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Review

Series in Le Roy Type Functions: A Set of Results in the Complex Plane—A Survey

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Abstract: This study is based on a part of the results obtained in the author's publications. An enumerable family of the Le Roy type functions is considered herein. The asymptotic formula for these special functions in the cases of 'large' values of indices, that has been previously obtained, is provided. Further, series defined by means of the Le Roy type functions are considered. These series are studied in the complex plane. Their domains of convergence are given and their behaviour is investigated 'near' the boundaries of the domains of convergence. The discussed asymptotic formula is used in the proofs of the convergence theorems for the considered series. A theorem of the Cauchy–Hadamard type is provided. Results of Abel, Tauber and Littlewood type, which are analogues to the corresponding theorems for the classical power series, are also proved. At last, various interesting particular cases of the discussed special functions are considered.

Keywords: Le Roy functions and series in them; inequalities; asymptotic formula; convergence of power and functional series in complex plane; Cauchy–Hadamard, Abel, Tauber and Littlewood type theorems

MSC: 30D20; 33E12; 30A10; 40E10; 30D15; 40A30; 40G10; 40E05

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1. Introduction

In two recent papers, S. Gerhold [1] and, independently, R. Garra and F. Polito [2] introduced the new special function

$$F_{\alpha,\beta}^{(\gamma)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{[\Gamma(\alpha k+\beta)]^{\gamma}}, \quad z \in \mathbb{C}, \tag{1}$$

for complex values of the variable z and values of parameters $\alpha > 0$, $\beta > 0$, $\gamma > 0$. On a later stage its definition is extended by R. Garrappa, S. Rogosin and F. Mainardi [3] under more general conditions for the parameters. However, making sure that the coefficients $[\Gamma(\alpha k+\beta)]^{-\gamma}$ in the Expansion (1) exist, the values of the parameters have to be restricted. A natural restriction in this direction would be the following:

$$\alpha,\ \beta \in \mathbb{C}, \quad \gamma > 0. \tag{2}$$

As is established in [3], this function turns out to be an entire function of the complex variable z for all values of the parameters such that

$$\Re(\alpha) > 0,\ \beta \in \mathbb{C},\ \gamma > 0. \tag{3}$$

Actually, this function has been recently considered in [1–6] from various points of view. Some of its important properties can be seen therein. For example, different asymptotic formulae can be found in S. Gerhold [1] and R. Garrappa, S. Rogosin, F. Mainardi [3], for complete monotonicity see K. Gorska, A. Horzela, R. Garrappa [4] and T. Simon [5]. For studying its properties in relation to some integro-differential operators involving

Hadamard fractional derivatives or hyper-Bessel-type operators see Garra-Polito [2], different integral representations can be seen in [3] and Pogány [6].

The function (1) is a natural generalization of the so-called Le Roy function

$$F^{(\gamma)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{[\Gamma(k+1)]^{\gamma}} = \sum_{k=0}^{\infty} \frac{z^k}{[k!]^{\gamma}}, \quad z \in \mathbb{C},\ \gamma \in \mathbb{C}, \tag{4}$$

which was named after the great French mathematician Édouard Louis Emmanuel Julien Le Roy (1870–1954), and probably for that reason the authors of [3] use the name Le Roy type function for the function $F^{(\gamma)}_{\alpha,\beta}$.

Keeping with this, and for the sake of brevity, we often use in this paper the name Le Roy type function for the function $F^{(\gamma)}_{\alpha,\beta}$, defined by (1). In this paper, considering the Le Roy type functions (1), we discuss various earlier results which are needed here. These are results related to inequalities in the complex plane $\mathbb{C}$ and on its compact subsets and asymptotic formula for 'large' values of indices of the functions (1). Further, considering series in such a kind of functions, we provide results for their domains of convergence and investigate their behaviour 'near' the boundaries of their domains of convergence.

In the series of papers [7–10], as well as in the recent book [11], we studied series in systems of some representatives of the special functions of fractional calculus, which are fractional index analogues of the Bessel functions and also multi-index Mittag–Leffler functions (in the sense of [12–15]), and we have proved various results connected with their convergence in the complex domains.

2. Inequalities and an Asymptotic Formula

For our purpose we consider the family

$$F^{(\gamma)}_{\alpha,n}(z) = \sum_{k=0}^{\infty} \frac{z^k}{[\Gamma(\alpha k+n)]^{\gamma}}, \quad z \in \mathbb{C};\ n \in \mathbb{N},\ \alpha > 0,\ \gamma > 0, \tag{5}$$

where $\mathbb{N}$ means the set of positive integers.

We are going to deal with some analytical transformations of the function (5) for each value of the parameter n. The following results hold true (for the formulation and proof see Paneva-Konovska [16]).

Lemma 1. *Let $z \in \mathbb{C}$, $\alpha > 0$, $\gamma > 0$, $n \in \mathbb{N}$ and let $K \subset \mathbb{C}$ be a nonempty compact set. Then there exists an entire function $\vartheta^{(\gamma)}_{\alpha,n}$ such that*

$$F^{(\gamma)}_{\alpha,n}(z) = \frac{1}{[\Gamma(n)]^{\gamma}}\,(1 + \vartheta^{(\gamma)}_{\alpha,n}(z)). \tag{6}$$

The entire function $\vartheta^{\gamma}_{\alpha,n}$ satisfies the following inequality

$$\left|\vartheta^{(\gamma)}_{\alpha,n}(z)\right| \le \frac{[\Gamma(\alpha+1)]^{\gamma}\,[\Gamma(n)]^{\gamma}}{[\Gamma(\alpha+n)]^{\gamma}} \left(F^{(\gamma)}_{\alpha,1}(|z|) - 1\right), \quad z \in \mathbb{C}, \tag{7}$$

Moreover there exists a positive constant $C = C(K)$, such that

$$\max_{z\in K}\left|\vartheta^{(\gamma)}_{\alpha,n}(z)\right| \le C\,\frac{[\Gamma(n)]^{\gamma}}{[\Gamma(\alpha+n)]^{\gamma}}, \tag{8}$$

for all the positive integers n.

