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STRICHARTZ TYPE ESTIMATES FOR OSCILLATORY PROBLEMS FOR SEMILINEAR WAVE EQUATION

Stefania Di Pomponio*

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Abstract. We treat the oscillatory problem for semilinear wave equation. The oscillatory initial data are of the type

$$\begin{aligned} u(0,x) &= h(x) + \varepsilon^{\sigma+1} e^{il(x)/\varepsilon} b_0(\varepsilon, x) \\ \partial_t u(0,x) &= \varepsilon^{\sigma} e^{il(x)/\varepsilon} b_1(\varepsilon, x). \end{aligned}$$

By using suitable variants of Strichartz estimate we extend the results from [6] on a priori estimates of the approximations of geometric optics.The main improvement is the fact that we can obtain a priori estimates for the case $\sigma = 1$, while in [6] we could treat only the case $\sigma > \dfrac{n}{2} - 1$.

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1. Introduction. We consider the following Cauchy problem for semilinear wave equation

$$\left\{\begin{array}{l}(\partial_{tt}-\Delta)u(t,x)=F_\lambda(u(t,x)) \quad (t,x)\in\mathbb{R}^{1+n}_+\\ u(0,x)=f(x)\\ \partial_t u(0,x)=g(x),\end{array}\right. \tag{1.1}$$

where f and g belong to suitable Sobolev spaces, $F_\lambda(u)=u|u|^{\lambda-1}$, $n\geq 2$ and $\lambda=\dfrac{n+3}{n-1}$ is the conformally invariant exponent. In order to obtain a priori estimate of the solution to this classical equation, among the most important techniques we recall the standard energy estimate, that gives a control on L^2-norm of derivatives of the solution, the estimate of Von Wahl, that controles the L^∞-norm of the solution and the Strichartz type estimate, that allows one to evaluate L^q-norm of the solution for a suitable $2\leq q<\infty$.

Combining energy type estimate with Sobolev embedding, we studied in [6] the existence of local solutions to problem (1.1) for the case when

(i) the initial data $(u(0,x),\partial_t u(0,x))\in H^{s+1}(\mathbb{R}^n)\times H^s(\mathbb{R}^n)$ are of oscillatory type (see (1.2) and (1.3)) and satisfy the minimal regularity assumption $s>\dfrac{n}{2}$ coming from the classical Sobolev embedding;

(ii) $|F^{(k)}(u)|\leq C(1+|u|)^{\lambda-k}$, for $\dfrac{n}{2}\leq k\leq\lambda$.

In this paper we apply a suitable generalization of Strichartz type estimate to control L^q-norm of solution to problem (1.1) in the case of oscillatory initial data of the form

$$f(x) \;=\; h(x)+\varepsilon^{\sigma+1}e^{il(x)/\varepsilon}b_0(\varepsilon,x) \tag{1.2}$$

$$g(x) \;=\; \varepsilon^{\sigma}e^{il(x)/\varepsilon}b_1(\varepsilon,x), \tag{1.3}$$

with $b_0,b_1\in O(\varepsilon^0)$ supported in $K=\{x\in\mathbb{R}^n:|x|\leq R\}$, $R>0$ (see Section 3). More precisely, we use the method of geometric optics and construct a solution of the form

$$u_\varepsilon(t,x)=H(t,x)+a(t,x,\theta;\varepsilon), \tag{1.4}$$

where $a(t,x,\theta;\varepsilon)=\sum\limits_{j=1}^{N}\varepsilon^{\sigma+j}a_j\,(t,x,\theta)+r_N(t,x,\theta;\varepsilon)$, $\theta=\dfrac{\varphi(t,x)}{\varepsilon}$ and the profiles $a_j(t,x,\theta)$ are smooth and 2π-periodic in θ (see [13]). Our main goal is to estimate the smallnest of L^q-norm of the remainder $r_N(t,x,\theta;\varepsilon)$ as $\varepsilon\to 0$.

A similar problem for $q=2$ has been considered by J. L. Joly, G. Métivier and J. Rauch in [12] in the semilinear dissipative case $F=F(\nabla_{t,x}u)$, for which they studied some nonlinear effects of focusing. Nonlinear geometric optics was applied also by D. Ludwig to construct an asymptotic solution of the reduced wave equation near a smooth convex caustic and near a cusped caustic ([14]). Nonlinear phenomena of caustics in geometric optics were studied recently by R. Carles, which provided a precise description of radial solution of wave equations for space dimension three (see [3] and [4]). For the case $F = F_\lambda(u) = u|u|^{\lambda-1}$ the problem for $q = 2$ have been studied in [6] under the condition $\lambda \geq \frac{n}{2}$, since a combination between energy type estimate and classical Sobolev embedding is used. This restriction is not satisfactory, because implies $\sigma \geq s > \frac{n}{2} - 1$ and this is far away from the desired value $\sigma = 1$.

To avoid this complicated restriction we use recent techniques to examine global existence of solutions based on Strichartz estimate. For this reason below we give a brief sketch of results in this direction.

In [10] F. John showed that when $n = 3$ there exists a critical value of λ, i. e. $\lambda_{cr}(3) = 1 + \sqrt{2}$, with the property that global existence of all small solutions holds if $\lambda > \lambda_{cr}(3)$ but no such result can occur if $\lambda < \lambda_{cr}(3)$. He assumed only that initial data were sufficiently smooth and compactly supported, but he not considered the critical case $\lambda = \lambda_{cr}(3)$.

The number $1 + \sqrt{2}$ appeared first in a paper of W. Strauss on scattering for semilinear Schrödinger equation (cf. [18, 19]). This led him to conjecture that if $n \geq 2$, then global solutions to problem (1.1) with small initial data should always exist if $\lambda > \lambda_{cr}(n)$. Here the critical exponent $\lambda_{cr}(n)$ is the positive solution to the quadratic equation

$$(n-1)\lambda_{cr}^2 - (n+1)\lambda_{cr} - 2 = 0. \tag{1.5}$$

R. T. Glassey proved this conjecture for $n = 2$ by showing that problem (1.1) has global solution for small data if $\lambda > \lambda_{cr}(2)$ (see [9]); the critical value $\lambda = \lambda_{cr}(n)$ was studied by J. Shaeffer who proved blow-up for $n = 2, 3$. In [17] T. C. Sideris generalised the blow-up result of F. John to dimensions $n \geq 4$; more precisely he showed that if $\lambda_{cr}(n)$ is the positive root of (1.5), for $1 < \lambda < \lambda_{cr}(n)$ and for suitable initial data the solution of problem (1.1) blows-up.

In [7] V. Georgiev, H. Lindblad and C. D. Sogge have considered problem (1.1) with initial data of the form $u(0,x) = \varepsilon f(x)$, $\partial_t u(0,x) = \varepsilon g(x)$ and have determined, for a given n and $\varepsilon > 0$ small enough, the sharp range of λ for wich one has global weak solutions. For this purpose a suitable weighted Strichartz estimate is proposed in [7].

In this work we apply the classical Strichartz estimate combined with space-time decay results due to I. Segal (cf. [16]) and prove that, under suitable assumptions on λ and F_λ and in the case of small initial data, the L^q-norm of the solution to problem (1.1) satisfies

$$\|u\|_{L^q(\mathbb{R}^{1+n}_+)} \le C\delta,$$

for $q = 2\dfrac{n+1}{n-1}$ and some constant $C > 0$ independent of δ (cf. Theorem 2.1). Here below small initial data condition means that

$$\|f\|_{H^{\frac{1}{2}}(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)} \le \delta, \tag{1.6}$$

for some sufficiently small $\delta > 0$.

