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SOME EXAMPLES OF RIGID REPRESENTATIONS**

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To the memory of my mother

Abstract. Consider the Deligne-Simpson problem: *give necessary and sufficient conditions for the choice of the conjugacy classes $C_j \subset GL(n,\mathbf{C})$ (resp. $c_j \subset gl(n,\mathbf{C})$) so that there exist irreducible $(p+1)$-tuples of matrices $M_j \in C_j$ (resp. $A_j \in c_j$) satisfying the equality $M_1 \ldots M_{p+1} = I$ (resp. $A_1 + \ldots + A_{p+1} = 0$).* The matrices M_j and A_j are interpreted as monodromy operators and as matrices-residua of fuchsian systems on Riemann's sphere.

We give new examples of existence of such $(p+1)$-tuples of matrices M_j (resp. A_j) which are *rigid*, i.e. unique up to conjugacy once the classes C_j (resp. c_j) are fixed. For rigid representations the sum of the dimensions of the classes C_j (resp. c_j) equals $2n^2 - 2$.

1. Fuchsian linear systems and the Deligne-Simpson problem. Consider the *fuchsian* system (i.e. with logarithmic poles) of n linear

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differential equations

$$\mathrm{d}X/\mathrm{d}t = \left(\sum_{j=1}^{p+1} A_j/(t-a_j)\right) X,\ t \in \mathbf{C}P^1 = \mathbf{C} \cup \infty$$

$A_j \in gl(n, \mathbf{C})$ being its *matrices-residua*. Assume that it has no pole at infinity, i.e.

$$A_1 + \ldots + A_{p+1} = 0 \ . \tag{1}$$

Fix a base point $a_0 \in S := \mathbf{C}P^1 \backslash \{a_1, \ldots, a_{p+1}\}$ and the value $B \in GL(n, \mathbf{C})$ of the solution X for $t = a_0$. Every pole a_j defines a conjugacy class γ_j in the fundamental group $\pi_1(S, a_0)$. The class γ_j is represented by a closed contour consisting of a segment $[a_0, a_j']$ (where the point a_j' is close to a_j), of a circumference centered at a_j and of radius $|a_j - a_j'|$ (containing inside no pole of the system other than a_j and circumventing a_j counterclockwise) and of the segment $[a_j', a_0]$. One enumerates the segments so that the index increases when one turns around a_0 clockwise.

Hence, $\pi_1(S, a_0)$ admits the presentation

$$\langle \gamma_1, \ldots, \gamma_{p+1} | \gamma_{p+1} \ldots \gamma_1 = e \rangle \ .$$

The *monodromy operator* M_j defined by the class γ_j is the one mapping the solution with initial data $X|_{t=a_0} = B$ onto the value at a_0 of its analytic continuation along the contour defining γ_j (i.e. $X \mapsto XM_j$). The monodromy operators of the system generate its *monodromy group* which is an antirepresentation $\pi_1(S, a_0) \to GL(n, \mathbf{C})$ because the monodromy operator corresponding to the class $\gamma_i\gamma_j$ equals M_jM_i. Thus for the matrices M_j one has

$$M_1 \ldots M_{p+1} = I. \tag{2}$$

Remark 1. If there are no non-zero integer differences between the eigenvalues of A_j, then the operator M_j is conjugate to $\exp(2\pi i A_j)$.

Remark 2. Fuchsian systems are a particular case of *regular* systems, i.e. linear systems whose solutions when restricted to sectors centered at the poles a_j grow no faster than some power of the distance to the pole a_j. Their monodromy groups are defined in the same way.

The *Deligne-Simpson problem (DSP)* is formulated like this: *give necessary and sufficient conditions for the choice of the conjugacy classes $C_j \subset GL(n, \mathbf{C})$ (resp. $c_j \subset gl(n, \mathbf{C})$) so that there exist irreducible $(p+1)$-tuples of* *matrices* $M_j \in C_j$ *satisfying* (2) (*resp. of matrices* $A_j \in c_j$ *satisfying* (1)). In the multiplicative version (i.e. for matrices M_j) it is stated by P. Deligne[1] and C. Simpson was the first to obtain a significant result towards its resolution, see [6].