Theorem 1. *Let $z \in \mathbb{C}$; $n \in \mathbb{N}$, $\alpha > 0$, $\gamma > 0$. Then the Le Roy type functions $F_{\alpha,n}^{(\gamma)}$ have the following asymptotic formula*

$$F_{\alpha,n}^{(\gamma)}(z) = \frac{1}{[\Gamma(n)]^{\gamma}}\,(1 + \vartheta_{\alpha,n}^{(\gamma)}(z)), \quad \vartheta_{\alpha,n}^{(\gamma)}(z) \to 0 \text{ as } n \to \infty. \tag{9}$$

The convergence is uniform in the nonempty compact subsets of the complex plane.

The results above allow us to write the next two remarks.

Remark 1. *According to the asymptotic Formula (9), it follows that there exists a natural number M such that the functions $[\Gamma(n)]^{\gamma}F_{\alpha,n}^{(\gamma)}(z)$ do not vanish for any n great enough (say $n > M$).*

Remark 2. *Note that each function $F_{\alpha,\,n}^{(\gamma)}(z)$ $(n \in \mathbb{N})$, being an entire function, no identically zero, has at most finite number of zeros in the closed and bounded set $|z| \le R$ ([17], p. 305). Moreover, because of Remark 1, at most finite number of these functions have some zeros.*

3. Series in Le Roy Type Functions

For the sake of simplicity, we introduce an auxiliary family of functions, related to the Le Roy type functions, adding $\widetilde{F}_{\alpha,0}^{(\gamma)}(z)$ just for completeness, namely:

$$\widetilde{F}_{\alpha,0}^{(\gamma)}(z) = 1,\ \widetilde{F}_{\alpha,n}^{(\gamma)}(z) = z^n\,[\Gamma(n)]^{\gamma}\,F_{\alpha,n}^{(\gamma)}(z),\ n \in \mathbb{N};\ \alpha > 0,\ \gamma > 0, \tag{10}$$

and we study the series with complex coefficients a_n ($n \in \mathbb{N}_0$, i.e., $n = 0, 1, 2, ...$) in these functions for $z \in \mathbb{C}$, namely:

$$\sum_{n=0}^{\infty} a_n\,\widetilde{F}_{\alpha,n}^{(\gamma)}(z). \tag{11}$$

Our major goal is to study the convergence of the series (11) in the complex plane. We give results, corresponding to the classical Cauchy–Hadamard theorem and Abel lemma for the power series and more precise results, giving the behaviour of the series 'near' the boundary of the domain of convergence, as well. Such kind of results may be useful for studying the solutions of some fractional order differential and integral equations, expressed in terms of series (or series of integrals) in special functions of the type (10) and their special cases (as for example in Kiryakova et al., in [18]—for the Mittag–Leffler functions; in [19]—for the hyper-Bessel functions; in [14,20]—for the multi-index Mittag–Leffler functions). Convergence theorems are also obtained for series in other special functions, for example, for series in Laguerre and Hermite polynomials (the results are obtained in a number of publications and they can be seen in Rusev [21]), and respectively by the author for series in Bessel and Mittag–Leffler types functions in the previous papers [7–10] and the book [11].

4. Cauchy–Hadamard Type Theorem and Corollaries

Let us denote by $D(0;R)$ the open disk with the radius R and centred at the origin, and let the circle $C(0;R)$ be its boundary, i.e.

$$D(0;R) = \{z:\ |z| < R\} \quad \text{and} \quad C(0;R) = \{z:\ |z| = R\} \quad (z \in \mathbb{C}).$$

In the beginning, we give a theorem of the Cauchy–Hadamard type for the series (11).

Theorem 2 (of Cauchy–Hadamard type)**.** *Let $z \in \mathbb{C}$, $n \in \mathbb{N}_0$, $\alpha > 0$, $\gamma > 0$. Then the domain of convergence of the series (11) with complex coefficients a_n is the disk $D(0;R)$ with a radius of convergence*

$$R = 1/\limsup_{n\to\infty}\,(\,|a_n|\,)^{1/n}. \tag{12}$$

The cases $R = 0$ and $R = \infty$ are included in the general case.

Let us note that the series (11) absolutely converges in the open disk $D(0; R)$ with the radius R, given by (12), and it diverges in its outside (i.e., for $z \in \mathbb{C}$ with $|z| > R$), like in the classical theory of the power series. These facts are established in the process of proving this basic theorem. Further, three corollaries are formulated. First of them is analogical to the classical Abel lemma.

Corollary 1. *Let $z \in \mathbb{C}$, $n \in \mathbb{N}_0$, $\alpha > 0$, $\gamma > 0$, and let the series (11) converge at the point $z_0 \neq 0$. Then it is absolutely convergent in the disk $D(0; |z_0|)$.*

Additionally, it turns out that the convergence of the discussed series is uniform inside the disk $D(0; R)$, i.e., on each closed disk $|z| \leq r < R$.

Corollary 2. *Let $z \in \mathbb{C}$, $n \in \mathbb{N}_0$, $\alpha > 0$, $\gamma > 0$. Then the convergence of the series (11) is uniform inside the disk $D(0; R)$, with R defined by (12), i.e., on each closed disk $[D(0; r)] = \{z \in \mathbb{C} : \ |z| \leq r < R\}$.*

The third corollary considers the behaviour of the series (11) outside the disk $D(0; |z_0|)$, described in Corollary 1.

Corollary 3. *Let $z \in \mathbb{C}$, $n \in \mathbb{N}_0$, $\alpha > 0$, $\gamma > 0$, and let the series (11) diverge at the point $z_0 \neq 0$. Then it is divergent for each z with $|z| > |z_0|$.*

Theorem 2 and Corollaries 1 and 2 are formulated and proved in [16]. The formulation and proof of Corollary 3 can be found in author's paper [22].

Thus, the series (11) absolutely converges in the open disk $D(0; R)$ and it diverges in the region $\{z \in \mathbb{C} : \ |z| > R\}$. Inside the open disk $D(0; R)$, i.e., in each closed disk $|z| \leq r$ which is a subset of $D(0; R)$, the convergence of the discussed series is uniform. However, the very disk of convergence is not obligatorily a domain of uniform convergence and at the points on its boundary divergence cannot be excluded. More precise results, connected with the behaviour of the series (11) 'near' the boundary $C(0; R)$, are obtained and discussed in the next sections.

