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LAPLACE TRANSFORMS OF THE BROWNIAN MOTION'S FIRST EXIT FROM A STRIP

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Abstract

The aim of this paper is to investigate the Laplace transforms related to the first exit time from a strip of a Brownian motion. We suppose the existence of a deterministic terminal moment. If the exit time is before this moment, we know the corresponding value of the Brownian motion and hence we have to derive the Laplace transform of the exit time. Otherwise, if the Brownian motion stays in the strip till the terminal date, we determine the Brownian motion Laplace transform at this moment.

Key words: Brownian motion, stopping times, first exit, two sided boundaries, Laplace transform

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1. Introduction. There are many studies which examine the first exit time of a Brownian motion from a strip – see [1–3]. In the present research we assume that the strip boundaries are continuous piecewise linear functions and we introduce a terminal time for the exit. We examine also the case when one of the boundaries vanishes after some moment. The great importance of the Brownian motion is seen by its wide use in many practical problems. Some examples which lead to first exit problems can be found in the financial markets. First, the two-sided barrier derivatives are a large class of the insurance instruments. Another outstanding example includes derivatives which allow early exercising. For example, the American style options lead to a one-sided optimal stopping problem –

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see [4,5]. When the early exercise right is given to both contract participants we have a two-sided optimal exit problem. Such derivatives involve different game options (cancelable puts and calls, see [6–8]), convertible bonds, etc. Note that if the exit happens before the terminal date, we know the Brownian motion value, but the exit moment is unknown, and vice versa. Because of this we examine separately both cases. At last, but not least we have to mention that the derived results can be an implement to the numerous existing numerical methods – some examples of such algorithms are examined in [9–11].

2. Preliminaries. Let B_t be a Brownian motion; T be the terminal moment, $T \leq \infty$; and $0 \equiv t_0 < t_1 < \cdots < t_n \equiv T$ be a division of the time interval. Let α_i and β_i, $i = 0, 1, \ldots, n$, be some values such that $\alpha_0 < 0 < \beta_0$, $\alpha_i < \beta_i$, and $\gamma_i = \beta_i - \alpha_i$. The case when some last values of α_i or β_i are infinity is also admissible. Let the functions $a_i(t) = a_{i,1}t + a_{i,2}$ and $b_i(t) = b_{i,1}t + b_{i,2}$ be the linear functions which connect the points α_i with α_{i+1} and β_i with β_{i+1}, respectively. Let the piecewise linear functions $a(t) < b(t)$ be composed by these functions. We shall denote by τ_1 and τ_2 the first hitting times to the functions $a(\cdot)$ and $b(\cdot)$, respectively, and by τ the lower one – $\tau = \tau_1 \wedge \tau_2$. Let us notate by Λ_t the indicator processes $\Lambda_t = I_{\tau \leq T}$ and by $N(\cdot)$ the cumulative distribution function of the standard normal distribution.

3. Linear boundaries. We shall use several times the following lemma. The proof of its first statement is an immediate consequence of Theorem 4.2 from [1]. See also the proof of Theorem 3 from [2]. The second statement is proven in [12].

Lemma 3.1. *Let $a(T) < z < b(T)$. Then*

$$P(\tau > T | B_T = z) = 1 - \sum_{j=1}^{\infty} q_j(0, z; 1), \tag{3.1}$$

where

$$\begin{aligned} q_j(y, z; i) = \exp&\left(-\frac{2[j\gamma_{i-1} + \alpha_{i-1} - y][j\gamma_i + \alpha_i - z]}{t_i - t_{i-1}}\right) \\ - \exp&\left(-\frac{2j[j\gamma_{i-1}\gamma_i + \gamma_{i-1}(\alpha_i - z) - \gamma_i(\alpha_{i-1} - y)]}{t_i - t_{i-1}}\right) \\ + \exp&\left(-\frac{2[j\gamma_{i-1} - (\beta_{i-1} - y)][j\gamma_i - (\beta_i - z)]}{t_i - t_{i-1}}\right) \\ - \exp&\left(-\frac{2j[j\gamma_{i-1}\gamma_i - \gamma_{i-1}(\beta_i - z) + \gamma_i(\beta_{i-1} - y)]}{t_i - t_{i-1}}\right). \end{aligned} \tag{3.2}$$