The problem is formulated without using the notions of fuchsian system and monodromy operator, yet they explain the interest in the problem. The multiplicative version is more important because the monodromy operators are invariant under the changes $X \mapsto W(t)X$ (where W depends meromorphically on t and $\det W \not\equiv 0$) while the matrices-residua are not. In the multiplicative version the problem admits the interpretation: *for which* $(p+1)$*-tuples of local monodromies* c_j *does there exist an irreducible monodromy group with such local monodromies?*

The paper is structured as follows. In the next section we recall the basic results announced in [4] and proved in [5]. In Section 3 we define the case of rigid $(p+1)$-tuples. In Section 4 we give some new examples of existence of rigid $(p+1)$-tuples of diagonalizable matrices. In Section 5 we describe all rigid $(p+1)$-tuples of such matrices in which the multiplicities of all eigenvalues of one of the matrices are ≤ 2. In Section 6 we explain how the examples from the previous two sections give rise to other examples in which the matrices are not necessarily diagonalizable.

2. The Deligne-Simpson problem for generic eigenvalues.

Definition 3. *A* Jordan normal form (JNF) of size n *is a family* $J^n = \{b_{i,l}\}$ $(i \in I_l,\ I_l = \{1, \ldots, s_l\},\ l \in L)$ *of positive integers* $b_{i,l}$ *whose sum is* n*. Here* L *is the set of indices of the eigenvalues* λ_l *(all distinct) and* I_l *is the set of indices of Jordan blocks with eigenvalue* λ_l*;* $b_{i,l}$ *is the size of the* i*-th block with this eigenvalue. We assume that for each* l *fixed one has* $b_{1,l} \geq \ldots \geq b_{s_l,l}$*. An* $n \times n$*-matrix* Y *has the JNF* J^n *(notation:* $J(Y) = J^n$*) if to its distinct eigenvalues* λ_l*,* $l \in L$*, there belong Jordan blocks of sizes* $b_{i,l}$*.*

In what follows we presume the necessary condition $\prod \det(C_j) = 1$ (resp. $\sum \mathrm{Tr}(c_j) = 0$) to hold. In terms of the eigenvalues $\sigma_{k,j}$ (resp. $\lambda_{k,j}$) of the matrices from C_j (resp. c_j) repeated with their multiplicities, this condition reads

$$\prod_{k=1}^{n} \prod_{j=1}^{p+1} \sigma_{k,j} = 1, \text{ resp. } \sum_{k=1}^{n} \sum_{j=1}^{p+1} \lambda_{k,j} = 0.$$

[1] It seems that the author of the present paper was the first to state the problem in the additive version

Definition 4. *An equality of the form*

$$\prod_{j=1}^{p+1}\prod_{k\in\Phi_j}\sigma_{k,j}=1,\ \textit{resp.}\ \sum_{j=1}^{p+1}\sum_{k\in\Phi_j}\lambda_{k,j}=0,$$

is called a non-genericity relation*; the sets Φ_j contain one and the same number κ ($1<\kappa<n$) of indices for all j. Eigenvalues satisfying none of these relations are called* generic. *Reducible $(p+1)$-tuples exist only for non-generic eigenvalues. Indeed, a reducible $(p+1)$-tuple can be conjugated to a block upper-triangular form and the eigenvalues of the restriction of the $(p+1)$-tuple to each diagonal block must satisfy condition* (2) *or* (1) *which is a non-genericity relation.*

For a conjugacy class C in $GL(n,\mathbf{C})$ or $gl(n,\mathbf{C})$ denote by $d(C)$ its dimension. Remind that $d(C)$ is always even. For a matrix Y from C set $r(C):=\min_{\lambda\in\mathbf{C}}\operatorname{rank}(Y-\lambda I)$. The integer $n-r(C)$ equals the maximal number of Jordan blocks of $J(Y)$ with one and the same eigenvalue.

