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DYNAMICS OF MODIFIED LOTKA–VOLTERRA MODEL WITH POLYNOMIAL INTERVENTION FACTORS. METHODOLOGICAL ASPECTS. III

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ABSTRACT: In the present article is considered a modification of the classical Lotka–Volterra model. The 'input functions $\alpha(t)$ and $\gamma(t)$ are algebraic polynomials of degree n, $\alpha(t)$ is monotonically increasing (with "saturation") and $\gamma(t) > 0$ is monotonically decreasing in the considered interval. This model is very sensitive with respect to the coefficients of the polynomials a_i and b_i. Therefore it is attractive for conducting computer simulations, including other modified predator-prey models and activator-inhibitor mechanisms. Numerical examples, illustrating our results are given using *CAS Mathematica*.

AMS Subject Classification: 65L07, 34A34

Key Words: Lotka–Volterra model, predator–prey systems, polynomial (P)–type control, numerical simulations, Murray's reaction differential system

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1. INTRODUCTION

As it is well known, Lotka–Volterra system is the simplest model of predator-prey interactions based on linear per capita growth rate, see [1] and [2].

The basic model is described by the following nonlinear system of ordinary differential equations

$$\begin{cases} \dfrac{dx(t)}{dt} & = & \alpha x(t) - \beta x(t)y(t) \\ \dfrac{dy(t)}{dt} & = & -\gamma y(t) + \delta x(t)y(t). \end{cases} \tag{1}$$

where x is the number of prey and y is the number of some predator.

The predator and the prey share the same habitat that is supposed a bounded domain in the space R^2 or R^3, depending on the species and the model. Time interval $[t_0, t_1]$, $t_0 > 0$, could be finite or infinite.

Controlled models are more sophisticated than the basic one (1), i.e. models of the type

$$\begin{cases} \dfrac{dx(t)}{dt} & = & (\alpha - u_{prey}(t))x(t) - \beta x(t)y(t) \\ \dfrac{dy(t)}{dt} & = & -(\gamma + u_{pred}(t))y(t) + \delta x(t)y(t), \end{cases} \tag{2}$$

where $u_{pred}(t)$ is the control variable for the pray population, acting on the predator dynamics, and similarly for $u_{prey}(t)$. Other models and results one may find in [3]–[14].

Another generalisation of Lotka - Volterra system (1) is the following one:

$$\begin{cases} \dfrac{dx(t)}{dt} & = & \alpha(t)x(t) - \beta(t)x(t)y(t) \\ \dfrac{dy(t)}{dt} & = & -\gamma(t)y(t) + \delta(t)x(t)y(t). \end{cases} \tag{3}$$

In this article we consider a particular case of the model (3), namely for $\beta(t) = \delta(t) = 1$.

Furthermore, we suppose that intervention factors $\alpha(t)$ and $\gamma(t)$ are both polynomials of degree n. Function

$$\alpha(t) = \sum_{i=0}^{n} a_i t^i$$

is called polynomial growth rate of the prey and we assume that $\alpha(t)$ is monotonically increasing, i.e. with saturation, in the considered interval $[t_0, t_1]$. Function

$$\gamma(t) = \sum_{i=0}^{n} b_i t^i$$

is polynomial death rate of the predator and we assume that $\gamma(t) > 0$ is monotonically decreasing in the considered interval $[t_0, t_1]$.

Therefore in this article are studies systems of the type

$$\begin{cases} \dfrac{dx(t)}{dt} &= -x(t)y(t) + \alpha(t)x(t) \\ \dfrac{dy(t)}{dt} &= x(t)y(t) - \gamma(t)y(t) \end{cases} \tag{4}$$

Briefly we say that we apply Polynomial (P)–type control to the classical Lotka–Volterra model.

2. MAIN RESULTS.

One of the key features of Lotka - Volterra system is the equilibrium of the number of prey and predator.

2.1. EQUILIBRIUM

The equilibrium of a system ODEs is the steady state of the system. When the ODEs are of first order ones, the equilibrium is easy to calculate - the left hand side of the system (4) should be 0.