If $a(T) = -\infty$, then formula (3.1) *turns to*

$$P(\tau > T | B_T = z) = 1 - \exp\left(-\frac{2b_2(b(T) - z)}{T}\right). \tag{3.3}$$

Theorems 4.3 and 5.1 from [1] give the probabilities and the corresponding densities the hitting time to be before T. We give them as the following lemma.

Lemma 3.2.

(3.4)

$$P_1^l(T;a_1,a_2,b_1,b_2) \equiv P((\tau_1 \wedge \tau_2) < T, \tau_1 < \tau_2)$$
$$= N\left(\frac{a_1T+a_2}{\sqrt{T}}\right)$$
$$+\sum_{j=1}^{\infty}\left\{\begin{array}{l} e^{-2[-ja_2+(j-1)b_2][-ja_1+(j-1)b_1]}N\left(\frac{-a_1T-2(j-1)b_2+(2j-1)a_2}{\sqrt{T}}\right)\\ -e^{-2\left[j^2(a_1a_2+b_1b_2)-j(j-1)a_2b_1-j(j+1)b_2a_1\right]}N\left(\frac{-a_1T-2jb_2+(2j-1)a_2}{\sqrt{T}}\right)\\ -e^{-2[-(j-1)a_2+jb_2][-(j-1)a_1+jb_1]}N\left(\frac{a_1T-2jb_2+(2j-1)a_2}{\sqrt{T}}\right)\\ +e^{-2\left[j^2(a_1a_2+b_1b_2)-j(j-1)b_2a_1-j(j+1)a_2b_1\right]}N\left(\frac{a_1T+(2j+1)a_2-2jb_2}{\sqrt{T}}\right)\end{array}\right\}$$

$$P_2^l(T;a_1,a_2,b_1,b_2) \equiv P((\tau_1 \wedge \tau_2) < T, \tau_2 < \tau_1)$$
$$= 1-N\left(\frac{b_1T+b_2}{\sqrt{T}}\right)$$
$$+\sum_{j=1}^{\infty}\left\{\begin{array}{l} e^{-2[jb_2-(j-1)a_2][jb_1-(j-1)a_1]}N\left(\frac{b_1T+2(j-1)a_2-(2j-1)b_2}{\sqrt{T}}\right)\\ -e^{-2\left[j^2(b_1b_2+a_1a_2)-j(j-1)b_2a_1-j(j+1)a_2b_1\right]}N\left(\frac{b_1T+2ja_2-(2j-1)b_2}{\sqrt{T}}\right)\\ -e^{-2[(j-1)b_2-ja_2][(j-1)b_1-ja_1]}N\left(\frac{-b_1T+2ja_2-(2j-1)b_2}{\sqrt{T}}\right)\\ +e^{-2\left[j^2(b_1b_2+a_1a_2)-j(j-1)a_2b_1-j(j+1)b_2a_1\right]}N\left(\frac{-b_1T-(2j+1)b_2+2ja_2}{\sqrt{T}}\right)\end{array}\right\}.$$

