $z \in \mathrm{cl}_Z B$ and $B \subset O_1 \subset O$. We shall show that $B \cup \{z\} = \mathrm{cl}_Z B$. Indeed, let

$y'' \in Y \setminus \{y\}$. Then there exists a $\gamma \in W \setminus \{\lambda\}$ such that $V_{\gamma, y} \cap \cap V_{\gamma, y''} = \emptyset$ . By the construction of the points $b_\alpha$, we have that $b_\alpha \in V_\gamma \cap O_1$ for every $\alpha > \gamma$. Let $\gamma' = \sup\{ \xi_\alpha : \alpha \leq \gamma \}$. Then $\gamma' < \lambda$, $\gamma' \geq \gamma$ and $V_{\gamma', y''} \times (\gamma', \lambda]$ is a neighbourhood of $(y'', \lambda)$ in $Z$ which has no common points with $B$. Hence, $\mathrm{cl}_Z B = B \cup \{z\}$. Q.E.D.

2.31.Theorem. All of the inclusions $g\mathcal{F} \subset \mathcal{K} \subset \mathcal{K}' \subset \mathcal{K}''$ are strong even in the class of Hausdorff spaces.

Proof. A). Construction of a space $Z \in (\mathcal{K} \setminus g\mathcal{F}) \cap T_2$.

Let $X_0 = (0,0.5] \cup \{1\} \cup \{1+\frac{1}{n} : n \in \mathbb{N}\}$ with the topology of a subspace of $\mathbb{R}$ (see 2.27 for $\mathbb{N}$ and $\mathbb{R}$) and let $E$ be the equivalence relation on $X_0$ defined by letting $x E y$ iff either $x = y$ or $|x-y| = 1$. Let $X = X_0/E$ and $q: X_0 \dashrightarrow X$ be the natural quotient mapping. Then $q$ is not hereditarily quotient (= pseudo-open) mapping (see [E, 2.4.17 and 2.4.F]). Since $\chi(X_0) = \aleph_0$ we have, by 2.22, that $X_0 \in g\mathcal{F}$. These facts, together with [DIT, Theorem 3.48], show that $X \notin g\mathcal{F}$. Let $Y$ be the set $[0,0.5]$ endowed with the topology of a subspace of $\mathbb{R}$. Then the map $f: X \dashrightarrow Y$, defined by the formula $f(q(t)) = t$, for any $t \in (0,0.5]$, and $f(q(1)) = 0$, is, obviously a continuous bijection. Since $Y \in T_2$ and $\chi(y, Y) = \aleph_0$ for every $y \in Y$, we obtain, using Theorem 2.30, that there exist a Hausdorff $K$-space $Z$ and a homeomorphic embedding $\varphi: X \dashrightarrow Z$. Then $\varphi(X)$ is a subspace of $Z$ which is not a $g\mathcal{F}$-space. This implies, by [DIT, 3.46e)], that $Z \notin g\mathcal{F}$. So, $Z \in (\mathcal{K} \setminus g\mathcal{F}) \cap T_2$.

B) Construction of a space $X \in (\mathcal{K}' \setminus \mathcal{K}) \cap T_2$.

Let $X = \beta\mathbb{N}$ - the Stone-Čech compactification of $\mathbb{N}$. Then $X$ is a normal space and hence, by Definition 2.23ii), $X \in \mathcal{K}'$. We shall prove that $X \notin \mathcal{K}$. Indeed, let $U = \mathbb{N} \subset \beta\mathbb{N}$ and $x \in (\mathrm{cl}_X U) \setminus$

$\setminus U = X \setminus U$. Suppose that $X \in \mathcal{K}$. Then there exists a subset $B$ of $\mathbb{N} = U$ such that $\mathrm{cl}_X B = B \cup \{x\}$. Since, obviously, $B$ is an infinite set, we have that $\mathrm{cl}_X B$ is homeomorphic with $X = \beta\mathbb{N}$. Hence $|\mathrm{cl}_X B| > \aleph_0$ and $|\{x\}| = |(\mathrm{cl}_X B) \setminus B| > \aleph_0$ – a contradiction. So, $X \in (\mathcal{K}' \setminus \mathcal{K}) \cap T_2$.

C) Construction of a space $X \in (\mathcal{K}'' \setminus \mathcal{K}') \cap T_2$.