5. Abel Type Theorem

Let $z_0 \in \mathbb{C}$, $0 < R < \infty$, $|z_0| = R$ and g_φ be an arbitrary angular region with size $2\varphi < \pi$ and with a vertex at the point $z = z_0$. Let additionally this region be symmetric with respect to the straight line passing through the points 0 and z_0 and d_φ be its part, bounded by the arms of the angle g_φ and the arc of the circle centred at the point 0 and touching the arms of g_φ. The following inequality can be verified for $z \in d_\varphi$ [11] (p. 21):

$$|z - z_0| \cos \varphi < 2(|z_0| - |z|). \tag{13}$$

The next theorem refers to the uniform convergence of the series (11) in the set d_φ and the existence of the limit of its sum at the point z_0, provided $z \in D(0; R) \cap g_\varphi$.

Theorem 3 (of Abel type)**.** *Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of complex numbers, R be the real number defined by (12) and $0 < R < \infty$. If $f(z; \alpha, \gamma)$ is the sum of the series (11) in the open disk $D(0; R)$, i.e.,*

$$f(z; \alpha, \gamma) = \sum_{n=0}^{\infty} a_n \widetilde{F}_{\alpha,n}^{(\gamma)}(z), \quad z \in D(0; R),$$

and this series converges at the point z_0 of the boundary $C(0; R)$, then:

(i) *The following relation holds*

$$\lim_{z\to z_0} f(z;\,\alpha,\,\gamma) = \sum_{n=0}^{\infty} a_n \widetilde{F}^{(\gamma)}_{\alpha,n}(z_0), \tag{14}$$

provided $z \in D(0;R) \cap g_\varphi$.

(ii) *The series (11) is uniformly convergent in the region* d_φ.

Proof. The proofs of the two assertions (i) and (ii) are separately given.

(i) Beginning with (i) we only note that the detailed idea of its proof is given in [22] and that is why the proof is omitted here.

(ii) In order to prove (ii), we use the inequality (13) which is a key point of the proof. So, letting $z \in d_\varphi$ and setting for convenience

$$S_k(z) = \sum_{n=0}^{k} a_n \widetilde{F}^{(\gamma)}_{\alpha,n}(z), \quad S_k(z_0) = \sum_{n=0}^{k} a_n \widetilde{F}^{(\gamma)}_{\alpha,n}(z_0), \quad \lim_{k\to\infty} S_k(z_0) = s, \tag{15}$$

we obtain

$$S_{k+p}(z) - S_k(z) = \sum_{n=0}^{k+p} a_n \widetilde{F}^{(\gamma)}_{\alpha,n}(z) - \sum_{n=0}^{k} a_n \widetilde{F}^{(\gamma)}_{\alpha,n}(z) = \sum_{n=k+1}^{k+p} a_n \widetilde{F}^{(\gamma)}_{\alpha,n}(z).$$

According to Remark 2, there exists a natural number N_0 such that $\widetilde{F}^{(\gamma)}_{\alpha,n}(z_0) \neq 0$ when $n > N_0$. Let $k > N_0$ and $p > 0$. Then, using the denotation

$$\gamma_n(z;z_0) = \widetilde{F}^{(\gamma)}_{\alpha,n}(z) / \widetilde{F}^{(\gamma)}_{\alpha,n}(z_0),$$

the difference $S_{k+p}(z) - S_k(z)$ can be written as follows:

$$S_{k+p}(z) - S_k(z) = \sum_{n=k+1}^{k+p} a_n \widetilde{F}^{(\gamma)}_{\alpha,n}(z_0) \frac{\widetilde{F}^{(\gamma)}_{\alpha,n}(z)}{\widetilde{F}^{(\gamma)}_{\alpha,n}(z_0)} = \sum_{n=k+1}^{k+p} a_n \widetilde{F}^{(\gamma)}_{\alpha,n}(z_0)\gamma_n(z;z_0).$$

Now, by the Abel transformation (see in [17]),

$$\sum_{n=k+1}^{k+p} (\beta_n - \beta_{n-1})\gamma_n = \beta_{k+p}\gamma_{k+p} - \beta_k\gamma_{k+1} - \sum_{n=k+1}^{k+p-1} \beta_n(\gamma_{n+1} - \gamma_n),$$

and additionally denoting $\beta_n = S_n(z_0) - s$, we obtain consecutively:

$$S_{k+p}(z) - S_k(z) = \sum_{n=k+1}^{k+p} (\beta_n - \beta_{n-1})\gamma_n(z;z_0)$$

$$= \beta_{k+p}\gamma_{k+p}(z;z_0) - \beta_k\gamma_{k+1}(z;z_0) - \sum_{n=k+1}^{k+p-1} \beta_n(\gamma_{n+1}(z;z_0) - \gamma_n(z;z_0)),$$

and

$$|S_{k+p}(z) - S_k(z)| \leq |S_{k+p}(z_0) - s||\gamma_{k+p}(z;z_0)| + |S_k(z_0) - s||\gamma_{k+1}(z;z_0)|$$

$$+ \sum_{n=k+1}^{k+p-1} |S_n(z_0) - s| \times \left| \frac{\widetilde{F}^{(\gamma)}_{\alpha,n}(z)}{\widetilde{F}^{(\gamma)}_{\alpha,n}(z_0)} - \frac{\widetilde{F}^{(\gamma)}_{\alpha,n+1}(z)}{\widetilde{F}^{(\gamma)}_{\alpha,n+1}(z_0)} \right|. \tag{16}$$