Using the above estimate we avoid the application of energy method by a combination of Hölder and Young inequalities, that yields the following estimate for the L^q-norm of $a(t,x,\theta;\varepsilon) = u_\varepsilon(t,x) - H(t,x)$

$$\|a(\cdot,\cdot,\theta;\varepsilon)\|_{L^q(\mathbb{R}^{1+n}_+)} \le C\varepsilon^\sigma.$$

Our main goal is to justify the asymptotic expansion (1.4); in particular, by a standard application of Hölder inequality and using local estimates of Strichartz type in the case of variable coefficients (see [2]) we prove the following

Theorem 1.1. *For any $N \ge 1$ the remainder $r_N(t,x;\varepsilon)$ satisfies the estimate*

$$\|r_N(\cdot,\cdot,\theta;\varepsilon)\|_{L^q([0,T]\times\{|x|\le R+t\})} \le C_N\varepsilon^{\sigma+N+1}$$

where the constant $C_N > 0$ is independent of ε.

The plan of the work is the following.

In Section 2 we obtain a global a priori space-time estimate on the solution to problem (1.1) in the assumption that initial data $(f,g) \in H^{\frac{1}{2}}(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ satisfy (1.6).

In Section 3 we construct an approximate solution to problem (1.1), (1.2), (1.3) trough the method of geometric optics; see [15] and [20] for a complete reference on this subject. Then in last section we use the same arguments of Section 2 to estimate L^q-norm of the remainder $r_N(t,x,\theta;\varepsilon)$ as $\varepsilon \to 0$.

2. Space-time estimate of Strichartz type. We consider solutions to the following Cauchy problem for semilinear wave equation

$$\begin{cases} (\partial_{tt} - \Delta)u(t,x) = F_\lambda(u(t,x)) \quad (t,x) \in \mathbb{R}^{1+n}_+ \\ u(0,x) = f(x) \\ \partial_t u(0,x) = g(x); \end{cases} \tag{2.1}$$

we assume that

$(\mathbf{h_1})$ $\qquad (f,g) \in H^{\frac{1}{2}}(\mathbb{R}^n) \times L^2(\mathbb{R}^n);$

$$(\mathbf{h_2}) \qquad F_\lambda(u) = u\ |u|^{\lambda-1}, \text{ with } \lambda = \frac{n+3}{n-1}.$$

We start with the following standard result

Theorem 2.1. *Let $u(t,x)$ be the solution to problem (2.1). Suppose the assumptions $(\mathbf{h_1})$ and $(\mathbf{h_2})$ are satisfied and the initial data satisfy the estimate*

$$\|f\|_{H^{\frac{1}{2}}(\mathbb{R}^n)} + \|g\|_{L^2(\mathbb{R}^n)} \le \delta,$$

where δ is sufficiently small. Then

$$\|u\|_{L^q(\mathbb{R}^{1+n}_+)} \le C\delta,$$

for $q = 2\dfrac{n+1}{n-1}$ and some constant $C > 0$ independent of δ.

Proof. Using classical Strichartz estimate and its generalization due to M. Bezard (see [1]) and results of I. Segal about space-time decay for solutions to wave equation (see [16]), we get for u the following estimate

$$\|u\|_{L^q(\mathbb{R}^{1+n}_+)} \le C\ \|F_\lambda\|_{L^p(\mathbb{R}^{1+n}_+)} + C\ \|f\|_{H^{\frac{1}{2}}(\mathbb{R}^n)} + C\ \|g\|_{L^2(\mathbb{R}^n)} \tag{2.2}$$

for $p = 2\dfrac{n+1}{n+3}$. In particular, by our assumption on initial data, we have

$$\|u\|_{L^q(\mathbb{R}^{1+n}_+)} \le C\|F_\lambda\|_{L^p(\mathbb{R}^{1+n}_+)} + C\delta. \tag{2.3}$$

Since $F_\lambda(u) = u\ |u|^{\lambda-1}$, the definition of L^p-norm yields

$$\begin{aligned}\|F_\lambda\|_{L^p(\mathbb{R}^{1+n}_+)} &= \left(\int_{\mathbb{R}^{1+n}_+} \left|u\ |u|^{\lambda-1}\right|^p dt\ dx\right)^{\frac{1}{p}} \\ &= \left(\int_{\mathbb{R}^{1+n}_+} |u|^{\lambda p} dt\ dx\right)^{\frac{1}{\lambda p}\lambda} = \|u\|^\lambda_{L^{\lambda p}(\mathbb{R}^{1+n}_+)} = \|u\|^\lambda_{L^q(\mathbb{R}^{1+n}_+)},\end{aligned}$$

because by our choice is $\lambda p = q$; then

$$\|u\|_{L^q(\mathbb{R}^{1+n}_+)} \le C_1\|u\|^\lambda_{L^q(\mathbb{R}^{1+n}_+)} + C_2\delta. \tag{2.4}$$

Now we need the following

Lemma 2.2. *Let $u_{-1}(t,x) = 0$ and for $k = 0, 1, 2, \ldots$ let $u_k(t,x)$ defined recursively by requiring*

$$\begin{cases} (\partial_{tt} - \Delta)u_k = F_\lambda(u_{k-1}), \\ u_k(0,x) = f(x), \\ \partial_t u_k(0,x) = g(x). \end{cases}$$

Set $X_k = \|u_k\|_{L^q(\mathbb{R}^{1+n}_+)}$ *and* $Y_k = \|u_k - u_{k-1}\|_{L^q(\mathbb{R}^{1+n}_+)}$. *Then*

(**a**) $X_k \le 2X_0$ *and* (**b**) $2Y_{k+1} \le Y_k$ *for* $k = 0, 1, 2, \ldots$

Proof. From (2.4) it follows that $X_k \le C_1 X_{k-1}^\lambda + C_2\delta$. Using this, we can prove inductively that

(**c**) $$X_k \le 2C_2\delta.$$

For $k = 0$ it is obvious. Suppose (**c**) is true for $k-1$; then $X_k \le C_1(2C_2\delta)^\lambda + C_2\delta$. The inequality $X_k \le 2C_2\delta$ is true if $C_1(2C_2\delta)^\lambda \le C_2\delta$, i.e. $\delta^{\lambda-1} \le \dfrac{1}{2^\lambda C_1 C_2^\lambda}$.