Let $W$ be the space of all ordinal numbers less than or equal to the first uncountable ordinal number $\omega_1$ with the usual order topology, $Z$ be the subset of $W$ consisting of all isolated points in $W$ and $\mathbb{N}$ be the space of all natural numbers with the discrete topology. Let $X = (W \times \mathbb{N}) \cup \{p\}$, where $p \notin W \times \mathbb{N}$. Endow the set $X$ with the following topology $\mathcal{T}$: all sets which are open in the space $W \times \mathbb{N}$ belong to $\mathcal{T}$; the local base $\mathcal{B}(p)$ for $X$ at the point $p$ consists of all subsets of $X$ of the form $U_{A,i} = \{p\} \cup \bigcup \{ A \times \{j\} : j \geq i\}$, where $i \in \mathbb{N}$ and $A$ is a subset of $Z$ such that $|Z \setminus A| < \aleph_1$. The space $(X, \mathcal{T})$ is, obviously, a Hausdorff space, but $(X, \mathcal{T}) \notin T_3$. Indeed, let $H = \{ (\omega_1, i) \in W \times \mathbb{N} : i \in \mathbb{N}\}$. Then $H = \mathrm{cl}_X H$ and hence $O = X \setminus H \in \mathcal{T}$. The pair $(p, O)$ is a nonseparable pair in $X$ since for every $U_{A,i} \in \mathcal{B}(p)$ we have that $(\omega_1, j) \in H \cap \mathrm{cl}_X U_{A,i}$, for every $j \geq i$, i.e. $(\mathrm{cl}_X U_{A,i}) \setminus O \neq \emptyset$. So, $X \notin T_3$.

In order to prove that $X \notin \mathcal{K}'$, it is suffices to show that the nonseparable pair $(p, O)$ is not $N$-embedded in $X$.

Let $U = U_{A,i} \in \mathcal{B}(p)$. Then $p \in U \subset O$. We will show that, for every subset $B$ of $U$, either $(\mathrm{cl}_X B) \setminus U = \emptyset$ or $(\mathrm{cl}_X B) \setminus U$ is not a subset of $X \setminus O = H$. Indeed, let $B \subset U$ and $(\mathrm{cl}_X B) \setminus U \neq \emptyset$. Then there exists a $j \geq i$ such that $|B \cap (W \times \{j\})| \geq \aleph_0$. Let $C$ be an infinite countable subset of $B \cap (W \times \{j\})$. Then there exists an $\alpha \in (\mathrm{cl}_{W \times \{j\}} C) \setminus (U \cup H) = (\mathrm{cl}_X C) \setminus (U \cup H)$. Hence, $(\mathrm{cl}_X B) \setminus U$

is not a subset of $X \setminus O$. So, the pair $(p,O)$ is not $N$-embedded in $X$ and hence $X \notin \mathcal{K}'$.

Let $U = U_{A,i} \in \mathcal{B}(p)$. We will show that the pair $(p,U)$ is not $F$-embedded in $X$. Indeed, it is enough to prove that for every $U' = U_{A',i'} \in \mathcal{B}(p)$ with $U' \subset U$, there exists a $\Phi_{U'} \subset U'$ such that $\Phi_{U'} \in 2^U \setminus 2^X$. But, obviously, if, for every $U' \in \mathcal{B}(p)$ such that $U' \subset U$, we put $\Phi_{U'} = U'$, then we will get that $\Phi_{U'} \in \in 2^U \setminus 2^X$. Hence, the pair $(p,U)$ is not $F$-embedded in $X$, which implies that $X \notin \mathcal{FN}$. Now, we get, by Theorem 2.28b), that $X \in \mathcal{K}''$. So, $X \in (\mathcal{K}'' \setminus \mathcal{K}') \cap T_2$. Q.E.D.

The Hausdorff space $X \in \mathcal{K}'' \setminus \mathcal{K}'$ which was constructed in the proof of Theorem 2.31 seems to be a natural and simple example with such properties, but it has cardinality $\aleph_1$. Now, we will describe a countable $T_2$-space $Y \in \mathcal{K}'' \setminus \mathcal{K}'$.

2.32.Example. A countable, sequential, Hausdorff space $Y$ which is a $K''$-space and is not a $K'$-space.

