Theory and applications of convex VSIO*

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Abstract

This presentation deals with vector semi-infinite optimization problems which are defined by finitely many objective functions whose variables belong to a finite-dimensional space and whose feasible sets are defined by infinitely many inequality constraints. The objective is to show several fields of applications of vector semi-infinite optimization as well as some theoretical results in convex vector semi-infinite optimization (characterization of solutions, necessary and sufficient optimality conditions).

1 Introduction

This paper deals with vector (multiobjective) semi-infinite optimization problems which are defined by finitely many objective functions whose variables belong to a finite-dimensional space and whose feasible sets are defined by infinitely many inequality constraints. There is a wide range of applications of semi-infinite optimization and of vector optimization; both topics, their theory and numerical analysis, became very active research areas in the recent two decades. We refer to several recent books [5, 12, 13]; in particular to the standard books [4, 14] on vector optimization.

As a starting point of this work we consider a vector semi-infinite optimization (VSIO) problem of the form

$$\begin{array}{ll} \text{VSIO:} & \text{" min "}\ f(x) = (f_1(x), ..., f_p(x)) \\ & \text{s.t. } x \in M \end{array}$$

where $M = \{x \in \mathbb{R}^n \mid g(x,t) \leq 0, t \in T\}$, the set T is a compact infinite subset of $\mathbb{R}^m$, $f : \mathbb{R}^n \to \mathbb{R}^p$, $g : \mathbb{R}^n \times T \to \mathbb{R}$.

The objective is to show several fields of applications of vector semi-infinite optimization as well as some theoretical results in convex vector semi-infinite optimization (characterization of solutions, necessary and sufficient optimality conditions). In Section 2 we will show several fields of applications which can be modeled as a problem of type VSIO. In Section 3 we will present some recent results of the authors concerning the theory of convex VSIO.

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2 Applications

2.1 Robotics

As far as we know up to now, there are publications concerning standard (only one objective function) semi-infinite optimization in robotics and, on the other hand, some applications of standard (finitely many constrains) vector optimization models applied to robotics. It means it is time to consider vector semi-infinite optimization models in robotics, which will be an useful and interesting field of applications.

Concerning standard semi-infinite optimization in robotics we mention [9, 11, 16] and for standard vector optimization models applied to robotics, see e.g. [1, 2, 15].

In [9] design centering is used to determine lower bounds for the volumen of a complicated container set by inscribing ellipsoids in the *maneuverabilty problem of a robot* from [8].

In first instance, the problem of steering a robot in an optimal way is typically an optimal control problem. Restricting state and/or control functions to finite dimensional spaces, the number of variables becomes finite. However, restrictions on the movement typically are required in a whole time-interval or for all points of the robot, leading to semi-infinite programming problems. In [11] such models are discussed for the problem of optimally re-parametrizing given paths and the maneuverability problem.

How robot trajectory planning can be formulated as a semi-infinite optimization problem is shown in [16], where two of the robotics trajectory planning problems are formulated, namely, *Model 1,* where constants bounds on velocity, acceleration and jerk on each joint are considered, and *Model 2*, where constants bounds on each robot torque are imposed.

Some applications that consider vector optimization models in robotics have been done. For instance, such approach is proposed in [1, 2] in order to enhance the design of service robots. The proposed procedure has been applied to a robotic arm for service tasks, where several objective functions are considered, namely, robot reach, position workspace volume, orientation work space volume, path planning, lightweight design, stiffness, safety. Also, an application of vector optimization in order to estimate a solution to the multi-robot dynamic task allocation problem is presented in [15].

2.2 Simultaneous Chebyshev best approximation (see e.g. [10])

Necessary conditions for efficient solutions in simultaneous Chebyshev best approximation, derived from an abstract characterization theory of efficiency, are obtained in [3]. Let $\left\{\psi_i^{(k)}(\alpha)\right\}_{i=1}^{n}$, $k = 1, 2, \ldots, p$ $(p > 1)$, be p families of real-valued continuous functions on the interval $[a, b]$. Let $\psi_0^{(k)}(\alpha)$, $k = 1, 2, \ldots, p$, be p given real valued continuos functions on $[a, b]$. For $k = 1, 2, \ldots, p$ define the following functions

$$f_k(x) = \left\| \psi_0^{(k)} - \sum_{i=1}^{n} x_i \psi_i^{(k)} \right\|_\infty = \max_{\alpha\in[a,b]} \left| \psi_0^{(k)}(\alpha) - \sum_{i=1}^{n} x_i \psi_i^{(k)}(\alpha) \right|.$$