Trivial calculations give that in our case we have two sets of potential equilibrium - the trivial $x(t)y(t)) = 0$ and $(x(t), y(t)) = (\gamma(t), \alpha(t))$.

But in the second, nontrivial case, substituting $x(t) = \gamma(t)$, $y(t) = \alpha(t)$ in (4) we obtain $\frac{d\gamma(t)}{dt} = 0$, $\frac{d\alpha(t)}{dt} = 0$.

In other words, we have equilibrium for Lotka - Volterra system with polynomial nonlinearity only if there are points $t_k \in [t_0, t_1]$, $0 < k \leq n-1$, such that $\frac{d\gamma(t_k)}{dt} = \frac{d\alpha(t_k)}{dt} = 0$.

Since x and y are the numbers of the predator and the prey, they are both non-negative.

Therefore one can substitute $x = e^u$; $y = e^v$.

Then system (4) is equivalent to

$$\begin{cases} \dot{u} &= \alpha(t) - e^v \\ \dot{v} &= -\gamma(t) + e^u \end{cases} \tag{5}$$

Having in mind that in the time interval $[t_0, t_1]$ we have $\alpha(t) > 0$ and $\gamma(t) > 0$, one can rewrite (5) as

$$\begin{cases} \frac{1}{\alpha(t)}\dot{u} + e^{\ln\left(\frac{1}{\alpha(t)}\right)+v} & = & 1 \\ -\frac{1}{\gamma(t)}\dot{v} + e^{\ln\left(\frac{1}{\gamma(t)}\right)+u} & = & 1. \end{cases} \tag{6}$$

The substitution $U = \ln\left(\frac{1}{\gamma(t)}\right) + u$; $V = \ln\left(\frac{1}{\alpha(t)}\right) + v$ yields

$$\begin{cases} \frac{1}{\alpha(t)}\dot{U} + e^{V} & = & 1 + \frac{1}{\alpha(t)}\frac{d}{dt}\left(\ln\frac{1}{\gamma(t)}\right) = 1 + k_1 \\ -\frac{1}{\gamma(t)}\dot{V} + e^{U} & = & 1 - \frac{1}{\gamma(t)}\frac{d}{dt}\left(\ln\frac{1}{\alpha(t)}\right) = 1 - k_2. \end{cases} \tag{7}$$

This implies that the equilibrium of system (7) in terms of x and y is

$$(x, y) := ((1 - k_2)\gamma(t),\ (1 + k_1)\alpha(t))$$

or

$$(x, y) := \left(\gamma(t) + \frac{\alpha'(t)}{\alpha(t)},\ \alpha(t) - \frac{\gamma'(t)}{\gamma(t)}\right)$$

Another preliminary, expected feature of systems (4), (6) and (7) is instability.

The exponential nonlinearity in the coupling terms in systems (6) and (7) suggests non-stable solutions.

2.2. SIMULATIONS

Example 1. Let

$$\alpha(t) = 0.05 + 0.006t - 0.00015t^2 + 0.000001t^3,$$

$$\gamma(t) = 0.5 - 0.0002t - 0.000675t^2 + 0.00001t^3$$

(see, Fig. 1).

The solutions of the system of differential equations

$$\begin{cases} \dfrac{dx(t)}{dt} & = & -x(t)y(t) + \alpha(t)x(t) \\ \dfrac{dy(t)}{dt} & = & x(t)y(t) - \gamma(t)y(t) \end{cases}$$

with $x(0) = x_0 = 1$; $y(0) = y_0 = 0.001$ in interval $[0, 40]$ are depicted on Fig. 2 a.

For the phase plot, see, Fig 2 b.

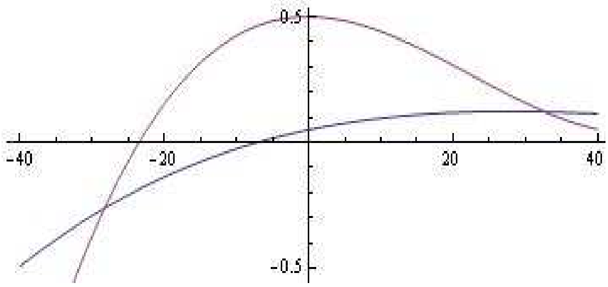

Figure 1: The polynomial factors $\alpha(t)$ and $\gamma(t)$ (Example 1.).