(3.5)

$$p_1^l(t;a(\cdot),b(\cdot)) \equiv \frac{dP((\tau_1 \wedge \tau_2) \in dt, \tau_1 < \tau_2)}{dt}$$
$$= \frac{1}{\sqrt{2\pi}t^{\frac{3}{2}}}e^{-\frac{(a_1t+a_2)^2}{2t}}\sum_{j=0}^{\infty}\left\{\begin{array}{l} e^{-\frac{2j[jb_2-(j+1)a_2][b(t)-a(t)]}{t}}[2jb_2-(2j+1)a_2]\\ -e^{-\frac{2(j+1)[(j+1)b_2-ja_2][b(t)-a(t)]}{t}}[2(j+1)b_2-(2j+1)a_2]\end{array}\right\}$$

$$p_2^l(t;a(\cdot),b(\cdot)) \equiv \frac{dP((\tau_1 \wedge \tau_2) \in dt, \tau_2 < \tau_1)}{dt}$$
$$= \frac{1}{\sqrt{2\pi}t^{\frac{3}{2}}}e^{-\frac{(b_1t+b_2)^2}{2t}}\sum_{j=0}^{\infty}\left\{\begin{array}{l} e^{-\frac{2j[(j+1)b_2-ja_2][b(t)-a(t)]}{t}}[(2j+1)b_2-(2j+1)a_2]\\ -e^{-\frac{2(j+1)[jb_2-(j+1)a_2][b(t)-a(t)]}{t}}[(2j+1)b_2-2ja_2]\end{array}\right\}.$$

Our first result is presented in the following theorem.

Theorem 3.1. *Let $\theta > 0$. The Laplace transforms of τ if it is before T are*

$$\begin{aligned} L_1(t,\theta;a(\cdot),b(\cdot)) &= E\left[e^{-\theta\tau}\Lambda_T I_{\tau=\tau_1}\right] \\ &= e^{a_2\left(\sqrt{a_1^2+2\theta}-a_1\right)} P_1^l\left(T;\sqrt{a_1^2+2\theta},a_2,b_1+\sqrt{a_1^2+2\theta}-a_1,b_2\right) \\ L_2(t,\theta;a(\cdot),b(\cdot)) &= E\left[e^{-\theta\tau}\Lambda_T I_{\tau=\tau_2}\right] \\ &= e^{b_2\left(\sqrt{b_1^2+2\theta}-b_1\right)} P_2^l\left(T;a_1+\sqrt{b_1^2+2\theta}-b_1,a_2,\sqrt{b_1^2+2\theta},b_2\right), \end{aligned} \tag{3.6}$$

where $P_1^l(\cdot)$ and $P_2^l(\cdot)$ are given by equations (3.4).

Proof. Using equation (3.5) and the form of the normal distribution density we can easily obtain

$$\begin{aligned} E\left[e^{-\theta\tau}\Lambda_T I_{\tau=\tau_1}\right] &= \int_0^\infty e^{-\theta t} I_{t<T} p_1^l(t;a_1,a_2,b_1,b_2)\,dt \\ &= e^{a_2\left(\sqrt{a_1^2+2\theta}-a_1\right)} P_1^l\left(T;\sqrt{a_1^2+2\theta},a_2,b_1+\sqrt{a_1^2+2\theta}-a_1,b_2\right). \end{aligned}$$

The Laplace transform L_2 can be derived analogously. □

Note that the four terms of q_j in formula (3.2) (for $i=1$) can be presented as exponent of linear functions $\lambda_{j,k}z+\xi_{j,k}$, $k=\{1,2,3,4\}$, divided by T. We shall use also the notations $s_2=s_4=1$ and $s_1=s_3=-1$. Using this presentation we can formulate our second result in the following way.

Theorem 3.2. *If $a(T) < z < b(T)$, then the Laplace transform of the Brownian motion if τ is after T is*

$$\begin{aligned} V(\theta,z,T,a(\cdot),b(\cdot)) &= E\left[e^{\theta B_T} I_{\tau>T,B_T>z}\right] \\ &= \exp\left(\frac{\theta^2 T}{2}\right)\left\{\begin{array}{l} \left(N\left(\frac{b(T)-\theta T}{\sqrt{T}}\right)-N\left(\frac{z-\theta T}{\sqrt{T}}\right)\right)+ \\ +\sum_{j=1}^{\infty}\sum_{k=1}^{4}\left[\begin{array}{l} s_k\exp\left(\lambda_{j,k}\theta+\left(\frac{\lambda_{j,k}^2+2\xi_{j,k}}{2T}\right)\right)\times \\ \left(N\left(\frac{b(T)-(\theta T+\lambda_{j,k})}{\sqrt{T}}\right)\right. \\ \left.-N\left(\frac{z-(\theta T+\lambda_{j,k})}{\sqrt{T}}\right)\right) \end{array}\right] \end{array}\right\}. \end{aligned} \tag{3.7}$$