Then, using the inequality (16), we intend to estimate the module of the difference $S_{k+p}(z) - S_k(z)$. Due to (8) and (9), along with the Γ-functions quotient property (see e.g., [11] (p. 101)) and the equalities $\lim_{n\to\infty} \frac{1}{n^{\alpha\gamma}} = 0$, $\lim_{n\to\infty} (1+\theta_n(z_0))^{-1} = 1$, we conclude that there exist numbers $A > 0$ and $N_1 > N_0$ such that $|1+\theta_n(z)| \le A/2$ for all the positive integers n and $|1+\theta_n(z_0)|^{-1} < 2$ for $n > N_1$, whence

$$|\gamma_n(z;z_0)| \le A \quad \text{for } n > N_1. \tag{17}$$

Further, denoting

$$f_n(z;z_0) = \gamma_n(z;z_0) - \gamma_{n+1}(z;z_0),$$

which is the same as

$$f_n(z;z_0) = \frac{\widetilde{F}^{(\gamma)}_{\alpha,n}(z)}{\widetilde{F}^{(\gamma)}_{\alpha,n}(z_0)} - \frac{\widetilde{F}^{(\gamma)}_{\alpha,n+1}(z)}{\widetilde{F}^{(\gamma)}_{\alpha,n+1}(z_0)},$$

and observing that $f_n(z_0;z_0) = 0$, we apply the Schwartz lemma for the function $f_n(z;z_0)$. So, we obtain that there exists a positive constant C such that:

$$|f_n(z;z_0)| = \left| \frac{\widetilde{F}^{(\gamma)}_{\alpha,n}(z)}{\widetilde{F}^{(\gamma)}_{\alpha,n}(z_0)} - \frac{\widetilde{F}^{(\gamma)}_{\alpha,n+1}(z)}{\widetilde{F}^{(\gamma)}_{\alpha,n+1}(z_0)} \right| \le C|z-z_0||z/z_0|^n,$$

whence, and according to (13) as well, we have:

$$\sum_{n=k+1}^{k+p+1} |f_n(z;z_0)| \le \sum_{n=0}^{\infty} C|z-z_0||z/z_0|^n = C|z_0| \times \frac{|z-z_0|}{|z_0|-|z|} < \frac{2C|z_0|}{\cos\varphi}. \tag{18}$$

Let ε be an arbitrary positive number. Taking in view the third relation (15), we deduce that there exists a positive number $N_2 > N_0$ so large that

$$|S_n(z_0) - s| < \min\left(\frac{\varepsilon}{3A}, \frac{\varepsilon\cos\varphi}{6C|z_0|} \right) \quad \text{for } n > N_2. \tag{19}$$

Now, let us take $N = N(\varepsilon) = \max(N_1, N_2)$ and $k > N$. Therefore the inequalities (16)–(19) give

$$|S_{k+p}(z) - S_k(z)| < \frac{2\varepsilon}{3} + \frac{\varepsilon\cos\varphi}{6C|z_0|} \sum_{n=k+1}^{k+p+1} |f_n(z;z_0)| < \frac{2\varepsilon}{3} + \frac{\varepsilon\cos\varphi}{6C|z_0|}\frac{2C|z_0|}{\cos\varphi} = \varepsilon,$$

that completes the proof of (ii).

Thus, the theorem is completely proved. □

6. Tauber Type Theorem

It is established in Section 5 that the convergence of the considered series in Le Roy type functions at the point z_0 from the boundary of $D(0;R)$ implies the existing of the limit of its sum when z tends to z_0, provided $z \in D(0;R) \cap g_\varphi$. It turns out that under additional conditions on the coefficients of the considered series, the inverse proposition is also valid.

Now, let $z_0 \in \mathbb{C}$, $\quad |z_0| = R$, $\quad 0 < R < \infty$, and let $\widetilde{F}^{(\gamma)}_{\alpha,n}(z_0) \neq 0 \quad$ for $n = 0, 1, 2, \ldots$. Note that, the last condition is fulfilled due to Remark 2, since each function $\widetilde{F}^{(\gamma)}_{\alpha,n}(z)$ $(n \in \mathbb{N})$, being an entire function, no identically zero, has at most a finite number of zeros in the closed and bounded set $|z| \le R$, and moreover, no more than a finite number of these functions have some zeros.

For the sake of brevity, denote

$$F^*_{n,\alpha,\gamma}(z;z_0) = \frac{\widetilde{F}^{(\gamma)}_{\alpha,n}(z)}{\widetilde{F}^{(\gamma)}_{\alpha,n}(z_0)}.$$

Let the series $\sum_{n=0}^{\infty} a_n F^*_{n,\alpha,\gamma}(z;z_0)$, with $a_n \in \mathbb{C}$, be convergent for $|z| < R$, and

$$F(z) = \sum_{n=0}^{\infty} a_n F^*_{n,\alpha,\gamma}(z;z_0), \quad |z| < R. \tag{20}$$

Then the following theorem can be formulated.

Theorem 4 (of Tauber type)**.** *If $\{a_n\}_{n=0}^{\infty}$ is a sequence of complex numbers with*

$$\lim\{na_n\} = 0, \tag{21}$$

and there exists

$$\lim_{z\to z_0} F(z) = S \quad (|z| < R, z \to z_0 \text{ radially}), \tag{22}$$

then the numerical series $\sum_{n=0}^{\infty} a_n$ is convergent and $\sum_{n=0}^{\infty} a_n = S$.

Proof. Let z belong to the segment $[0, z_0]$. By using the asymptotic Formula (9) for the Le Roy type functions, we obtain:

$$a_n F^*_{n,\alpha,\gamma}(z;z_0) = a_n \left(\frac{z}{z_0}\right)^n \frac{1+\vartheta^{(\gamma)}_{\alpha,n}(z)}{1+\vartheta^{(\gamma)}_{\alpha,n}(z_0)} = a_n \left(\frac{z}{z_0}\right)^n \left(1+\widetilde{\vartheta}^{(\gamma)}_{\alpha,n}(z;z_0)\right), \tag{23}$$

where $\widetilde{\vartheta}^{(\gamma)}_{\alpha,n}(z;z_0) = \dfrac{\vartheta^{(\gamma)}_{\alpha,n}(z) - \vartheta^{(\gamma)}_{\alpha,n}(z_0)}{1+\vartheta^{(\gamma)}_{\alpha,n}(z_0)}$. Then, due to (8) and the Γ-functions quotient property, $\widetilde{\vartheta}^{(\gamma)}_{\alpha,n}(z;z_0)$ satisfies the following relation

$$\widetilde{\vartheta}^{(\gamma)}_{\alpha,n}(z;z_0) = O\left(\frac{1}{n^{\alpha\gamma}}\right). \tag{24}$$

Writing $\sum_{n=0}^{\infty} a_n F^*_{n,\alpha,\gamma}(z;z_0)$ in the form

$$\begin{aligned}\sum_{n=0}^{\infty} a_n F^*_{n,\alpha,\gamma}(z;z_0) &= \sum_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n \frac{1+\vartheta^{(\gamma)}_{\alpha,n}(z)}{1+\vartheta^{(\gamma)}_{\alpha,n}(z_0)} \\ &= \sum_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n \left(1+\widetilde{\vartheta}^{(\gamma)}_{\alpha,n}(z;z_0)\right),\end{aligned} \tag{25}$$

and denoting $w_n(z) = a_n \left(\frac{z}{z_0}\right)^n \widetilde{\vartheta}^{(\gamma)}_{\alpha,n}(z;z_0)$, we consider the series $\sum_{n=0}^{\infty} w_n(z)$.