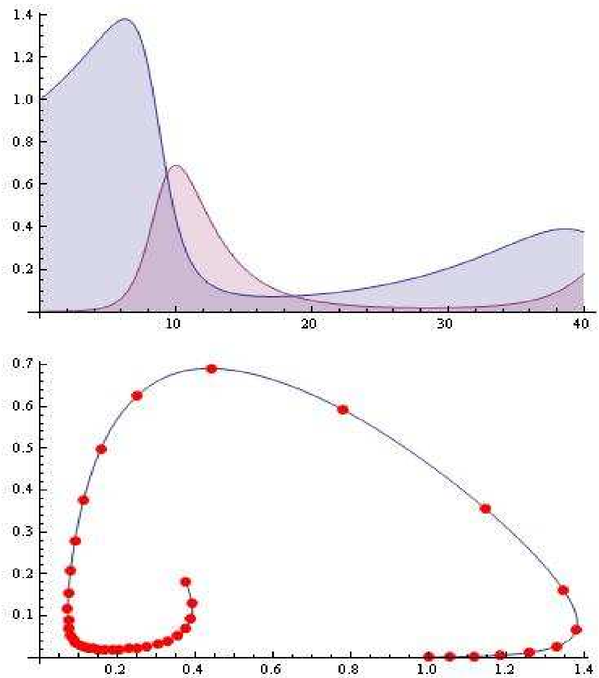

Figure 2: a) The solutions of the system of differential equations (Example 1); b) The phase plot.

Example 2. Let

$$\alpha(t) = 0.05 + 0.0061t - 0.00013t^2 + 0.00000092t^3,$$

$$\gamma(t) = 0.5 - 0.02t + 0.0002t^2 + 0.00000123t^3 - 0.00000002t^4$$

(see, Fig. 3).

The solutions of the system of differential equations with $x(0) = x_0 = 1$; $y(0) =$

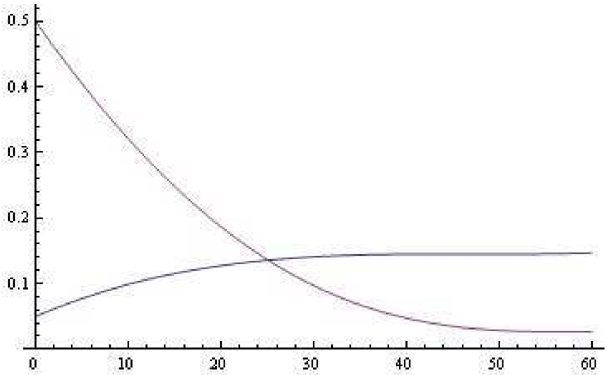

Figure 3: The polynomial factors $\alpha(t)$ and $\gamma(t)$ (Example 2.).

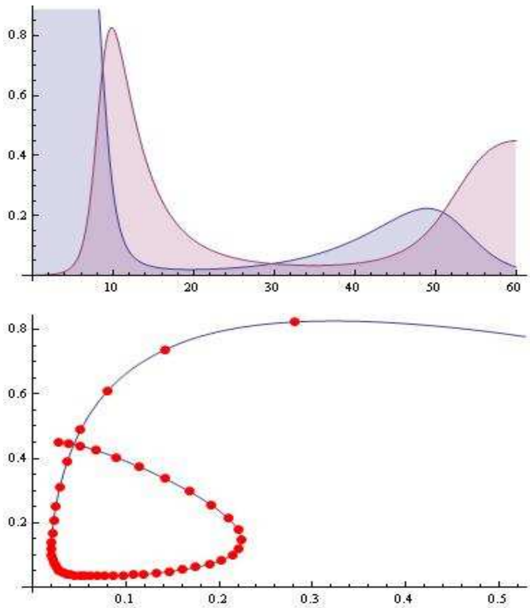

Figure 4: a) The solutions of the system of differential equations (Example 2); b) The phase plot .