The proof of (**b**) is similar; the only difference is to replace (2.4) by

$$\|F_\lambda(u_k) - F_\lambda(u_{k-1})\|_{L^p(\mathbb{R}^n)} \le \|u_k - u_{k-1}\|_{L^q(\mathbb{R}^n)} \left(\|u_k\|^{\lambda-1}_{L^q(\mathbb{R}^n)} + \|u_{k-1}\|^{\lambda-1}_{L^q(\mathbb{R}^n)}\right). \ \square$$

Since $u_k \to u$ in $L^q(\mathbb{R}^n)$, the previous lemma completes the proof of the Theorem. $\square$

3. Construction of the approximate solution. In this section we consider problem (2.1) from the oscillatory point of view, i.e. we impose the following initial data

$$f(x) = h(x) + \varepsilon^{\sigma+1} e^{il(x)/\varepsilon} b_0(\varepsilon, x) \tag{3.1}$$

$$g(x) = \varepsilon^{\sigma} e^{il(x)/\varepsilon} b_1(\varepsilon, x); \tag{3.2}$$

here $\sigma \ge 1$ is an integer, the amplitudes b_0, b_1 are supported in $K = \{x \in \mathbb{R}^n : |x| \le R\}$ and belong to the class $O(\varepsilon^0)$, namely are $C^\infty([0,\varepsilon_0] \times \mathbb{R}^n)$ functions, for some small ε_0, so that for every compact set $M \subseteq \mathbb{R}^n$ and for every multiindex α there is a constant $C(\alpha, M) > 0$ such that $|\partial_x^\alpha b_j(\varepsilon, x)| \le C(\alpha, M)\varepsilon^{|\alpha|}$, for every $\varepsilon \in [0, \varepsilon_0]$ and for each $x \in M$, $j = 0, 1$. Note that in this assumption we have for $b_j(\varepsilon, x)$ an asymptotic expansion of the form $b_j(\varepsilon, x) \sim b_{j,0}(x) + \varepsilon b_{j,1}(x) + \varepsilon^2 b_{j,2}(x) + \cdots$, for $j = 0, 1$; moreover we assume that $\nabla l \ne 0$ on a neighbourhood of K.

We look for solutions to problem (2.1) with initial data (3.1) and (3.2) of the form

$$u_\varepsilon(t, x) = H(t, x) + a\,(t, x, \theta; \varepsilon)\,, \tag{3.3}$$

where $a(t, x, \theta; \varepsilon) = \sum_{j=1}^{N} \varepsilon^{\sigma+j} a_j\,(t, x, \theta) + r_N(t, x, \theta; \varepsilon)$, $\theta = \dfrac{\varphi(t, x)}{\varepsilon}$ and the profiles $a_j(t, x, \theta)$ are smooth and 2π-periodic in θ (see [13]). For u_ε we compute

$$(\partial_{tt} - \Delta)u_\varepsilon = (\partial_{tt} - \Delta)H + \varepsilon^{\sigma-1} a_{1,\theta\theta}(\varphi_t^2 - |\nabla\varphi|^2) + \varepsilon^\sigma [2\partial_\mu\varphi\partial^\mu a_{1,\theta} +$$

$$a_{1,\theta}(\partial_{tt}-\Delta)\varphi + a_{2,\theta\theta}(\varphi_t^2 - |\nabla\varphi|^2)] + \sum_{j=1}^{N}\varepsilon^{\sigma+j}[(\partial_{tt}-\Delta)a_j + 2\partial_\mu\varphi\partial^\mu a_{j+1,\theta}+$$

$$a_{j+1,\theta}(\partial_{tt}-\Delta)\varphi + a_{j+2,\theta\theta}(\varphi_t^2 - |\nabla\varphi|^2)] + \varepsilon^{-1}(2\partial_\mu\varphi\partial^\mu r_{N,\theta}+$$

$$r_{N,\theta}(\partial_{tt}-\Delta)\varphi) + \varepsilon^{-2}r_{N,\theta\theta}(\varphi_t^2 - |\nabla\varphi|^2) + (\partial_{tt}-\Delta)r_N, \tag{3.4}$$

where $a_j = 0$ if $j \neq 1,\ldots,N$. Now we need to compute $F_\lambda(u_\varepsilon)$; to do this we use Taylor's expansion of F_λ up to first order around H

$$F_\lambda(u_\varepsilon) = F_\lambda(H) + F_\lambda'(H)(u_\varepsilon - H) + R_1(u_\varepsilon; H).$$

In particular we can represent the remainder term R_1 in the integral form and we obtain

$$\begin{aligned} F_\lambda(u_\varepsilon) &= F_\lambda(H) + \sum_{j=1}^{N}\varepsilon^{\sigma+j}F_\lambda'(H)a_j + F_\lambda'(H)r_N + \sum_{j,k=1,j\neq k}^{N}\varepsilon^{2\sigma+j+k}a_ja_kA \\ &\quad + \sum_{j=1}^{N}\frac{1}{2}\varepsilon^{2\sigma+2j}a_j^2A + \sum_{j=1}^{N}\varepsilon^{\sigma+j}a_jAr_N + \frac{1}{2}Ar_N^2, \end{aligned} \tag{3.5}$$

with

$$A = \int_0^1 (1-\gamma)F_\lambda''(H+\gamma a)\,d\gamma$$

Therefore $(\partial_{tt}-\Delta)u_\varepsilon = F_\lambda(u_\varepsilon)$ iff (3.4) and (3.5) coincide; in such way we obtain an equation in which the coefficients of ε^0 and $\varepsilon^{\sigma+j}$, $j = -1, 0, 1, \ldots, N-2$, have the following expressions (for $\sigma \geq 0$ integer)

$$\varepsilon^0: \quad (\partial_{tt}-\Delta)H - F_\lambda(H); \tag{3.6}$$

$$\varepsilon^{\sigma-1}: \quad a_{1,\theta\theta}(\varphi_t^2 - |\nabla\varphi|^2); \tag{3.7}$$

$$\varepsilon^{\sigma}: \quad 2\partial_\mu\varphi\partial^\mu a_{1,\theta} + a_{1,\theta}(\partial_{tt}-\Delta)\varphi + a_{2,\theta\theta}(\varphi_t^2 - |\nabla\varphi|^2); \tag{3.8}$$

$$\begin{aligned} \varepsilon^{\sigma+j}: \quad & (\partial_{tt}-\Delta)a_j + 2\partial_\mu\varphi\partial^\mu a_{j+1,\theta} + a_{j+1,\theta}(\partial_{tt}-\Delta)\varphi + \\ & + a_{j+2,\theta\theta}(\varphi_t^2 - |\nabla\varphi|^2) - F_\lambda'(H)a_j - \sum_{\substack{h,k=1,h\neq k\\ h+k=j-\sigma}}^{N} a_ha_kA - \frac{1}{2}a_m^2A. \end{aligned} \tag{3.9}$$

If $\sigma \geq 0$ is not integer then there is no term of the form $\sum_{\substack{h,k=1,h\neq k\\ h+k=j-\sigma}}^{N} a_ha_kA - \frac{1}{2}a_m^2A$ while the terms of order ε^μ, with $\mu \geq \sigma + N - 1$, are given by

$$\varepsilon^{\sigma+N-1}: \quad \varepsilon^{\sigma+N-1}[(\partial_{tt}-\Delta)a_{N-1} + 2\partial_\mu\varphi\partial^\mu a_{N,\theta} + a_{N,\theta}(\partial_{tt}-\Delta)\varphi +$$

$$-F'_\lambda(H)a_{N-1}] + \varepsilon^{-2} r_{N,\theta\theta}(\varphi_t^2 - |\nabla\varphi|^2) +$$

$$- \sum_{\substack{h,k=1, h\neq k\\ h+k=N-1-\sigma}}^{N} \varepsilon^{2\sigma+h+k} a_h a_k A - \frac{1}{2}\varepsilon^{2\sigma+2m} a_m^2 A \tag{3.10}$$