According to condition (21), the numerical sequence $\{na_n\}_{n=0}^{\infty}$, being a convergent sequence, is bounded. Then, since $|w_n(z)| \le |a_n|\,|\widetilde{\vartheta}^{(\gamma)}_{\alpha,n}(z;z_0)|$ and having in view (8), there exists a constant C, such that $|w_n(z)| \le C/n^{1+\alpha\gamma}$ for all the positive integers n. Since $\sum_{n=1}^{\infty} 1/n^{1+\alpha\gamma}$ converges, the series $\sum_{n=0}^{\infty} w_n(z)$ also converges, even absolutely and uniformly on the segment $[0, z_0]$. Therefore, changing the order of the limit and summation, in view of the equality $\lim_{z\to z_0} w_n(z) = 0$, we deduce that

$$\lim_{z\to z_0} \sum_{n=0}^{\infty} w_n(z) = \sum_{n=0}^{\infty} \lim_{z\to z_0} w_n(z) = 0. \tag{26}$$

Then, bearing in mind that (20) can be written in the form

$$F(z) = \sum_{n=0}^{\infty} a_n F^*_{n,\alpha,\gamma}(z; z_0) = \sum_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n + \sum_{n=0}^{\infty} w_n(z),$$

along with the assumption (22), we conclude that the limit

$$\lim_{z\to z_0} \sum_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n \tag{27}$$

also exists and, moreover, in view of (26),

$$\lim_{z\to z_0} F(z) = \lim_{z\to z_0} \sum_{n=0}^{\infty} a_n F^*_{n,\alpha,\gamma}(z; z_0) = S = \lim_{z\to z_0} \sum_{n=0}^{\infty} a_n \left(\frac{z}{z_0}\right)^n. \tag{28}$$

Now, from (28) and the existence of the limit (27), by the classical Tauber theorem for the power series, it follows that the series $\sum_{n=0}^{\infty} a_n$ converges and $\sum_{n=0}^{\infty} a_n = S$. □

The conclusion of the above theorem is still valid even if the condition imposed on the coefficients a_n is weakened. Namely, the following theorem holds true.

Theorem 5 (of Littlewood type)**.** *If* $\{a_n\}_{n=0}^{\infty}$ *is a sequence of complex numbers with*

$$a_n = O(1/n), \tag{29}$$

$F(z)$ *is the function defined by* (20)*, and if there exists*

$$\lim_{z\to z_0} F(z) = S \quad (|z| < R, z \to z_0 \text{ radially}), \tag{30}$$

then the numerical series $\sum_{n=0}^{\infty} a_n$ *is convergent and* $\sum_{n=0}^{\infty} a_n = S$*.*

Proof. Let z belong to the segment $[0, z_0]$. The proof goes in the same way as the proof of Theorem 4, and using the same denotations. The only difference is in proving the estimation for $|w_n(z)|$. More especially, according to the relation (24) and the condition (29), it follows that there exists a constant C, such that $|w_n(z)| \le C/n^{1+\alpha\gamma}$ for all the positive integers n. Finally, the proof ends applying in the last step Littlewood's classical theorem instead of Tauber's theorem. The details are omitted. □

7. $(F_{\alpha,\gamma}, Z_0)$—Summation and (J, Z_0)—Summation

The theorems in the previous section can be formulated in alternative forms. For this purpose, two additional definitions are firstly given.

Let us consider the numerical series

$$\sum_{n=0}^{\infty} a_n, \quad a_n \in \mathbb{C}, \quad n = 0, 1, 2, \ldots \tag{31}$$

To define its Abel summability ([23], p. 20), we consider also the power series $\sum_{n=0}^{\infty} a_n z^n$.

Definition 1. *The series* (31) *is called A—summable if the series* $\sum_{n=0}^{\infty} a_n z^n$ *converges in the open unit disk* $D(0;1)$ *and moreover there exists*

$$\lim_{z\to 1-0} \sum_{n=0}^{\infty} a_n z^n = S.$$

The complex number S is called A-sum of the series (31) and the usual notation of that is

$$\sum_{n=0}^{\infty} a_n = S \quad (A).$$

Remark 3. *The A-summation is regular. It means that if the series (31) converges, then it is A-summable, and its A-sum is equal to its usual sum.*

Remark 4. *It is well known that in general, the A-summability of the series (31) does not imply its convergence. However, with additional conditions imposed on the growth of the general term of the series (31), the convergence can be provided.*

Let $z_0 \in \mathbb{C}$, $|z_0| = R$, $0 < R < \infty$ and $\widetilde{F}_{\alpha,n}^{(\gamma)}(z_0) \neq 0$ (note that, the last condition is again fulfilled due to Remark 2). For the sake of convenience, denote

$$F^*_{n,\alpha,\gamma}(z;z_0) = \frac{\widetilde{F}_{\alpha,n}^{(\gamma)}(z)}{\widetilde{F}_{\alpha,n}^{(\gamma)}(z_0)}. \tag{32}$$

Further, by analogy with the A-summability of the series (31), another definition is introduced, where the power series $\sum_{n=0}^{\infty} a_n z^n$ is replaced by the series in the Le Roy type functions (32) with the same coefficients.

Definition 2. *The numerical series (31) is said to be $(F_{\alpha,\gamma}, z_0)$—summable if the series*

$$\sum_{n=0}^{\infty} a_n F^*_{n,\alpha,\gamma}(z;z_0), \tag{33}$$

converges in the open disk $D(0;R)$ and, moreover, there exists the limit

$$\lim_{z \to z_0} \sum_{n=0}^{\infty} a_n F^*_{n,\alpha,\gamma}(z;z_0), \tag{34}$$

provided z remains on the segment $[0, z_0)$ (i.e., z radially tends to z_0).