Proof. Denote by $AF(x)$ the Arhangel'skii-Franklin space $S_\omega$ with basic point $x$ (see [AF]). Since we use it, we shall describe its construction for the convenience of the reader.

The set $AF(x)$ is of the form $AF(x)=\oplus\{ AF_i(x) : i \in \mathbb{N} \cup \{0\}\}$, where the set $AF_i(x)$ is called the $i^{\text{th}}$ level of the set $AF(x)$, for every $i \in \mathbb{N}$. The levels $AF_i(x)$ will be constructed by induction. Put $AF_0(x) = \{x\}$. Assuming that all levels $AF_i(x)$, for $i = 0,1,2,\ldots,k$, have already been defined, we will construct the set $AF_{k+1}(x)$. With every point $y \in AF_k(x)$, we associate an infinite countable set $M_y$ (called a **sequence corresponding to** $y$) in such a way that $M_y \cap \bigcup\{ AF_i(x) : i = 0,1,\ldots,k \} = \emptyset$ and $M_y \cap \cap M_z = \emptyset$ for $y,z \in AF_k(x)$, $y \neq z$. Then we put $AF_{k+1}(x) = \bigcup \{ M_y :$

$y \in AF_k(x)$ }.

For every point $z \in AF(x)$ we denote by $\Phi_z$ the Frechet filter on the sequence $M_z$ corresponding to $z$.

The topology $\mathcal{T}_x$ on the set $AF(x)$ is defined as follows. Let $y \in AF(x)$. Then there exists a unique $i_y \in \mathbb{N} \cup \{0\}$ such that $y \in AF_{i_y}(x)$. The local base $\mathcal{B}_y$ for $AF(x)$ at the point $y$ consists of all subsets $U$ of $AF(x)$ which satisfy the following two conditions:

1) $\{y\} = U \cap \bigcup\{ AF_j(x) : j \le i_y \}$;

2) $U \cap AF_{k+1}(x) = \bigcup \{ A_z : z \in U \cap AF_k(x) \}$, for every $k \ge i_y$ (where, for every $z \in U \cap AF_k(x)$, $A_z$ is some element of $\Phi_z$).

It is easy to check that in such a way we define a topology $\mathcal{T}_x$ on the set $AF(x)$ and that the space $(AF(x),\mathcal{T}_x)$ is a countable, Hausdorff, sequential, zero-dimensional space.

Let now $\{ x_i : i \in \mathbb{N} \}$ be a sequence such that $x_i \neq x_j$ for $i \neq j$, $i,j \in \mathbb{N}$. We put $Y' = \oplus\{ AF(x_i) : i \in \mathbb{N} \}$, $Y = Y' \cup \{c\}$, where $c \notin Y'$, and $T_{i,k} = \bigcup \{ AF_s(x_i) : s \ge k \}$, for every $i,k \in \mathbb{N}$. Obviously, the sets $T_{i,k}$ are open subsets of $(AF(x_i),\mathcal{T}_{x_i})$, for every $i,k \in \mathbb{N}$. Let's introduce a topology $\mathcal{T}$ on the set $Y$. The local base $\mathcal{B}_c$ for $(Y,\mathcal{T})$ at the point $c$ consists of all subsets $U_{i,j}$, $i,j \in \mathbb{N}$, of $Y$ which have the form: $U_{i,j} = \{c\} \cup \bigcup \{ T_{m,j} : m \ge i \}$. Further, for every point $y \in Y'$ there exists a unique $i \in \mathbb{N}$ such that $y \in AF(x_i)$. Then the local base for $(Y,\mathcal{T})$ at the point $y$ coincides with the local base $\mathcal{B}_y$ for $(AF(x_i),\mathcal{T}_{x_i})$ at $y$. It is easy to see that in such a way we define a topology $\mathcal{T}$ on $Y$ and that the space $(Y,\mathcal{T})$ is a countable, Hausdorff, sequential space. We are going to show that $Y \in \mathcal{K}'' \setminus \mathcal{K}'$.

Let us first prove that $Y \in \mathcal{K}''$.