Proof. Using equation (3.1) we obtain

$$P(B_T<u,\tau>T) = \frac{1}{\sqrt{2\pi T}}\int_{a(T)}^{u}\sum_{j=1}^{\infty}\sum_{k=0}^{4} s_k\exp\left(\frac{\lambda_{j,k}v+\xi_{j,k}}{T}\right)\exp\left(-\frac{v^2}{2T}\right)dv.$$

Therefore

$$E\left[e^{\theta B_T} I_{\tau>T,B_T<z}\right] = \int\limits_z^{b(T)} e^{\theta u} dP\left(B_T < u, \tau > T\right)$$

$$= \frac{1}{\sqrt{2\pi T}} \sum_{j=1}^{\infty} \sum_{k=0}^{4} s_k \int\limits_z^{b(T)} e^{\theta u} \exp\left(\frac{\lambda_{j,k} u + \xi_{j,k}}{T}\right) \exp\left(-\frac{u^2}{2T}\right) dy,$$

which after rearranging leads to formula (3.7). □

4. Piecewise linear boundaries. The following theorem holds if the boundaries are piecewise linear.

Theorem 4.1. *Let* $\theta > 0$. *If the functions* $L_{1,2}(\cdot)$ *and* $V(\cdot)$ *are given by equations* (3.6) *and* (3.7), *then the Laplace transforms are*

(4.1)

$$E\left[e^{-\theta\tau} I_{\tau\in(t_{m-1},t_m),\tau=\tau_{1,2}}\right]$$
$$= \int\limits_{\alpha_1,\ldots,\alpha_{m-1}}^{\beta_1,\ldots,\beta_{m-1}} \left(\begin{array}{l} \prod\limits_{i=1}^{m-1} \left(1 - \sum\limits_{j=1}^{\infty} q_j\left(x_{i-1}, x_i; i\right)\right) \dfrac{\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)}{\sqrt{2\pi\left(t_i-t_{i-1}\right)}} \\ e^{-\theta t_{m-1}} L_{1,2}\left(\begin{array}{l} t_m - t_{m-1}, \theta; \\ a_m(\cdot) - x_{m-1}, b_m(\cdot) - x_{m-1} \end{array}\right) \end{array} \right) dx_1 \cdots dx_{m-1}$$

(4.2)

$$E\left[e^{\theta B_T} I_{B_T>z,\tau>T}\right]$$
$$= \int\limits_{\alpha_1,\ldots,\alpha_{n-1}}^{\beta_1,\ldots,\beta_{n-1}} \left(\begin{array}{l} \prod\limits_{i=1}^{n-1} \left(1 - \sum\limits_{j=1}^{\infty} q_j\left(x_{i-1}, x_i; i\right)\right) \dfrac{\left(\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)\right)}{\sqrt{2\pi\left(t_i-t_{i-1}\right)}} \\ e^{\theta x_{n-1}} V\left(\begin{array}{l} \theta, z - x_{n-1}, t_n - t_{n-1}; \\ a_{n-1}(\cdot) - x_{n-1}, b_{n-1}(\cdot) - x_{n-1} \end{array}\right) \end{array} \right) dx_1 \cdots dx_{n-1}.$$