Remark 5. *The $(F_{\alpha,\gamma}, z_0)$—summation is regular, and this property is merely a particular case of Theorem 3.*

Taking into account the latest definitions and remarks, Theorems 4 and 5 can be formulated in the following alternative ways.

Theorem 6 (of Tauber type). *If the numerical series (31) is $(F_{\alpha,\gamma}, z_0)$—summable and*

$$\lim\{n a_n\} = 0, \tag{35}$$

then it is convergent.

Theorem 7 (of Littlewood type). *If the numerical series (31) is $(F_{\alpha,\gamma}, z_0)$—summable and*

$$a_n = O(1/n), \tag{36}$$

then it is convergent.

Remark 6. *We observe that all the functions of the family*

$$(F_{\alpha,\gamma}; z_0) = \{F^*_{n,\alpha,\gamma}(z;z_0), \quad n = 0, 1, \dots\} \tag{37}$$

are entire functions satisfying the condition $F^*_{n,\alpha,\gamma}(z_0;z_0) = 1$.

For convenience, in order to make Definition 2 more universal and usable for various considerations, we intend to paraphrase it in the way, given in [11] (p. 35). For this purpose, we firstly introduce one more denotation.

Let $z_0 \in \mathbb{C},\ z_0 \neq 0,\ |z_0| = R,\ 0 < R < \infty$ and let $(J;z_0)$ be the following family of functions

$$(J;z_0) := \{j_n :\ j_n - \text{entire} \quad \text{function}, \quad j_n(z_0) = 1\}_{n\in\mathbb{N}_0}. \tag{38}$$

Now, considering the series given below

$$\sum_{n=0}^{\infty} a_n j_n(z), \quad j_n \in (J;z_0), \tag{39}$$

Definition 2 can be expanded as follows.

Definition 3. *The numerical series (31) is said to be* (J,z_0)*-summable, if the series (39) converges in the disk* $D(0;R)$*, and moreover, there exists the limit*

$$\lim_{z\to z_0} \sum_{n=0}^{\infty} a_n j_n(z), \tag{40}$$

provided z *remains on the segment* $[0,z_0)$.

Remark 7. *Let us note that using this definition must necessarily take into account of the regularity of the summation.*

Ending this section we are going to make one more remark.

Remark 8. *Taking* $j_n(z) = F^*_{n,\alpha,\gamma}(z;z_0)$*, the family (38) of entire functions reduces to the family (37). Therefore, in this case the* (J,z_0)*—summation and* $(F_{\alpha,\gamma},z_0)$*—summation are the same. Thus Theorems 6 and 7 can be written in equivalent ways, using the notion* $(J,z_0)-$*summation (with* $j_n(z) = F^*_{n,\alpha,\gamma}(z;z_0)$*), instead of* $(F_{\alpha,\gamma},z_0)$*—summation. That means that the theorems of Tauber and Littlewood type are statements relating the* (J,z_0)*—summability and the usual convergence of a numerical series by means of some assumptions imposed on the general term of the numerical series under consideration.*

8. Special Cases

In this section we consider some interesting special cases of the Le Roy type function $F^{(\gamma)}_{\alpha,\beta}$, given by (1), taking the parameters

$$\alpha,\ \beta \in \mathbb{C},\ \Re(\alpha) > 0 \text{ and } \gamma > 0,$$

when (1) is an entire function.

Case 1. If γ is an arbitrary positive number, $\alpha = 1$ and $\beta = 1$, then the function (1) coincides with the Le Roy function (confer with (4)), i.e.,

$$F^{(\gamma)}(z) = F^{(\gamma)}_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{[\Gamma(k+1)]^{\gamma}}, \quad z \in \mathbb{C}. \tag{41}$$

We have to note that, studying the asymptotics of the analytic continuation of the sum of power series, Le Roy himself used it in [24]. This reason for the origin of (41) sounds somehow similar to the Mittag–Leffler's idea to introduce the function $E_\alpha(z)$ for the aims of analytic continuation (it have to be noted that Mittag–Leffler and Le Roy were working on this idea in competition). The Le Roy function is involved in the solution of various problems; in particular it has been recently used in the construction of a Conway–Maxwell–

Poisson distribution [25] which is important due to its ability to model count data with different degrees of over- and under-dispersion [26,27].

Case 2. If $\gamma = 1$, then the function (1) gives the Mittag–Leffler function $E_{\alpha,\beta}$, namely

$$E_{\alpha,\beta}(z) = F^{(1)}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C}. \tag{42}$$

In addition, when $\beta = 1$, the function (1) reduces to E_α, and to the exponential function, if $\alpha = \beta = 1$, i.e.,

$$E_\alpha(z) = F^{(1)}_{\alpha,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \ \exp z = F^{(1)}_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}; \quad z \in \mathbb{C}. \tag{43}$$

The functions (42) and (43) are named after the great Swedish mathematician Gösta Magnus Mittag–Leffler (1846–1927) who defined the 1-parametric function $E_\alpha(z)$ by a power series (given by (43)) and he studied its properties in 1902–1905 (detailed description can be seen in [28]). Actually, Mittag–Leffler introduced the function $E_\alpha(z)$ for the purposes of his method for summation of divergent series. Later, the function (43) was recognized as the 'Queen function of fractional calculus' [29–31], see also [11], for its basic role for analytic solutions of fractional order integral and differential equations and systems. In the recent decades successful applications of the Mittag–Leffler function and its generalizations in problems of physics, biology, chemistry, engineering and other applied sciences made it better known among scientists. A considerable literature is devoted to the investigation of the analytical properties of these functions; among the references of [11,28,32], where are quoted several authors who, after Mittag–Leffler, have investigated such kinds of functions from a pure mathematical, applied and numerical oriented point of view as well.

Case 3. If $\gamma = 1/2$ and $\alpha = \beta = 1$, then the function (1) becomes the function $R(z)$, given by the series (see Kolokoltsov [33] Formula (50))

$$R(z) = F^{(1/2)}_{1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\sqrt{k!}}, \quad z \in \mathbb{C}. \tag{44}$$

The function (44) is used by Kolokoltsov in [33] to estimate the solution of initial stochastic differential equations. As he comments in his paper, the function $R(z)$ plays the same role for stochastic equations as the exponential and the Mittag–Leffler functions for deterministic equations.