Put $H = \{ x_i : i \in \mathbb{N} \}$. Then the pair $(H, Y')$ in $Y$ is not $F$-embedded in $Y$. Indeed, let $V = \bigcup \{ U_i : i \in \mathbb{N} \}$, where $U_i \in \mathcal{B}_{x_i}$ for every $i \in \mathbb{N}$. Putting $\Phi = V$, we obtain that $\Phi \in 2^{Y'}$, $\Phi \subseteq V$ and $\Phi \notin 2^Y$. Since the open sets like $V$ form a local base for $(Y,\mathcal{T})$ at the set $H$, we get that the pair $(H, Y')$ is not $F$-embedded in $Y$. Hence, $Y \notin \mathcal{FN}$. This implies, by 2.28b), that $Y \in \mathcal{K}''$.

Next, let us show that $Y \notin \mathcal{K}'$.

Put $O = Y \setminus H$. Then the pair $(c,O)$ in $Y$ is nonseparable one since any pair of neighbourhoods of $c$ and $H$ has nonvoid intersection. Further, the pair $(c,O)$ in $Y$ is not $N$-embedded in $Y$. Indeed, put $V = U_{1,2} \in \mathcal{B}_c$. Then $c \in V \subset O$. We will show that there is no subset $B_V$ of $V$ such that $\emptyset \neq (\mathrm{cl}_Y B_V) \setminus V \subseteq H$. For proving this, consider a subset $B$ of $V$ such that $(\mathrm{cl}_Y B) \setminus V \neq \emptyset$. We have that $(\mathrm{cl}_Y B) \setminus V \subset Y \setminus V = H \cup \bigcup\{ AF_1(x_i) : i \in \mathbb{N} \}$. Suppose that $(\mathrm{cl}_Y B) \setminus V \subseteq H$ and let $x_i \in (\mathrm{cl}_Y B) \setminus V$. Then, for every $y \in AF_1(x_i)$, there exists a $U_y \in \mathcal{B}_y$ such that $U_y \cap B = \emptyset$. Let $W = \{x_i\} \cup \bigcup\{ U_y : y \in AF_1(x_i) \}$. Then $W$ is a neighbourhood of $x_i$ in $Y$ and, hence, $W \cap B \neq \emptyset$.. But $W \cap B \subseteq \{x_i\}$ and $x_i \notin B$ since $x_i \notin V$. Hence, $W \cap B = \emptyset$. This is a contradiction, showing that $(\mathrm{cl}_Y B) \setminus V \subseteq H$. So, the nonseparable pair $(c,O)$ in $Y$ is not $N$-embedded in $Y$. This implies that $Y \notin \mathcal{K}'$. Q.E.D.

**2.33. Example.** A Hausdorff non-normal space $(Z,\mathcal{T}) \in \mathcal{K}''$ such that, for each pair $(H,U)$ in $Z$, every local base $\mathcal{B}_H$ for $Z$ at $H$ has non-clopen in $U$ elements (in contrast with the Hausdorff spaces $X$ and $Y$ constructed in the part C) of the proof of 2.31 and in 2.32, respectively).

Let $\mathcal{T}'$ be the natural Euclidean topology on the real line $\mathbb{R}$ and $\mathcal{T}''$ be the cocountable topology on $\mathbb{R}$. Let $Z$ coincides with $\mathbb{R}$ as a set and $\mathcal{T}$ be the suprema of $\mathcal{T}'$ and $\mathcal{T}''$. We will show that $(Z,\mathcal{T})$ is the desired example.