Proof. We have

$$L_1 = \int\limits_{t_{m-1}}^{t_m} e^{-\theta u} dP\left(\tau < u, \tau = \tau_1\right)$$

$$= \int\limits_{t_{m-1}}^{t_m} \left(\begin{array}{l} e^{-\theta u} \prod\limits_{i=1}^{m-1} \dfrac{\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)}{\sqrt{2\pi\left(t_i - t_{i-1}\right)}} \\ dP\left(\tau < u, \tau = \tau_1 \middle| B_{t_1=x_1}, \ldots, B_{t_{m-1}=x_{m-1}}\right) \end{array} \right) dx_1 \cdots dx_{m-1}$$

$$= \int_{t_{m-1}}^{t_m} e^{-\theta u} d \left(\int_{\alpha_1,\dots,\alpha_{m-1}}^{\beta_1,\dots,\beta_{m-1}} \left(\begin{array}{l} \prod_{i=1}^{m-1} \frac{\exp\left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)}{\sqrt{2\pi (t_i - t_{i-1})}} \\ \prod_{i=1}^{m-1} P^{t_{i-1},x_{i-1}} \left(t_i < \tau | B_{t_i = x_i}\right) \\ P^{t_{m-1},x_{m-1}} \left(\tau < u, \tau = \tau_1\right) \end{array} \right) dx_1 \cdots dx_{m-1} \right)$$

$$= \int_{\alpha_1,\dots,\alpha_{m-1}}^{\beta_1,\dots,\beta_{m-1}} \left(\begin{array}{l} \prod_{i=1}^{m-1} \frac{\exp\left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)}{\sqrt{2\pi (t_i - t_{i-1})}} \\ \prod_{i=1}^{m-1} \left(1 - \sum_{j=1}^{\infty} q_j (x_{i-1}, x_i; i)\right) \\ e^{-\theta t_{m-1}} \int_0^{t_m - t_{m-1}} e^{-\theta v} p_1^l \left(\begin{array}{l} v, a_{m-1}(\cdot) - x_{m-1}, \\ b_{m-1}(\cdot) - x_{m-1} \end{array} \right) dv \end{array} \right) dx_1 \cdots dx_{m-1}.$$

We have used Lemma 3.1 and the Markovian property of the Brownian motion. It remains to use Theorem 3.1. The form of L_2 can be obtained analogously. We prove formula (4.2) in a similar way

$$E\left[e^{\theta B_T} I_{B_T > z, \tau > T}\right] = \int_z^{b_{n(T)}} e^{\theta u} dP\left(B_T < u, \tau > T\right)$$

$$= \int_z^{b_{n(T)}} e^{\theta u} d \left(\int_{\alpha_1,\dots,\alpha_{n-1}}^{\beta_1,\dots,\beta_{n-1}} \left(\begin{array}{l} P\left(B_T < u, \tau > T | B_{t_1} = x_1, \dots, \right. \\ \left. B_{t_{n-1}} = x_{n-1}\right) \\ \prod_{i=1}^{n-1} \frac{\left(\exp\left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)\right)}{\sqrt{2\pi (t_i - t_{i-1})}} \end{array} \right) dx_1 \cdots dx_{n-1} \right)$$

$$= \int_{-\infty}^{\beta_1,\dots,\beta_{n-1}} \left(\begin{array}{l} \prod_{i=1}^{n-1} \left(1 - \sum_{j=1}^{\infty} q_j (x_{i-1}, x_i; i)\right) \\ \prod_{i=1}^{n-1} \frac{\left(\exp\left(-\frac{(x_i - x_{i-1})^2}{2(t_i - t_{i-1})}\right)\right)}{\sqrt{2\pi (t_i - t_{i-1})}} \\ \int_z^{b_{n(T)}} e^{\theta u} dP^{t_{n-1},x_{n-1}} \left(\tau > T, B_T < u\right) \end{array} \right) dx_1 \cdots dx_{n-1}.$$

We finish the proof using Theorem 3.2. □

Finally, we present a modification of Theorem 4.1 concerning the case when one of the boundaries is infinitely large (small) after some moment. We shall present only the case when the lower boundary does not exist.