$y_0 = 0.001$ in interval $[0, 60]$ are depicted on Fig. 4 a.

For the phase plot, see, Fig 4 b.

2.3. MODIFICATION OF MURRAY'S REACTION DIFFERENTIAL SYSTEM

Another important application of modified Lotka–Volterra systems is modelling of biochemical reactions.

Such reactions are continually taking place in all living organisms and most of them involve enzymes - proteins that are remarkably efficient catalysts.

Enzymes are important in regulating biological processes, for instance, as activators or inhibitors in a reaction.

One model of two dimensionless activator–inhibitor mechanisms is given by [15].

It describes the reaction kinetics of two dimensionless activator–inhibitor mechanisms with the following system:

$$\begin{cases} \dfrac{dx(t)}{dt} &= a - x(t) + x^2(t)y(t) \\ \dfrac{dy(t)}{dt} &= b - x^2(t)y(t) \end{cases} \tag{8}$$

where a and b are positive constants.

Let us consider the following modification of the model (8) with "intervention polynomial factors" $a(t)$ and $b(t)$:

$$\begin{cases} \dfrac{dx(t)}{dt} &= a(t) - x(t) + x^2(t)y(t) \\ \dfrac{dy(t)}{dt} &= b(t) - x^2(t)y(t) \end{cases} \tag{9}$$

Simulations using model (8) (Murrey's approach) for fixed $a = 0.05$; $b = 0.5$ and new model (9) for fixed

$$a(t) = 0.05 + 0.006t - 0.00015t^2 + 0.000001t^3;$$

$$b(t) = 0.5 - 0.0002t - 0.000675t^2 + 0.00001t^3$$

are presented on Fig 5–Fig.6.

The reader may sketch the null clines following the ideas given in this paper.

Remark. Our considerations can be applied to the following modified Lotka–Volterra competition with a non-linear relationship between species:

$$\begin{cases} \dfrac{dx(t)}{dt} &= ax(t) - bx^2(t) - cx(t)y^2(t) \\ \dfrac{dy(t)}{dt} &= dy(t) - ey^2(t) - fx^2(t)y(t). \end{cases} \tag{10}$$

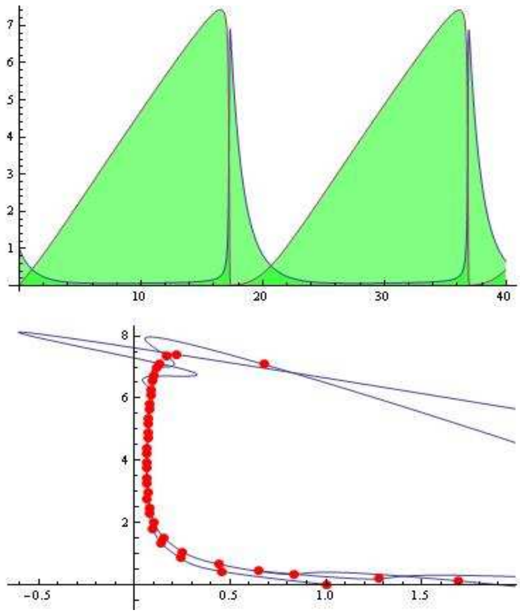

Figure 5: Simulations using model (8) for fixed $a = 0.05$; $b = 0.5$.

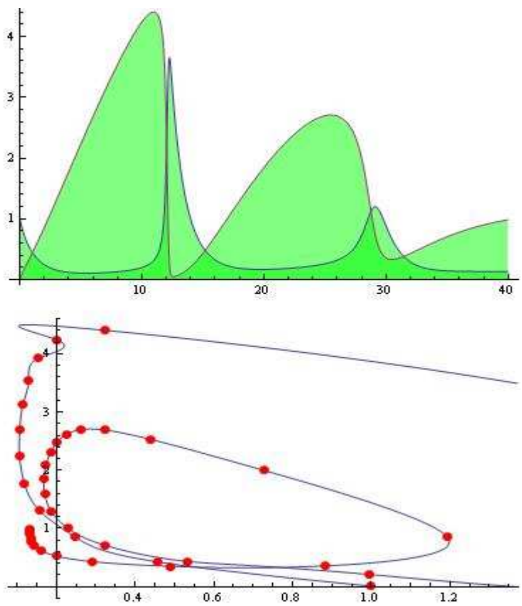

Figure 6: Simulations using model (9) with "intervention polynomial factors" $a(t)$ and $b(t)$.