The simultaneous Chebyshev best approximation problem is to find $x = (x_1, \dots, x_n)^\top \in \mathbb{R}^n$ that will solve the non-differentiable multiobjective optimization problem

$$\text{"min"}\ (f_1(x), \dots, f_p(x)) \ \text{ subject to } x \in Q \subset \mathbb{R}^n, \tag{1}$$

where Q stands for the feasible set of the problem. The epigraph reformulation of (1) yields the equivalent problem

$$\text{"}\min_{(x,q)\in Q\times\mathbb{R}^p}\text{"}\ (q_1, \dots, q_p) \ \text{ s. t. } \left\| \psi_0^{(k)} - \sum_{i=1}^{n} x_i \psi_i^{(k)} \right\|_\infty \le q_k, \ k = 1, \dots, p,$$

which can be written as the following differentiable multiobjective semi-infinite optimization problem

$$\begin{array}{lll} \text{"}\min_{(x,q)\in Q\times\mathbb{R}^p}\text{"}\ (q_1, \dots, q_p) & \text{s.t.} & \psi_0^{(k)}(\alpha) - \sum_{i=1}^{n} x_i \psi_i^{(k)}(\alpha) \le q_k, \\ & & k = 1, \dots, p, \\ & & \text{for all } \alpha \in [a, b], \\ & & -\psi_0^{(k)}(\alpha) + \sum_{i=1}^{n} x_i \psi_i^{(k)}(\alpha) \le q_k, \\ & & k = 1, \dots, p, \\ & & \text{for all } \alpha \in [a, b]. \end{array}$$

2.3 Robust linear vector optimization (see e.g. [6])

Let

$$P_0 : \text{"min"}\ \ f(x) = \left(c_1' x, ..., c_p' x\right) \ \ \text{s.t. } a_i' x \ge b_i, i = 1, ..., q,$$

be a linear vector optimization problem with uncertain data.

Firstly, we assume that the uncertainty is confined to the cost vectors, i.e., the constraints remain fixed. Let c_i range on certain set $C_i \subset \mathbb{R}^n$, $i = 1, ..., p$. The robust perspective corresponds to a pessimistic decision maker, for which the cost vector of deciding $x \in \mathbb{R}^n$ such that $a_i' x \ge b_i, i = 1, ..., q$, will be worst possible, i.e., $\left(\max_{c_1\in C_1} c_1' x, ..., \max_{c_p\in C_p} c_p' x\right)$. Then, the robust counterpart of P_0 is the linear VSIO problem,

$$\begin{array}{lll} P_1 : & \text{"min"} & f(x, z) = (z_1, ..., z_p) \\ & \text{s.t.} & z_j \ge c_j' x, c_j \in C_j, j = 1, ..., p, \\ & & a_i' x \ge b_i, i = 1, ..., q, \end{array}$$

whose decision space is $\mathbb{R}^{n+p}$ (so, the initial problem is embedded into a higher dimensional decision space and the number of constraints turns out to be infinite). If $\{C_j, j = 1, ..., p\}$, is a family of compact sets, then P_1 is a particular case of VSIP.

Secondly, let us assume that the uncertain data in P_0 are the coefficient vectors $(a_i, b_i), i = 1, ..., q$. Let us assume that each vector (a_i, b_i) ranges on a given set $S_i \subset \mathbb{R}^{n+1}$, $i = 1, ..., q$, whereas c remains fixed. The robust approach tries to guarantee the feasibility of the selected decision under any conceivable circumstance. Thus, we must solve the linear VSIO problem

$$P_2 : \text{"min"} \quad f(x) = \left(c_1' x, ..., c_p' x\right) \quad \text{s.t. } a'x \geq b, (a, b) \in \bigcup_{i=1}^{q} S_i,$$

whose decision space coincides with the initial one. If $S_1, ..., S_q$ are compact, then P_2 is a particular case of VSIP.

Finally, we assume that all the data in P_0 are uncertain. Combining the previous arguments we get the following robust counterpart of P_0 :

$$\begin{array}{lll} P_3 : & \text{"min"} & f(x, z) = (z_1, ..., z_p) \\ & \text{s.t.} & z_j \geq c_j' x, c_j \in C_j, j = 1, ..., p, \\ & & a'x \geq b, (a, b) \in \bigcup_{i=1}^{q} S_i. \end{array}$$

If the sets $C_1, ..., C_p$, $S_1, ..., S_q$ are compact, then P_3 is a particular case of VSIP.

2.4 Convex vector optimization (see [6], [7])

Consider the vector optimization problem

$$P_0 : \text{"min"} \quad h(x) = (h_1(x), ..., h_p(x)) \quad \text{s.t. } g(x, t) \leq 0, t \in T,$$

where the functions h_i, $i = 1, \ldots, p$ and $g(\cdot, t)$ are proper, lower semicontinuous, and convex from $\mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$, whereas the cardinality of T is irrelevant. It is possible to reformulate P_0 as a linear VSIO problem by means of the Fenchel conjugates of the involved functions.