Theorem 4.2. *Let $\alpha_i = -\infty$ for all $i \geq k$, $\theta > 0$, and $m > k$. Then*

$$L_2 = E\left[e^{-\theta\tau} I_{\tau\in(t_{m-1},t_m),\tau=\tau_{1,2}}\right]$$

$$= \int\limits_{\alpha_1,\ldots,\alpha_{k-1},-\infty}^{\beta_1,\ldots,\beta_{m-1}} \begin{pmatrix} \prod_{i=1}^{k-1}\left(1-\sum_{j=1}^{\infty} q_j\left(x_{i-1},x_i;i\right)\right) \\ \prod_{i=k}^{m-1}\left(1-\exp\left(-\frac{2\left(\beta_{i-1}-x_{i-1}\right)\left(\beta_i-x_i\right)}{t_i-t_{i-1}}\right)\right) \\ \prod_{i=1}^{m-1}\frac{\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)}{\sqrt{2\pi\left(t_i-t_{i-1}\right)}} \\ e^{-\theta t_{m-1}}\overline{L}\left(t_m-t_{m-1},\theta;b_{m,1},\beta_{m-1}-x_{m-1}\right) \end{pmatrix} dx_1\cdots dx_{m-1}$$

$$E\left[e^{\theta B_T} I_{B_T>z,\tau>T}\right]$$

$$= \int\limits_{\alpha_1,\ldots,\alpha_{k-1},-\infty}^{\beta_1,\ldots,\beta_{n-1}} \begin{pmatrix} \prod_{i=1}^{k-1}\left(1-\sum_{j=1}^{\infty} q_j\left(x_{i-1},x_i;i\right)\right) \\ \prod_{i=k}^{n-1}\left(1-\exp\left(-\frac{2\left(\beta_{i-1}-x_{i-1}\right)\left(\beta_i-x_i\right)}{t_i-t_{i-1}}\right)\right) \\ \prod_{i=1}^{n-1}\frac{\left(\exp\left(-\frac{(x_i-x_{i-1})^2}{2(t_i-t_{i-1})}\right)\right)}{\sqrt{2\pi\left(t_i-t_{i-1}\right)}} \\ e^{\theta x_{n-1}}\overline{V}(\theta,z-x_{n-1},t_n-t_{n-1}; \\ b_{1,n-1},b_{2,n-1}-x_{n-1}) \end{pmatrix} dx_1\cdots dx_{n-1}.$$

The functions $\overline{L}(\cdot)$ and $\overline{V}(\cdot)$ are obtained in Theorems 3.1 *and* 3.2 *from* [13] *and they are*

(4.3)

$$\overline{L}\left(T,\theta;b_1,b_2\right) = E\left[e^{-\theta\tau}\Lambda_T\right] = e^{b_2\left(\sqrt{b_1^2+2\theta}-b_1\right)} g\left(T;\sqrt{b_1^2+2\theta},b_2\right)$$

$$\overline{V}\left(\theta,z,T;b_1,b_2\right) \equiv E\left[e^{\theta B_T} I_{B_T>z,\tau>T}\right] =$$

$$= \exp\left(\frac{T\theta^2}{2}\right)\begin{bmatrix} N\left(\frac{b\left(T\right)-T\theta}{\sqrt{T}}\right)-N\left(\frac{z-T\theta}{\sqrt{T}}\right) \\ +e^{2b_2(\theta-b_1)}\left(N\left(\frac{z-T\theta-2b_2}{\sqrt{T}}\right)-N\left(\frac{b\left(T\right)-T\theta-2b_2}{\sqrt{T}}\right)\right) \end{bmatrix}.$$

Proof. The proof is similar to the proof of Theorem 4.1. Note that when the lower boundary is infinity we have to use the second statement of Lemma 3.1. □

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