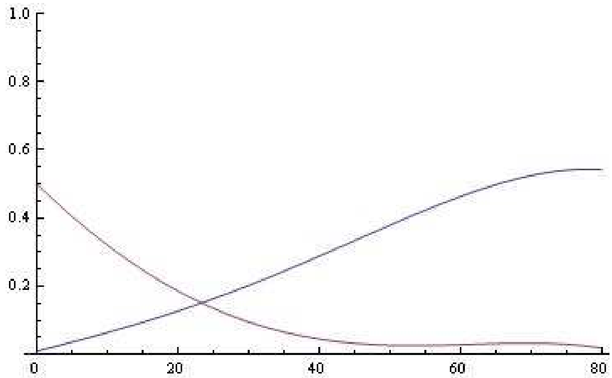

Figure 7: The polynomial factors $\alpha(t)$ and $\gamma(t)$ for the model (11).

For example, consider

$$\begin{cases} \dfrac{dx(t)}{dt} & = & \alpha(t)x(t) - x^2(t) - x(t)y^2(t) \\ \dfrac{dy(t)}{dt} & = & \gamma(t)y(t) - y^2(t) - x^2(t)y(t). \end{cases} \tag{11}$$

The simulation using model (11) for fixed

$$\alpha(t) = 0.01 + 0.0053t - 0.00000075t^2 + 0.0000019t^3 - 0.000000021t^4;$$
$$\gamma(t) = 0.5 - 0.002t + 0.0002t^2 + 0.00000111t^3 - 0.0000000178t^4$$

(see, Fig. 7) is presented on Fig 8.

2.4. CONCLUDING REMARKS

Some modifications of the SIR/SEIR/GSEIR/SIRD epidemic models with intervention polynomial factor are considered in [16]–[18]. Note that the choice of "input functions", especially for the SIRD model, is quite specific and is almost subject to the requirement for these functions to be increasing and decreasing respectively in a fixed time interval.

Influenced by these results, in the present article is considered a modification of the classical Lotka-Volterra model with input functions $\alpha(t)$ and $\gamma(t)$ are algebraic polynomials of degree n and also $\alpha(t)$ is monotonically increasing (with "saturation") and $\gamma(t) > 0$ is monotonically decreasing in the considered interval.

Analysing the simulations using the model (4) (see, Fig. 2 - Fig. 4) one conclude that the new model is very sensitive with respect to the coefficients a_i, b_i and number and distribution of the zeros of polynomials [19].

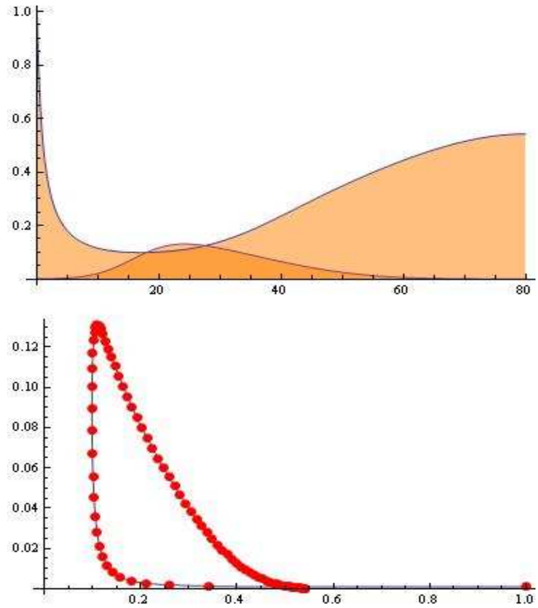

Figure 8: Simulations using model (11) with "intervention polynomial factors" $\alpha(t)$ and $\gamma(t)$.

This makes it attractive for conducting computer simulations, including other modified predator–prey models and activator-inhibitor mechanisms.

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