The *effective domain* of $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ is $\operatorname{dom} f = \{x \in \mathbb{R}^n : f(x) < +\infty\}$. The *conjugate function* of f, $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$, is defined by

$$f^*(v) = \sup\{\langle v, x\rangle - f(x) : x \in \operatorname{dom} f\}.$$

It is well-known that, if f is a proper lower semicontinuous convex function, then f^* enjoys the same properties and its conjugate, denoted by $f^{**} : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ and defined by

$$f^{**}(x) = \sup\{\langle v, x\rangle - f^*(v) : v \in \operatorname{dom} f^*\},$$

coincides with f.

From the assumptions on P_0 we get two consequences:
1st, each objective function is the supremum of affine functions because

$$h_i(x) = h_i^{**}(x) = \sup_{u \in \text{dom } h_i^*} \{x'u - h_i^*(u)\}, i = 1, ..., p.$$

2nd, since all the constraints in P_0 can be expressed as $g_t(x) \leq 0$ $(g_t(x) := g(x,t))$, and $g_t^{**} = g_t$, we have

$$\begin{array}{l} g_t(x) \leq 0 \iff g_t^{**}(x) \leq 0 \\ \iff x'u - g_t^*(u) \leq 0 \; \forall u \in \text{dom} g_t^* \\ \iff u'x \leq g_t^*(u) \; \forall u \in \text{dom} g_t^*. \end{array}$$

Thus P_0 is equivalent to the linear VSIO problem, whose decision space is $\mathbb{R}^{n+p}$:

$$\begin{array}{lll} P_1 : & \text{"min"} & f(z,x) = (z_1, ..., z_p) \\ & \text{s.t.} & -z_i + u'x \leq h_i^*(u), \; u \in \text{dom } h_i^*, i = 1, ..., p, \\ & & u'x \leq g_t^*(u), \; (u,t) \in \text{dom} g_t^* \times T, \end{array} .$$

2.5 Portfolio management (see e.g. [6], [7])

We can invest a capital M in n shares. For $i \in \{1, ..., n\}$, we denote by x_i the amount to be invested in the $i-$th share, and by r_i its rate of return. Obvious constraints are $\sum_{i=1}^{n} x_i = M$ and $x_i \geq 0$, $i = 1, ..., n$. We express these, and possibly other linear constraints, in a compact way as $a_i'x \geq b_i, i = 1, ..., q$. In the (unrealistic) absence of uncertainty, the problem to be solved is the linear optimization one

$$P_0 : \max r'x \text{ s.t. } a_i'x \geq b_i, i = 1, ..., q.$$

Unfortunately, r is in practice an uncertain vector. The uncertain problem P_0 can be modeled in a variety of ways, taking into account that the decision maker intends to maximize its return at a minimum risk. If the probability distribution of r is unknown, the first objective for a pessimistic decision maker consists of maximizing $\inf_{r \in R} r'x$ (or, equivalently, minimizing $\sup_{r \in R}(-r'x)$), where $R \subset \mathbb{R}^n$ denotes the set of conceivable values of r. Concerning the risk, it is usually identified with the variance of the portfolio x, i.e., the uncertain number $x'Vx$, where V denotes the (positive definite) matrix of variances-covariances of r. So, the second objective consists of minimizing the quadratic convex function $h(x) := x'Vx$. Consequently, we have a biobjective convex optimization problem that can be reformulated as a linear VSIO problem as in the latter subsection. Indeed, since $h^*(u) = \frac{1}{4}u'V^{-1}u$ for all $u \in \mathbb{R}^n$, the equivalent problem is P_1, whose decision space is $(z_1, z_2, x) \in \mathbb{R}^{n+2}$:

$$\begin{array}{lll} P_1 : & \text{"min"} & f(z,x) = (z_1, z_2) \\ & \text{s.t.} & r'x + z_1 \geq 0, r \in R, \\ & & -u'x + z_2 \geq -\frac{1}{4}u'V^{-1}u, u \in \mathbb{R}^n, \\ & & a_i'x \geq b_i, i = 1, ..., q, \end{array} .$$

2.6 Functional approximation (see [6], [7])

Let $h, v_1, ..., v_n$ be Riemann integrable functions on $T = [\alpha, \beta]$, $\alpha < \beta$. Consider the problem consisting of computing a "good" approximation to h from above by means of linear combinations of $v_1, ..., v_n$, but it is not obvious how to measure the approximation error. If we choose the L_∞ and the L_1 norms, the problem consists of the simultaneous minimization of the L_∞ and the L_1 errors. If we consider a linear combination $\sum_{i=1}^{n} v_i(t)x_i$ such that $h(t) \leq \sum_{i=1}^{n} v_i(t)x_i$, for all $t \in T$ (a feasible decision of the approximation problem), the corresponding L_∞ and the L_1 errors are