Case 4. If the parameter $\gamma = 2$ and $\alpha = \beta = 1$, then the function (1) can be presented as Bessel function of the first kind and related to it, and as 2-parametric Bessel–Maitland function, as well. Namely, the function (1) can be written in the following alternative forms:

$$F^{(2)}_{1,1}(z) = J_0(2i\sqrt{z}) = I_0(2\sqrt{z}) = C_0(-z) = J^1_0(-z) = \sum_{k=0}^{\infty} \frac{z^k}{(k!)^2}, \quad z \in \mathbb{C}. \tag{45}$$

In this relation J_0 and I_0 are respectively the classical Bessel function of the first kind J_ν and its modified function I_ν with an index $\nu = 0$, C_0 is the Bessel–Clifford function C_ν with an index $\nu = 0$, and J^1_0 is its 2-parametric Bessel–Maitland generalization J^μ_ν (named after Sir Edward Maitland Wright and also known as Bessel–Wright function) with indices $\nu = 0$ and $\mu = 1$.

Case 5. If the number m is a positive integer, $\gamma = m + 1$, $\beta = \lambda + 1$ $(\lambda \neq 0)$, and $\alpha = 1$, then the function (1) can be expressed with $3-$index generalization, as well as by the $4-$index generalization of the Bessel function of the first kind. More especially if $m = 1$, then the special function (1) turns, with an exactness to a power function, into the generalized Bessel–Maitland (or Wright's) function $J^{\mu}_{\nu,\lambda}$ (with $\nu = 0$ and $\mu = 1$) of the Bessel function $J_\nu(z)$, introduced by Pathak (for details see [14]):

$$J^{\mu}_{\nu,\lambda}(z) = (z/2)^{\nu+2\lambda}\,\widetilde{J}^{\mu}_{\nu,\lambda}(z) = (z/2)^{\nu+2\lambda}\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{2k}}{\Gamma(k+\lambda+1)\Gamma(\mu k+\nu+\lambda+1)}. \tag{46}$$

More precisely,

$$\widetilde{J}^{1}_{0,\lambda}(2i\sqrt{z}) = F^{(2)}_{1,\lambda+1}(z) = \sum_{k=0}^{\infty}\frac{z^k}{[\Gamma(k+\lambda+1)]^2}. \tag{47}$$

The special case (for $m \geq 2$) is expressed by the generalized Lommel–Wright function $J^{\mu,m}_{\nu,\lambda}$ with 4 indices (with $\nu = 0$ and $\mu = 1$), introduced by de Oteiza, Kalla and Conde (for details see [14]):

$$J^{\mu,m}_{\nu,\lambda}(z) = (z/2)^{\nu+2\lambda}\widetilde{J}^{\mu,m}_{\nu,\lambda}(z) = (z/2)^{\nu+2\lambda}\sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{2k}}{(\Gamma(k+\lambda+1))^m\Gamma(\mu k+\nu+\lambda+1)}. \tag{48}$$

Especially,

$$\widetilde{J}^{1,m}_{0,\lambda}(2i\sqrt{z}) = F^{(m+1)}_{1,\lambda+1}(z) = \sum_{k=0}^{\infty}\frac{z^k}{[\Gamma(k+\lambda+1)]^{m+1}}. \tag{49}$$

Just to mention that $J^{\mu,1}_{\nu,\lambda} = J^{\mu}_{\nu,\lambda}$, as well as $\widetilde{J}^{\mu,1}_{\nu,\lambda} = \widetilde{J}^{\mu}_{\nu,\lambda}$.

Case 6. If the number m is a positive integer $m \geq 2$, then the function (1) can be presented as the multi-index extensions of (42) (with $2m$ and $3m$ parameters, $m = 1, 2, \ldots$, [11,13,34–36]), i.e., the so-called multi-index Mittag–Leffler functions. The first one was introduced by Yakubovich and Luchko [37] and studied in details by Kiryakova [12,34]. It is defined by the formula

$$E_{(\alpha_i),(\beta_i)}(z) = E^{m}_{(\alpha_i),(\beta_i)}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha_1 k+\beta_1)\ldots\Gamma(\alpha_m k+\beta_m)}, \tag{50}$$

for $z \in \mathbb{C}$ and $m > 1$. The parameters α_i, β_i are all complex for $i = 1, 2, \ldots m$ and $\Re(\alpha_i) > 0$. The second one has m additional complex parameters γ_i. It was introduced and studied in details by Paneva-Konovska (for its properties see e.g., [11]). It is defined by the formula

$$E^{(\gamma_i),m}_{(\alpha_i),(\beta_i)}(z) = \sum_{k=0}^{\infty}\frac{(\gamma_1)_k\ldots(\gamma_m)_k}{\Gamma(\alpha_1 k+\beta_1)\ldots\Gamma(\alpha_m k+\beta_m)}\,\frac{z^k}{(k!)^m}, \tag{51}$$

where $(\gamma)_k$ is the Pochhammer symbol: $(\gamma)_k = \gamma(\gamma+1)\ldots(\gamma+k-1)$, $k = 1, 2, \ldots$, $(\gamma)_0 = 1$. More precisely, in this case the function (1) turns into the above multi-index Mittag–Leffler functions, with indices $\alpha_i = \alpha$, $\beta_i = \beta$ and $\gamma_i = 1$ ($i = 1, 2, \ldots m$), namely

$$E_{(\alpha),(\beta)}(z) = E^{(1),m}_{(\alpha),(\beta)}(z) = F^{(m)}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{[\Gamma(\alpha k+\beta)]^m}. \tag{52}$$

Case 7. If the number m is a positive integer $m \geq 2$, $\alpha = 1$ and $\beta = 1$ then the function (1) is the hyper-Bessel function

$$J^{(m-1)}_{\nu_1,\ldots,\nu_{m-1}}(z) = \left(\frac{z}{m}\right)^{\sum_{i=1}^{m-1}\nu_i}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{z}{m}\right)^{km}}{\Gamma(k+\nu_1+1)\ldots\Gamma(k+\nu_{m-1}+1)}\,\frac{1}{k!}.$$

introduced by Delerue in 1953 [38]. It is a generalization of the Bessel function of the first type J_ν with vector indices $\nu = (\nu_1, \nu_2, \ldots, \nu_{m-1})$. The hyper-Bessel function of Delerue is closely related to the hyper-Bessel differential operators of arbitrary order $m > 1$, introduced by Dimovski [39]. The function (1) is represented as the hyper-Bessel function with parameters $\nu_i = 0$ ($i = 1, 2, \ldots m-1$), i.e.,

$$J^{(m-1)}_{0,\ldots,0}\left(m(-z)^{1/m}\right) = F^{(m)}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{[\Gamma(k+1)]^m}. \tag{53}$$