$$\varepsilon^{\sigma+N}: \quad \varepsilon^{\sigma+N}(\partial_{tt} - \Delta)a_N + \varepsilon^{-1}[2\partial_\mu\varphi\partial^\mu r_{N,\theta} + r_{N,\theta}(\partial_{tt} - \Delta)\varphi] +$$

$$-\varepsilon^{\sigma+N} F'_\lambda(H)a_N - \theta \sum_{\substack{h,k=1, h\neq k\\ h+k=N-\sigma}}^{N} \varepsilon^{2\sigma+h+k} a_h a_k A - \frac{1}{2}\varepsilon^{2\sigma+2m} a_m^2 A, \tag{3.11}$$

$$\varepsilon^{\mu}: \quad (\partial_{tt} - \Delta)r_N - F'_\lambda(H)r_N - \sum_{\substack{j,k=1, j\neq k\\ h+k\geq N-1-\sigma}}^{N} \varepsilon^{2\sigma+h+k} a_h a_k A +$$

$$- \sum_{\substack{j=1\\ j\geq \frac{N-1-\sigma}{2}}}^{N} \frac{1}{2}\varepsilon^{2\sigma+2j} a_j^2 A - \sum_{j=1}^{N} \varepsilon^{\sigma+j} a_j A r_N + \frac{1}{2} A r_N^2; \tag{3.12}$$

we remark that in (3.9), (3.10) and (3.11), m is given respectiveli by $\frac{j-\sigma}{2}$, $\frac{N-1-\sigma}{2}$, $\frac{N-\sigma}{2}$ for $j \in \{1,\dots,N\}$ and $a_m = 0$ if $m \neq 1,\dots,N$.

We impose that all the coefficients of different powers of ε vanish. To determine $H(t,x)$ we use the following nonlinear Cauchy problem

$$\begin{cases} (\partial_{tt} - \Delta)H = F_\lambda(H), \\ H(0,x) = h(x), \\ \partial_t H(0,x) = 0. \end{cases} \tag{3.13}$$

This particular choice of the initial data in (3.13) is motivated by (3.1) and (3.2) and for the existence of local solution to (3.13) see results in [17]. In this way we reduce the initial data (3.1), (3.2) to the case $h = 0$.

The term (3.7) is zero provided $\varphi(t,x)$ satisfies the *eikonal equation*

$$\varphi_t^2 - |\nabla\varphi|^2 = 0. \tag{3.14}$$

The initial data (3.1) hints that we have to impose the following initial condition

$$\varphi(0,x) = l(x). \tag{3.15}$$

To solve this Cauchy problem for φ we use the classical results for the existence of solution to first order nonlinear PDE's (see for istance [20], sec. 15, Chap. I). More precisely, these results guarantee that there is a $T > 0$ and a neighborhood U of K so that (3.14) and (3.15) has a unique pair of solutions

$\varphi^{\pm}(t,x) \in C^{\infty}((0,T)\times U)$, satisfying

$$\varphi^{\pm}(0,x) = l(x), \qquad \partial_t \varphi^{\pm}(0,x) = \pm|\nabla l(x)|$$

and $\varphi^{\pm}(t,x) \neq 0$ for t small enough. We shall choose one of them, for example $\varphi(t,x) = \varphi_{+}(t,x)$; in such way the terms (3.8), (3.9), (3.10) and (3.10) simplify; the first one will be zero if $a_{1,\theta}$ satisfies the *first transport equation*

$$2\partial_{\mu}\varphi\partial^{\mu} a_{1,\theta}(t,x,\theta) + a_{1,\theta}(t,x,\theta)(\partial_{tt} - \Delta)\varphi = 0, \quad \text{for all } \theta \in [0,2\pi]. \tag{3.16}$$

The initial data for (3.16) can be deduced from (3.1) and (3.2); more precisely we have that $u_{\varepsilon}(0,x) = h(x) + \varepsilon^{\sigma+1} e^{il(x)/\varepsilon} b_0(\varepsilon,x)$ and $\partial_t u_{\varepsilon}(0,x) = \varepsilon^{\sigma} e^{il(x)/\varepsilon} b_1(\varepsilon,x)$ if and only if

$$\begin{aligned} &H(0,x) + \sum_{j=1}^{N} \varepsilon^{\sigma+j} a_j\left(0,x,l(x)/\varepsilon\right) + r_N(0,x,l(x)/\varepsilon;\varepsilon) = \\ &= h(x) + \varepsilon^{\sigma+1} b_{0,0}(x) e^{il(x)/\varepsilon} + \varepsilon^{\sigma+2} b_{0,1}(x) e^{il(x)/\varepsilon} + \cdots \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} &\partial_t H(0,x) + \varepsilon^{\sigma} a_{1,\theta}\left(0,x,l(x)/\varepsilon\right)\partial_t\varphi(0,x) + \sum_{j=1}^{N-1} \varepsilon^{\sigma+j}(a_{j,t}\left(0,x,l(x)/\varepsilon\right) + \\ &+ a_{j+1,\theta}\left(0,x,l(x)/\varepsilon\right)\partial_t\varphi(0,x)) + \varepsilon^{\sigma+N} a_{N,t}\left(0,x,l(x)/\varepsilon\right) + \partial_t r_N(0,x,l(x)/\varepsilon;\varepsilon) + \\ &\varepsilon^{-1} r_{N,\theta}(0,x,l(x)/\varepsilon;\varepsilon)\varphi_t(0,x) = \varepsilon^{\sigma} b_{1,0}(x) e^{il(x)/\varepsilon} + \varepsilon^{\sigma+1} b_{1,1}(x) e^{il(x)/\varepsilon} + \cdots \end{aligned} \tag{3.18}$$

respectively. If we require that in the previous expressions the coefficients of the same powers of ε of both sides of (3.17) and (3.18) coincide, then we obtain for $a_{1,\theta}$ the following initial data

$$a_{1,\theta}(0,x,l(x)/\varepsilon)\varphi_t(0,x) = b_{1,0}(x) e^{il(x)/\varepsilon}. \tag{3.19}$$

Note that by (3.17) we obtain $a_1(t,x,\theta)|_{t=0} = b_{0,0}(x)e^{i\theta}$; therefore $a_{1,\theta}(0,x,l(x)/\varepsilon) = i b_{0,0}(x) e^{il(x)/\varepsilon}$. Moreover, by (3.19) we have the following relation between for $b_{0,0}(x)$ and $b_{1,0}(x)$

$$i b_{0,0}(x)\varphi_t(0,x) = b_{1,0}(x).$$

From Cauchy problem (3.16), (3.19) for first transport equation we can recover $a_{1,\theta}$; the existence of local solutions to this problem is a well known result (see [11]). Then it remains to determine $a_1(t,x,\theta)$ from $a_{1,\theta}$; denote by ∂_{θ}^{-1} the operator of the primitive of $a_{1,\theta}$ that has mean zero (see [13]); namely $a_1 = \partial_{\theta}^{-1} a_{1,\theta}$

such that

$$\int_0^{2\pi} a_1(t,x,\theta)d\theta = 0. \tag{3.20}$$

We obtain

$$a_1(t,x,\theta) = C + \int_0^{\theta} a_{1,\eta}(t,x,\eta)d\eta,$$

therefore (3.20) gives

$$C = -\frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^{\theta} a_{1,\eta}(t,x,\eta)d\eta\right)d\theta.$$