It is easy to see that: 1) a set $O$ is open in $(Z,\mathcal{T})$ iff $O = U \setminus A$, where $U \in \mathcal{T}'$ and $|A| \le \aleph_0$, and 2) if $O = U \setminus A$, where $U \in \mathcal{T}'$ and $|A| \le \aleph_0$, then $\mathrm{cl}_{(Z,\mathcal{T})}O = \mathrm{cl}_{(\mathbb{R},\mathcal{T}')}U$ (see [SS, Example 63]). Using these two facts and the local connectedness of $(\mathbb{R},\mathcal{T}')$, one easily realizes that $(Z,\mathcal{T})$ has the desired local base property described above. Since $(Z,\mathcal{T})$ is obviously a Hausdorff space and $(Z,\mathcal{T}) \notin T_3$ (see [SS]), we have only to prove that $Z \in \mathcal{K}''$. We will show that the pair $(\sqrt{2},\mathbb{P})$, where $\mathbb{P}$ is the set of irrationals (see 2.27 for the notations), is not $F$-embedded in $(Z,\mathcal{T})$, which, by 2.18, will imply that $(\sqrt{2},\mathbb{P})$ is $N$-embedded in $(Z,\mathcal{T})$, i.e. that $Z \in \mathcal{K}''$. For doing this, it is enough to prove that for every $n \in \mathbb{N}$ and for every countable set $A \subset Z$ such that $\mathbb{Q} \cap V_n \subset A \subset V_n$, where $V_n = (\sqrt{2}-\frac{1}{n}, \sqrt{2}+\frac{1}{n})$, there exists a subset $\Phi$ of $V_n \setminus A$ which is closed in $(\mathbb{P},\mathcal{T}|\mathbb{P})$, but which is not closed in $(Z,\mathcal{T})$. So, let $n \in \mathbb{N}$ and $V_n \cap \mathbb{Q} \subset A$. If $A \cap \mathbb{P}$ is not dense in $(V_n,\mathcal{T}'|V_n)$, then we can find a closed interval $[r_1,r_2]$, where $r_1$, $r_2 \in \mathbb{Q}$, such that $\mathbb{P} \cap [r_1,r_2] \subset V_n \setminus A$, and put $\Phi = [r_1,r_2] \cap \mathbb{P}$. This $\Phi$ will do the job (see 2) above).

Let now $A \cap \mathbb{P}$ be dense in $(V_n,\mathcal{T}'|V_n)$. Then $|A \cap \mathbb{P}| = \aleph_0$, so we can let: $A \cap \mathbb{P} = \{ a_i : i \in \mathbb{N} \}$. To obtain the subset $\Phi$ of $V_n \setminus A$ under question, we exploit a construction similar to that of the Cantor set. First of all, let $l_0$, $r_0 \in \mathbb{Q}$ be such that $\sqrt{2} - \frac{1}{2n} < l_0 < \sqrt{2} < r_0 < \sqrt{2} + \frac{1}{2n}$. Then we start with $a_1$ and find $l_1$, $r_1 \in \mathbb{Q}$ such that $l_0 < l_1 < a_1 < r_1 < r_0$ and $l_1 - l_0 < \frac{1}{3n}$, $r_0 - r_1 < \frac{1}{3n}$. We put $F_1 = [l_0,l_1] \cup [r_1,r_0]$. Let $i_2 = \min\{ i \in \mathbb{N} : a_i \notin (l_1,r_1)\}$, $i_{2,1} = \min\{ i \in \mathbb{N} : a_i \in (l_0,l_1)\}$

and $i_{2,2} = \min\{ i \in \mathbb{N} : a_i \in (r_1, r_0) \}$. Then $i_2 \in \{ i_{2,1}, i_{2,2} \}$.

Obviously, there exist $l_{2,1}, r_{2,1}, l_{2,2}, r_{2,2} \in \mathbb{Q}$ such that $l_0 < l_{2,1} < a_{i_{2,1}} < r_{2,1} < l_1$, $r_1 < l_{2,2} < a_{i_{2,2}} < r_{2,2} < r_0$ and $l_{2,1} - l_0 < \frac{1}{3^2 n}$, $l_1 - r_{2,1} < \frac{1}{3^2 n}$, $l_{2,2} - r_1 < \frac{1}{3^2 n}$, $r_0 - r_{2,2} < \frac{1}{3^2 n}$. We put $F_2 = [l_0, l_{2,1}] \cup [r_{2,1}, l_1] \cup [r_1, l_{2,2}] \cup [r_{2,2}, r_0]$. Further, we define $F_j$, for every $j \in \mathbb{N}$, in the similar way. Put $\Phi' = \bigcap \{ F_j : j \in \mathbb{N} \}$. Then $\Phi'$ is homeomorphic to the Cantor set and, hence, $|\Phi'| = 2^{\aleph_0}$. Thus $\Phi = \Phi' \cap \mathbb{P}$ is a nonvoid closed subset of $(\mathbb{P}, \mathcal{T}|\mathbb{P})$ and $\Phi \subset V_n \setminus A$. It is well known that every point of the Cantor set is a complete accumulation point of it. Hence, $l_0 \in \mathrm{cl}_{(Z,\mathcal{T})}\Phi$. Thus $\Phi$ is not closed in $(Z,\mathcal{T})$. So, we proved that $(Z,\mathcal{T}) \in \mathcal{K}''$. Q.E.D.

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