At last, let us note that if $\gamma = m$ is a positive integer, then the Le Roy function $F^{(m)}_{\alpha,\beta}$ is the Wright generalized hypergeometric function with $2 \times m$ indices $\alpha_i = \alpha$, $\beta_i = \beta$ ($i = 1, \ldots, m$), namely

$$F^{(m)}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{[\Gamma(\alpha k+\beta)]^m} = {}_1\Psi_m\left[\begin{array}{c} (1,\ 1) \\ (\beta_i,\ \alpha_i)_1^m \end{array}\middle| z\right] = {}_1\Psi_m\left[\begin{array}{c} (1,\ 1) \\ (\beta,\ \alpha)_1^m \end{array}\middle| z\right],$$

and it is a particular case of the Wright generalized hypergeometric function with $2 \times (p+q)$ indices $a_i,\ A_i$ ($i = 1, \ldots, p$), and $b_j,\ B_j$ ($j = 1, \ldots, q$), defined by the formula

$${}_p\Psi_q\left[\begin{array}{c} (a_1,\ A_1)\ldots(a_p,\ A_p) \\ (b_1,\ B_1)\ldots(b_q,\ B_q) \end{array}\middle| \sigma\right] = \sum_{k=0}^{\infty} \frac{\Gamma(a_1+kA_1)\ldots\Gamma(a_p+kA_p)}{\Gamma(b_1+kB_1)\ldots\Gamma(b_q+kB_q)} \frac{\sigma^k}{k!}.$$

9. Conclusions

Letting the parameter β in the condition (2) be a positive integer, we consider the family of Le Roy type functions (5) with parameters as follows:

$$\alpha > 0,\ \gamma > 0,\ \text{and}\ \beta = n \in \mathbb{N}.$$

In Section 2 we provide an asymptotic formula for these functions for large values of the parameter n (Theorem 1). We also give upper estimates for the moduli of their remainder terms in the nonempty compact subsets of the complex plane and in the whole complex plane as well (Lemma 1). Further, in order to summarize the results obtained here, we consider the family of the type

$$\left\{\widetilde{j}_n(z)\right\}_{n\in\mathbb{N}}, \tag{54}$$

with the functions $\widetilde{j}_n$ as in (10), and the series

$$\sum_{n=0}^{\infty} a_n\, \widetilde{j}_n(z), \tag{55}$$

in this case coinciding with the series (11) in Le Roy type functions with complex coefficients a_n ($n = 0, 1, 2, \ldots$) and for $z \in \mathbb{C}$.

It turns out that the series (55) absolutely converges in the open disk $D(0; R)$ with the corresponding radius R, given by the Formula (12) and it diverges in its outside, i.e., for $z \in \mathbb{C}$ with $|z| > R$. Moreover, inside the disk $D(0; R)$, i.e., in each closed disk $[D(0;r)] = \{z : z \in \mathbb{C},\ |z| \le r\}$ with $r < R$, the convergence is uniform. Near the boundary $C(0; R)$ the series (55) satisfies Theorem 3 of Abel type. At last, the series fulfills the theorem of Tauber and Littlewood types, which are inverse of the Abel type theorem.

Now, let us consider the functions from the Section 8 with the same types of parameters. Since in this case they are of the types (5), then all of them satisfy Lemma 1 and the inequalities therein. Further, paying attention to the fact, that the functions (42), (52), (47) and (49) can be considered as representatives of different families of the types (5), we have to note that the functions of each family, discussed above, have the asymptotic Formula (9) with the corresponding values of the parameters α and γ. Further, taking the family of the type (54) with the functions $\widetilde{j}_n$ as follows:

$$\widetilde{j}_n(z) = z^n\ [\Gamma(n)]^m\, E_{((\alpha),(n))}(z) = z^n\ [\Gamma(n)]^m\, F^{(m)}_{\alpha,n}(z), \quad m,\, n \in \mathbb{N}, \tag{56}$$

in the case (52) (in particular $m = 1$ in the case (42)), and respectively

$$\widetilde{j}_n(z) = z^n\ [\Gamma(n)]^{m+1}\, \widetilde{J}^{1,m}_{0,n-1}(2i\sqrt{z}) = z^n\ [\Gamma(n)]^{m+1}\, F^{(m+1)}_{1,n}(z),\ m,\, n \in \mathbb{N}, \tag{57}$$

in the case (49) ($m = 1$ in the case (47)), and adding, just for completeness $\widetilde{j}_0(z) = 1$, we consider the corresponding series (55) with complex coefficients a_n $(n = 0, 1, 2, \dots)$ for $z \in \mathbb{C}$, namely the series $\sum\limits_{n=0}^{\infty} a_n\, \widetilde{j}_n(z)$.

Taking into account that the series (55) is of the type (11) (however with special values of the parameters), it has the same behaviour. That means that the series (55) absolutely converges in the open disk $D(0;R)$ with the corresponding radius R, and it diverges in its outside, i.e., for $z \in \mathbb{C}$ with $|z| > R$. Moreover, inside the disk $D(0;R)$, i.e., in each closed disk $[D(0;r)]$ with $r < R$, the convergence is uniform. Replacing the parameter γ with the corresponding value in Theorem 3, it is reduced to the Abel type theorem for the series (55), referring to the behaviour of (55) near the boundary $C(0;R)$. At last, the series (55) fulfills the theorem of Tauber and Littlewood types, which are inverse of the Abel type theorem.

Thus, generally speaking, the described behaviour of the series (11) in Le Roy type functions, as well as in particular the behaviour of the corresponding series (55) (in the functions of the families (56), respectively (57)), and that of the classical power series are the same. Moreover, the results discussed here are analogues to the Cauchy–Hadamard, Abel, Tauber and Littlewood theorems for the widely used power series.

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