In such way, we get for a_1 the following expression

$$a_1(t,x,\theta) = -\frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^{\theta} a_{1,\eta}(t,x,\eta)d\eta\right)d\theta + \int_0^{\theta} a_{1,\eta}(t,x,\eta)d\eta. \tag{3.21}$$

In general, the term (3.9) vanishes provided $a_{j+1,\theta}$ is solution to the following $(j+1)-$th transport equation

$$\begin{aligned} 2\partial_\mu\varphi\partial^\mu a_{j+1,\theta} + a_{j+1,\theta}(\partial_{tt}-\Delta)\varphi = & -(\partial_{tt}-\Delta)a_j + F'_\lambda(H)a_j - \\ & - \sum_{\substack{h,k=1,h\neq k\\ h+k=j-\sigma}} a_h a_k A - \frac{1}{2}a_m^2 A \end{aligned} \tag{3.22}$$

for $m = \dfrac{j-\sigma}{2}, j = 1,\dots,N-1$; note that the last two terms in the above equation disappear if the conditions $h+k=j-\sigma$, $m=\dfrac{j-\sigma}{2}\in\{1,\dots,j+1\}$ are not satisfied for any $j,k=1,\dots,N$. Having in mind (3.18) we impose to this equation the following initial data

$$a_{j+1,\theta}(0,x,l(x)/\varepsilon)\varphi_t(0,x) = b_{1,j}(x)e^{il(x)/\varepsilon} - a_{j,t}(0,x,l(x)/\varepsilon). \tag{3.23}$$

Again, for the existence of local solution $a_{j+1,\theta}$ for these Cauchy problems for higher order transport equations we recall results in [11]; with the same argument discussed for a_1 we determine a_{j+1} from $a_{j+1,\theta}$. Moreover, by (3.17) we obtain $a_{j+1}(0,x,l(x)/\varepsilon) = b_{0,j}(x)e^{il(x)/\varepsilon}$. Therefore, if we combine this identity with (3.23), we deduce to the following compatibility condition for $b_{0,j}(x)$ and $b_{1,j}(x)$

$$ib_{0,j}(x)e^{il(x)/\varepsilon}\varphi_t(0,x) = b_{1,j}(x)e^{il(x)/\varepsilon} - a_{j,t}(0,x,l(x)/\varepsilon).$$

4. Justification of the oscillatory expansion. Using results in Section 2 we want to justify the asymptotic expansion in the previous Section 3.

By Theorem 2.1, in the assumption $\|h\|_{H^{\frac{1}{2}}(\mathbb{R}^n)} \leq C$, we derive for H the following estimate

$$\|H\|_{L^q(\mathbb{R}^{1+n}_+)} \leq C_H, \tag{4.1}$$

with $q = 2\dfrac{n+1}{n-1}$; moreover note that $a(t,x,\theta;\varepsilon)$ solves the following problem

$$\begin{cases} (\partial_{tt} - \Delta)a = F_\lambda(H+a) - F_\lambda(H) \\ a(0,x,\theta;\varepsilon) = \varepsilon^{\sigma+1} e^{il(x)/\varepsilon} b_0(\varepsilon,x) \\ \partial_t a(0,x,\theta;\varepsilon) = \varepsilon^\sigma e^{il(x)/\varepsilon} b_1(\varepsilon,x) \end{cases} \tag{4.2}$$

and the initial data for $a(t,x,\theta;\varepsilon)$ satisfy the assumptions of Theorem 2.1. Therefore the same argument of this theorem leads to the proof of the following

Corollary 4.1. *Let $a(t,x,\theta;\varepsilon)$ be the solution to problem* (4.2). *Then there exists a constant $C>0$ independent of ε such that*

$$\|a(\cdot,\cdot,\theta;\varepsilon)\|_{L^q(\mathbb{R}^{1+n}_+)} \leq C\ \varepsilon^\sigma$$

for $q = 2\dfrac{n+1}{n-1}$.

Proof. In this proof we denote $\|a(\cdot,\cdot,\theta;\varepsilon)\|_{L^q(\mathbb{R}^{1+n}_+)}$ simply by $\|a\|_{L^q(\mathbb{R}^{1+n}_+)}$. We observe that

$$|F_\lambda(H+a) - F_\lambda(H)| \leq C|a|\left(|a|^{\lambda-1} + |H|^{\lambda-1}\right);$$

then

$$\begin{aligned} &\|F_\lambda(H+a) - F_\lambda(H)\|_{L^p(\mathbb{R}^{1+n}_+)} \leq \\ &\leq C\|a\|_{L^p(\mathbb{R}^{1+n}_+)}\left(\||a|^{\lambda-1}\|_{L^p(\mathbb{R}^{1+n}_+)} + \||H|^{\lambda-1}\|_{L^p(\mathbb{R}^{1+n}_+)}\right) \\ &\leq C\|a|a|^{\lambda-1}\|_{L^p(\mathbb{R}^{1+n}_+)} + C\|a\|_{L^p(\mathbb{R}^{1+n}_+)}\||H|^{\lambda-1}\|_{L^p(\mathbb{R}^{1+n}_+)} \\ &\leq C\|a\|^\lambda_{L^q(\mathbb{R}^{1+n}_+)} + C\|a\|_{L^q(\mathbb{R}^{1+n}_+)}\||H|^{\lambda-1}\|_{L^r(\mathbb{R}^{1+n}_+)}; \end{aligned} \tag{4.3}$$

here we used the fact that $\lambda p = q$ and we applied Hölder inequality with $\dfrac{1}{r} + \dfrac{1}{q} = \dfrac{1}{p}$. Moreover by (4.1) we have

$$\||H|^{\lambda-1}\|_{L^r(\mathbb{R}^{1+n}_+)} = \|H\|^{\lambda-1}_{L^{r(\lambda-1)}(\mathbb{R}^{1+n}_+)} = \|H\|^{\lambda-1}_{L^q(\mathbb{R}^{1+n}_+)} \leq C_H^{\lambda-1};$$

now we can use Young inequality to estimate the term $\|a\|_{L^q(\mathbb{R}^{1+n}_+)} C_H^{\lambda-1}$; we obtain that

$$\|a\|^{\frac{1}{\lambda}}_{L^q(\mathbb{R}^{1+n}_+)}(C_H)^{\frac{\lambda-1}{\lambda}} \leq C\left(\|a\|_{L^q(\mathbb{R}^{1+n}_+)} + C_H\right)$$

and this implies that

$$\|a\|_{L^q(\mathbb{R}^{1+n}_+)}(C_H)^{\lambda-1} \leq C\|a\|^{\lambda}_{L^q(\mathbb{R}^{1+n}_+)} + CC^{\lambda}_H. \tag{4.4}$$

Then by (4.3) and (4.4) we conclude that

$$\|F_\lambda(H+a) - F_\lambda(H)\|_{L^p(\mathbb{R}^{1+n}_+)} \leq C\|a\|^{\lambda}_{L^q(\mathbb{R}^{1+n}_+)} + C.$$

Finally we observe that

$$\|a(0,\cdot,l(x)/\varepsilon;\varepsilon)\|_{H^{\frac{1}{2}}(\mathbb{R}^n)} + \|\partial_t a(0,\cdot,l(x)/\varepsilon;\varepsilon)\|_{L^2(\mathbb{R}^n)} \leq C\varepsilon^{\sigma},$$

then with the same argument of Theorem 2.1 we conclude the proof. □

At this point we must estimate the remainder $r_N(t,x,\theta;\varepsilon)$ in the above approximation (3.3). We start with the simplest case $N=1$, then we shall generalize the result for $N \geq 1$.