$$\left\| h - \sum_{i=1}^{n} x_i v_i \right\|_\infty = \max_{t \in T} \left\{ \sum_{i=1}^{n} x_i v_i(t) - h(t) \right\}$$

and

$$\begin{aligned} \left\| h - \textstyle\sum_{i=1}^{n} x_i v_i \right\|_1 &= \textstyle\int_\alpha^\beta \left[\sum_{i=1}^{n} v_i(t) x_i - h(t) \right] dt \\ &= \textstyle\sum_{i=1}^{n} \left(\int_\alpha^\beta v_i(t) dt \right) x_i - \int_\alpha^\beta h(t) dt, \end{aligned}$$

respectively. So, we have to solve

$$\begin{aligned} P_1: \quad \text{"min"} \quad & f(x, x_{n+1}) = \left(x_{n+1}, \textstyle\sum_{i=1}^{n} \left(\int_\alpha^\beta v_i(t) dt \right) x_i \right) \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^{n} v_i(t) x_i \geq h(t), t \in T, \\ & x_{n+1} \geq h(t) - \textstyle\sum_{i=1}^{n} v_i(t) x_i \geq -x_{n+1}, \ t \in T. \end{aligned}$$

The approximating function $\sum_{i=1}^{n} v_i(t)x_i$ can be forced to satisfy conditions such as the requirement to be non-decreasing or convex on T, that can be expressed through the linear systems $\left\{ \sum_{i=1}^{n} \frac{dv_i}{dt} x_i \geq 0, t \in T \right\}$ and $\left\{ \sum_{i=1}^{n} \frac{d^2 v_i}{dt^2} x_i \geq 0, t \in T \right\}$ (assuming that $v_1, ..., v_n \in \mathcal{C}^1(T)$ and $v_1, ..., v_n \in \mathcal{C}^2(T)$, respectively).

References

[1] Castejon, C., Carbone, G., Garcia-Prada, J.C., Ceccarelli, M., Computational multi-objective optimization to design service robots, Proceedings of EUCOMES 08 (2008) 139-147.

[2] Castejon, C., Carbone, G., Garcia-Prada, J.C., Ceccarelli, M., A multiobjective optimization of a robotic arm for service tasks, Strojniški vestnik - Journal of Mechanical Engineering 56 (2010) 5, 316-329.

[3] Censor, Y., Necessary conditions for Pareto optimality in simultaneous Chebyshev best approximation, J. Approx. Theory 27 (2), (1979) 127-134.

[4] Ehrgott, M.: Multicriteria Optimization (2nd ed). Springer-Verlag, Berlin (2005).

[5] Goberna, M.A., López, M.A. (eds.) Semi-infinite Programming - Recent Advances. Kluwer, Boston (2001).

[6] Goberna, M.A., Guerra-Vazquez, F., Todorov, M.I., Constraint qualifications in linear vector semi-infinite optimization, Eur. J. Oper. Res. 227 (2013) 12–21.

[7] Goberna, M.A., Guerra-Vazquez, F., Todorov, M.I., Constraint qualifications in convex vector semi-infinite optimization, Eur. J. Oper. Res. 249 (2016) 32–40.

[8] Graettinger, T.J., Krogh, B.H., The acceleration radius: a global performance measure for robotic manipulators, IEEE J. Robot. Automat. 4 (1988) 60–69.

[9] Guerra-Vazquez, F., Rückmann, J.-J., Stein, O., Still, G., Generalized semi-infinite programming: A tutorial, J. Comput. Appl. Math. 217 (2008) 394-419.

[10] Guerra-Vazquez, F., Rückmann, J.-J., On proper efficiency in multiobjective semi-infinite optimization, in: Xu, H. et al. (eds.), Optimization and Control Techniques and Applications. Springer, Berlin (2014) 115-135.

[11] Hettich, R., Still, G., Semi-infinite programming models in Robotics, in Guddat et al. (eds.), Parametric Optimization and Related Topics II, Akademie Verlag, Berlin (1991).

[12] Polak, E.: Optimization. Algorithms and Consistent Approximations. Springer, New York (1997).

[13] Reemtsen, R., Rückmann, J.-J. (eds.) Semi-Infinite Programming. Kluwer, Boston (1998).

[14] Sawaragi, Y., Nakayam, H., Tanino, T.: Theory of multiobjective optimization. Burlington, MA : Elsevier (1985).

[15] Tolmidis, A. Th., Petrou, L., Multi-objective optimization for dynamic allocation in a multi-robot system, Engineering Applications of Artificial Intelligence, Vol. 26, I. 5-6, (2013), 1458-1468.

[16] Vaz, A.I.F., Fernandes, E.M.G.P., Gomes, M.P.S.F., Robot trayectory planning with semi-infinite programming, Eur. J. Oper. Res. (02/2004), DOI: 10.1016/S0377-2217(03)00266-2.