For $N=1$ the approximation (3.3) takes the form

$$u_\varepsilon(t,x) = H(t,x) + \varepsilon^{\sigma+1}a_1(t,x,\theta) + r_1(t,x,\theta;\varepsilon);$$

from (3.4) and (3.5) we obtain

$$\begin{aligned}
&(\partial_{tt}-\Delta)H + \varepsilon^{\sigma-1}a_{1,\theta\theta}\left(\varphi_t^2 - |\nabla\varphi|^2\right) + \varepsilon^{\sigma}[2\partial_\mu\varphi\partial^\mu a_{1,\theta} + \\
&+ a_{1,\theta}(\partial_{tt}-\Delta)\varphi] + \varepsilon^{\sigma+1}(\partial_{tt}-\Delta)a_1 + \varepsilon^{-2}r_{1,\theta\theta}\left(\varphi_t^2 - |\nabla\varphi|^2\right) + \\
&\varepsilon^{-1}[2\partial_\mu\varphi\partial^\mu r_{1,\theta} + r_{1,\theta}(\partial_{tt}-\Delta)\varphi] + (\partial_{tt}-\Delta)r_1 =
\end{aligned}$$

$$= F(H) + \varepsilon^{\sigma+1}F'(H)a_1 + F'(H)r_1 + \frac{1}{2}\varepsilon^{2\sigma+2}a_1^2A + 2\varepsilon^{\sigma+1}a_1Ar_1 + \frac{1}{2}Ar_1^2, \tag{4.5}$$

where

$$A = \int_0^1 (1-\gamma)F''(H+\gamma a)d\gamma;$$

note that in the previous expression the term r_1 is of order $\varepsilon^{\sigma+2}$. If we require that in (4.5) the terms of order $\varepsilon^{\sigma+2}$ are zero, then, having in mind (3.13), (3.14) and (3.16), we obtain for $r_{1,\theta}$ the following equation

$$\partial_t r_{1,\theta} - \sum_{j=1}^{n}\mu_j\partial_{x_j}r_{1,\theta} + \nu r_{1,\theta} = f, \tag{4.6}$$

with $\mu_j = \dfrac{\partial_{x_j}\varphi}{2\partial_t\varphi}$, $\nu = \dfrac{(\partial_{tt}-\Delta)\varphi}{2\partial_t\varphi}$ and $f = -\varepsilon^{\sigma+2}(\partial_{tt}-\Delta)a_1 + \varepsilon^{\sigma+2}F'(H)a_1$.

Moreover, if we impose that in (3.18) the terms of order $\varepsilon^{\sigma+1}$ must be equal, we get for $r_{1,\theta}$ the following initial condition

$$r_{1,\theta}(0,x,\frac{l(x)}{\varepsilon};\varepsilon)\varphi_t(0,x)=\varepsilon^{\sigma+2}b_{1,1}(x)e^{il(x)/\varepsilon}. \tag{4.7}$$

The existence of local solution to problem (4.6), (4.7) is well known (see [11]); to determine r_1 from $r_{1,\theta}$ we use the same argument discussed above for a_1.

At this point we note that if we set $B=\sum\limits_{j=1}^{n}\mu_j\partial_{x_j}$, then (4.6) can be written as follows

$$\partial_t r_{1,\theta}=Br_{1,\theta}-\nu r_{1,\theta}+f. \tag{4.8}$$

By differentiating with respect to t both sides of (4.8) we obtain

$$\partial_{tt}r_{1,\theta}-B^2r_{1,\theta}=(\nu^2-B\nu-\nu B+\partial_tB-\partial_t\nu)r_{1,\theta}+(B-\nu+\partial_t)f; \tag{4.9}$$

note that $B^2=\sum\limits_{j,k=1}^{n}\mu_j\mu_k\partial_{x_j}\partial_{x_k}$ and for $\mu=(\mu_1,\dots,\mu_n)$ by eikonal equation we have $\|\mu\|=\left|\frac{\nabla\varphi}{2\partial_t\varphi}\right|=\frac{1}{2}$.

For the initial data we impose in addition to (4.9) the condition (4.7) and by (4.8) we require that

$$(\partial_t r_{1,\theta})\mid_{t=0}=(Br_{1,\theta}-\nu r_{1,\theta}+f)\mid_{t=0}. \tag{4.10}$$

Since the operator $\nu^2-2B\nu+\partial_tB-\partial_t\nu$ is bounded, by local estimates of Strichartz type in the case of variable coefficients (cf. [2]) we obtain for $r_{1,\theta}$ the following inequality

$$\|r_{1,\theta}(\cdot,\cdot,\theta;\varepsilon)\|_{L^q([0,T]\times\{|x|\le R+t\})}\le C(R,T)\|(B-\nu+\partial_t)f\|_{L^p([0,T]\times\{|x|\le R+t\})}+$$
$$+\|r_{1,\theta}(0,\cdot,l(x)/\varepsilon;\varepsilon)\|_{\overset{\circ}{H}^{\frac{1}{2}}(\mathbb{R}^n)}+\|\partial_t r_{1,\theta}(0,\cdot,l(x)/\varepsilon;\varepsilon)\|_{\overset{\circ}{H}^{-\frac{1}{2}}(\mathbb{R}^n)}.$$

Note that for any $x\in K$ and for any $\theta\in[0,2\pi]$ there exists $\varepsilon=\varepsilon(x)>0$ such that $r_{1,\theta}(0,x,\theta;\varepsilon)=r_{1,\theta}(0,x,l(x)/\varepsilon;\varepsilon)$; then by our assumptions on b_1 and φ we deduce from (4.7) that

$$\|r_{1,\theta}(0,\cdot,l(x)/\varepsilon;\varepsilon)\|_{\overset{\circ}{H}^{\frac{1}{2}}(\mathbb{R}^n)}\|r_{1,\theta}(0,\cdot,l(x)/\varepsilon;\varepsilon)\|_{H^{\frac{1}{2}}(\mathbb{R}^n)}\le C\varepsilon^{\sigma+2}. \tag{4.11}$$

Similarly, from (4.10) and (4.11) we obtain

$$\|\partial_t r_{1,\theta}(0,\cdot,l(x)/\varepsilon;\varepsilon)\|_{\overset{\circ}{H}^{\frac{1}{2}}(\mathbb{R}^n)}\|\partial_t r_{1,\theta}0,\cdot,l(x)/\varepsilon;\varepsilon)\|_{L^2(\mathbb{R}^n)}\le C\varepsilon^{\sigma+2}.$$

Therefore our next step is to estimate the local L^p-norm of $(B-\nu+\partial_t)f$; for simplicity, here below we write L^p_{loc} instead of $L^p([0,T]\times\{|x|\le R+t\})$. Note that

$$\|Bf\|_{L^p_{loc}} \leq C\varepsilon^{\sigma+2}\|\nabla_x(F'(H)a_1)\|_{L^p_{loc}} + C\varepsilon^{\sigma+2}\|\nabla_x(\partial_{tt}-\Delta)a_1)\|_{L^p_{loc}};$$

in particular

$$\|\nabla_x(F'(H)a_1)\|_{L^p_{loc}} = C\|\nabla_x(|H|^{\lambda-2}Ha_1)\|_{L^p_{loc}} \leq$$

$$\leq C\|\nabla_x H|H|^{\lambda-2}\|_{L^r_{loc}}\|a_1\|_{L^q_{loc}} + C\|H|H|^{\lambda-2}\|_{L^r_{loc}}\|\nabla_x a_1\|_{L^q_{loc}},$$

with $\frac{1}{r}+\frac{1}{q}=\frac{1}{p}$. Applying again Hölder inequality, we get

$$\|\nabla_x H|H|^{\lambda-2}\|_{L^r_{loc}} \leq \|\nabla_x H\|_{L^q_{loc}}\|H\|^{\lambda-2}_{L^{r_2(\lambda-2)}_{loc}},$$

where $\frac{1}{r_2}+\frac{1}{q}=\frac{1}{r}$; this gives $r_2(\lambda-2)=q$, since $\lambda p = q$. Similarly,

$$\|H|H|^{\lambda-2}\|_{L^r_{loc}} \leq \|H\|_{L^q_{loc}}\|H\|^{\lambda-2}_{L^{r_2(\lambda-2)}_{loc}} = \|H\|^{\lambda-1}_{L^q_{loc}},$$

therefore we have

$$\|\nabla_x(F'(H)a_1)\|_{L^p_{loc}} \leq C\|\nabla_x H\|_{L^q_{loc}}\|H\|^{\lambda-2}_{L^q_{loc}}\|a_1\|_{L^q_{loc}} + C\|H\|^{\lambda-2}_{L^q_{loc}}\|\nabla_x a_1\|_{L^q_{loc}} \leq$$

$$\leq C\|\nabla_x H\|_{L^q_{loc}}\|a_1\|_{L^q_{loc}} + C\|\nabla_x a_1\|_{L^q_{loc}}.$$

Since H is the solution to (3.13), we conclude that $\nabla_x H$ satisfies the following problem

$$\begin{cases} (\partial_{tt}-\Delta)(\nabla_x H) = C|H|^{\lambda-2}\nabla_x H \\ \nabla_x H(0,x) = \nabla_x h(x) \\ \partial_t(\nabla_x H)(0,x) = 0; \end{cases}$$

therefore we derive from Theorem 2.1 the following estimate

$$\|\nabla_x H\|_{L^q_{loc}} \leq C.$$

Now it remains to estimate L^q_{loc}-norm of a_1 and $\nabla_x a_1$. First, recall that we can recover a_1 using (3.21); therefore we have Poincaré inequality for the periodic in θ function $a_1(\cdot,\cdot,\theta)$

$$\|a_1\|_{L^q_{loc}} \leq \sup_{0\leq\theta\leq 2\pi}\|a_{1,\theta}(\cdot,\cdot,\theta)\|_{L^q_{loc}}. \tag{4.12}$$

With the same notations and the same argument discussed above for $r_{1,\theta}$, differentiating with respect to t (3.16), we obtain for $a_{1,\theta}$ the following second order

Cauchy problem

$$(4.13)\qquad \begin{cases} (\partial_{tt} - B)a_{1,\theta} = (\nu^2 - \nu B - B\nu - \partial_t \nu)a_{1,\theta} \\ a_{1,\theta}(0, x, l(x)/\varepsilon)\varphi_t(0, x) = b_{1,0}(x)e^{il(x)/\varepsilon} \\ (\partial_t a_{1,\theta})(0, x, l(x)/\varepsilon) = (Ba_{1,\theta} - \nu a_{1,\theta})\mid_{t=0}; \end{cases}$$

therefore as for $r_{1,\theta}$ we have the following estimate

$$\|a_{1,\theta}\|_{L^q_{loc}} \le C\|a_{1,\theta}(0, \cdot, l(x)/\varepsilon)\|_{H^{\frac{1}{2}}(\mathbb{R}^n)} + C\|\partial_t a_{1,\theta}(0, \cdot, l(x)/\varepsilon)\|_{L^2} \le C.$$

Recall that also for $a_{1,\theta}$, for any $x \in K$ and for any $\theta \in [0, 2\pi]$ we have $a_{1,\theta}(0, x, \theta) = a_{1,\theta}(0, x, l(x)/\varepsilon)$ for some $\varepsilon = \varepsilon(x) > 0$; therefore (4.12) in combination with the last inequality leads to the following estimate for a_1

$$\|a_1(\cdot, \cdot, \theta)\|_{L^q_{loc}} \le C.$$

From (4.13) we compute that $\nabla_x a_{1,\theta}$ satisfies the following Cauchy problem

$$(4.14)\quad \begin{cases} (\partial_{tt} - \Delta)(\nabla_x a_{1,\theta}) = (\partial_t B - B\nu - \nu B - \partial_t \nu + \nu^2)(\nabla_x a_{1,\theta}) + \\ \qquad + \left(\nabla_x (B^2 + \partial_t B - B\nu - \nu B - \partial_t \nu + \nu^2)\right) a_{1,\theta} \\ \nabla_x a_{1,\theta}(0, x, l(x)/\varepsilon)\varphi_t(0, x) = \nabla_x \left(b_{1,0}(x)e^{il(x)/\varepsilon}\right) \\ \qquad - \nabla_x \varphi_t(0, x)\dfrac{b_{1,0}(x)e^{il(x)/\varepsilon}}{\varphi_t(0, x)} \\ \partial_t \left(\nabla_x a_{1,\theta}\right)\mid_{t=0} = \left(B\nabla_x a_{1,\theta} + \nabla_x B a_{1,\theta} - \nu \nabla_x a_{1,\theta} - \nabla_x \nu a_{1,\theta}\right)\mid_{t=0}. \end{cases}$$

Again, by results in [2] we derive for $\nabla_x a_{1,\theta}$ the following estimate

$$\|\nabla_x a_{1,\theta}(\cdot, \cdot, \theta)\|_{L^q_{loc}} \le C\|\nabla_x a_{1,\theta}(0, \cdot, l(x)/\varepsilon)\|_{H^{\frac{1}{2}}(\mathbb{R}^n)} + \\ + C\|\partial_t \nabla_x a_{1,\theta}(0, \cdot, l(x)/\varepsilon)\|_{L^2(\mathbb{R}^n)};$$

then by initial data in (4.14) and by our assumptions on b_1, φ and l we have

$$\|\nabla_x a_{1,\theta}(\cdot, \cdot, \theta)\|_{L^q_{loc}} \le C.$$

As for r_1 and a_1, from this last inequality we derive the following estimate for $\nabla_x a_{1,}$

$$\|\nabla_x a_{1,}(\cdot, \cdot, \theta)\|_{L^q_{loc}} \le C.$$

Recalling all these inequalities we conclude that

$$\|\nabla_x (F'(H)a_1)\|_{L^p_{loc}} \le C.$$

Exactly the same arguments allow us to get analogous results for the L^p_{loc}-norm of $\nabla_x((\partial_{tt}-\Delta)a_1)$; therefore we obtain that

$$\|Bf\|_{L^p_{loc}} \leq C\varepsilon^{\sigma+2},$$

for some costant C independent of ε.

The following estimate of the L^p_{loc}-norm of the term $(\partial_t - \nu)f$ is proved with the same tools

$$\|(\partial_t - \nu)f\|_{L^p_{loc}} \leq C\varepsilon^{\sigma+2},$$

therefore, if we combine all these results we get for $r_{1,\theta}$ the following estimate

$$\|r_{1,\theta}(\cdot,\cdot,\theta;\varepsilon)\|_{L^q_{loc}} \leq C\varepsilon^{\sigma+2}.$$

Since as for a_1, we can recover r_1 from $r_{1,\theta}$ by the formula

$$r_1(t,x,\theta;\varepsilon) = -\frac{1}{2\pi}\int_0^{2\pi}\left(\int_0^{\theta} r_{1,\eta}(t,x,\eta;\varepsilon)d\eta\right)d\theta + \int_0^{\theta} r_{1,\eta}(t,x,\eta;\varepsilon)d\eta,$$

we have

$$\begin{aligned}\|r_1(\cdot,\cdot,\theta;\varepsilon)\|_{L^q([0,T]\times\{|x|\leq R+t\})} &\leq \sup_{0\leq\theta\leq 2\pi}\|r_{1,\theta}(\cdot,\cdot,\theta;\varepsilon)\|_{L^q([0,T]\times\{|x|\leq R+t\})}\\ &\leq C(R,T)\varepsilon^{\sigma+2}.\end{aligned}$$

In general, for $N \geq 1$ the remainder $r_{N,\theta}$ satisfies the following Cauchy problem

$$\begin{cases} \partial_t r_{N,\theta} - \sum\limits_{j=1}^{n}\mu_j\partial_{x_j}r_{N,\theta} + \nu r_{N,\theta} = g,\\ \varepsilon^{-1}r_{N,\theta}(0,x,l(x)/\varepsilon;\varepsilon)\varphi_t(0,x) = \varepsilon^{\sigma+N}b_{1,N}(x)e^{il(x)/\varepsilon},\end{cases}$$

where μ_j and ν are as before and

$$\begin{aligned} g = &-\varepsilon^{\sigma+N+1}\frac{(\partial_{tt}-\Delta)a_N}{2\varphi_t}\varepsilon^{\sigma+N+1}F'(H)\frac{a_N}{2\varphi_t}+\\ &-\sum_{\substack{h,k=1,h\neq k\\ h+k=N-\sigma}}^{N}\varepsilon^{2\sigma+1+h+k}\frac{a_h a_k}{2\varphi_t}A - \frac{1}{2}\varepsilon^{2\sigma+1+2m}\frac{a_m^2}{2\varphi_t}A.\end{aligned}$$

With the same argument discussed above for $r_{1,\theta}$ we derive for $r_{N,\theta}$ the following second order Cauchy problem

$$\begin{cases} (\partial_{tt} - B^2)r_{N,\theta} = (\nu^2 - B\nu - \nu B + \partial_t B - \partial_t\nu)r_{N,\theta} + (B - \nu + \partial_t)g,\\ r_{N,\theta}(0,x,l(x)/\varepsilon;\varepsilon)\varphi_t(0,x) = \varepsilon^{\sigma+N+1}b_{1,N}(x)e^{il(x)/\varepsilon},\\ \partial_t r_{N,\theta}(0,x,l(x)/\varepsilon;\varepsilon) = (Br_{N,\theta} - \nu r_{N,\theta} + g)\mid_{t=0}.\end{cases}$$

Since again

$$\|(B - \nu + \partial_t)g\|_{L^p_{loc}} \leq C\varepsilon^{\sigma+N+1},$$

we can state in general the following

Theorem 4.2. *For any $N \geq 1$ the remainder $r_N(t, x; \varepsilon)$ satisfies the estimate*

$$\|r_N(\cdot, \cdot, \theta; \varepsilon)\|_{L^q([0,T]\times\{|x|\leq R+t\})} \leq C_N\varepsilon^{\sigma+N+1}$$

where the constant $C_N > 0$ is independent of ε.

Remark 4.3. In the previous corollary we have just local estimate since the Cauchy problems for eikonal equation and for transport equations have only local solutions (see [20]); in order to obtain global estimate on r_N it is necessary that $\varphi(t, x)$ and $a_j(t, x, \theta)$ should be defined for every $t \geq 0$.

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REFERENCES

[1] M. Bezard. Une version générale de l'inegalité de Strichartz. *C. R. Acad. Sci. Paris* **315** (1992), 1241–1244.

[2] P. Brenner. $L^p - L^{p'}$-Estimates for Fourier Integral Operators Related to Hyperbolic Equations. *Math. Z.* **152** (1977), 273–286.

[3] R. Carles. Precised behaviours near a caustic in nonlinear geometric optics. *Comm. Partial Differential Equations* **23**, *11–12* (1998), 1929–1967.

[4] R. Carles. Ph. D. Thesis, 1999 (available at `www.maths.univ-rennes1.fr/~carles`).

[5] P. D'Ancona, V. Georgiev, H. Kubo. L^p-weighted decay estimate for wave equation. *C. R. Acad. Sci. Paris* (to appear 2000).

[6] S. Di Pomponio. Nonlinear oscillatory problem for semilinear wave equation. Preprint, 1999.

[7] V. Georgiev, H. Lindblad, C. D. Sogge. Weighted Strichartz estimates and global existence for semilinear wave equation. *Amer. J. Math.* **119** (1997), 1291–1319.

[8] R. T. Glassey. Finite-time blow-up for solutions of nonlinear wave equations. *Math. Z.* **177** (1981), 323–340.

[9] R. T. Glassey. Existence in the large for $(\partial_{tt} - \Delta)u = F(u)$ in two space dimensions. *Math. Z.* **178** (1981), 233–261.

[10] F. John. Blow-up of solutions of nonlinear wave equations in three space dimensions. *Manuscripta Math.* **28** (1979), 235–265.

[11] F. John. Partial Differential Equations, 4-th ed. Springer-Verlag, 1982. Wien–New York.

[12] J. L. Joly, G. Métivier, J. Rauch. Focusing at a point and absorption of nonlinear oscillations. *Trans. Amer. Math. Soc.* **347** (1995), 3921–3969.

[13] J. L. Joly, J. Rauch. Justification of Multidimensional Single Phase Semilinear Geometric Optics. *Trans. Amer. Math. Soc.* **330**, *2* (1992), 599–623.

[14] D. Ludwig. Uniform asymptotic Expansions at a caustic. *Comm. Pure Appl. Math.* **19** (1966), 215–250.

[15] J. Rauch. Lectures on geometric optics, 1995 (available at `www.math.lsa.umich.edu/~rauch`).

[16] I. Segal. Space-time decay for solutions of wave equations. *Adv. Math.* **22** (1976), 305–311.

[17] T. C. Sideris. Nonexistence of global solutions to semilinear wave equation in high dimensions. *J. Differential Equations* **52** (1984), 378–406.

[18] W. Strauss. Nonlinear scattering theory in scattering theory in Mathematical Physics. *Reidel Dordrect* (1979), 53–79.

[19] W. Strauss. Nonlinear scattering theory at low energy. *J. Funct. Anal.* **41** (1981), 110–133.

[20] M. Taylor. Partial Differential Equations, basic theory. Springer-Verlag, 1997, Wien–New York.

Dipartimento di Matematica Pura ed Applicata
Università degli Studi di L'Aquila
via Vetoio, loc. Coppito – 67100 L'Aquila
Italy
e-mail `dipompon@univaq.it`

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