Set $d_j:=d(C_j)$ (resp. $d(c_j)$), $r_j:=r(C_j)$ (resp. $r(c_j)$). The quantities $r(C)$ and $d(C)$ depend only on the JNF $J(Y)=J^n$, not on the eigenvalues and we write sometimes $r(J^n)$ and $d(J^n)$.

Proposition 6 (C. Simpson, see [6])**.** *The following couple of inequalities is a necessary condition for the existence of irreducible $(p+1)$-tuples of matrices M_j satisfying* (2)*:*

(α_n) $$d_1+\ldots+d_{p+1}\geq 2n^2-2$$

(β_n) $$\text{for all } j,\ r_1+\ldots+\hat{r}_j+\ldots+r_{p+1}\geq n.$$

Definition 6. *Denote by $\{J_j^n\}$ a $(p+1)$-tuple of JNFs, $j=1,\ldots,p+1$. We say that the DSP is* solvable *(resp. is* weakly solvable*) for a given $\{J_j^n\}$ and given eigenvalues if there exists an irreducible $(p+1)$-tuple (resp. a $(p+1)$-tuple with a trivial centralizer) of matrices M_j satisfying* (2) *or of matrices A_j satisfying* (1)*, with $J(M_j)=J_j^n$ or $J(A_j)=J_j^n$ and with the given eigenvalues. By definition, the DSP is solvable and weakly solvable for $n=1$.*

Theorem 7. *The DSP is solvable for conjugacy classes C_j or c_j with generic eigenvalues and satisfying the condition*

(ω_n) $$(r_1+\ldots+r_{p+1})\geq 2n.$$

For a given $\{J_j^n\}$ with $n > 1$, which satisfies conditions (α_n) and (β_n) and doesn't satisfy condition (ω_n) set $n_1 = r_1 + \ldots + r_{p+1} - n$. Hence, $n_1 < n$ and $n - n_1 \leq n - r_j$. Define the $(p+1)$-tuple $\{J_j^{n_1}\}$ as follows: to obtain the JNF $J_j^{n_1}$ from J_j^n one chooses one of the eigenvalues of J_j^n with greatest number $n - r_j$ of Jordan blocks, then decreases by 1 the sizes of the $n - n_1$ *smallest* Jordan blocks with this eigenvalue and deletes the Jordan blocks of size 0. We use the notation $\Psi : \{J_j^n\} \mapsto \{J_j^{n_1}\}$.

Theorem 8. *Let $n > 1$. The DSP is solvable for the conjugacy classes C_j or c_j (with generic eigenvalues, defining the JNFs J_j^n and satisfying conditions (α_n) and (β_n)) if and only if either $\{J_j^n\}$ satisfies condition (ω_n) or the construction $\Psi : \{J_j^n\} \mapsto \{J_j^{n_1}\}$ iterated as long as it is defined stops at a $(p+1)$-tuple $\{J_j^{n'}\}$ either with $n' = 1$ or satisfying condition $(\omega_{n'})$* [2]*.*

In the case of diagonalizable matrices M_j or A_j the JNF of M_j or A_j is completely defined by the *multiplicity vector (MV)* of its eigenvalues. This is a vector Λ_j^n with positive integer components equal to the multiplicities of the eigenvalues of M_j or A_j; hence, their sum is n.

Remark 9. Set $\Lambda_j^n = (m_{1,j}, \ldots, m_{i_j,j})$ where $m_{1,j} \geq \ldots \geq m_{i_j,j}$. Hence, one has $r_j = m_{2,j} + \ldots + m_{i_j,j}$ and $d_j = n^2 - \sum_{i=1}^{i_j} (m_{i,j})^2$. In particular, the MV with greatest value of d_j is $(1, \ldots, 1)$, with $d_j = n^2 - n$.

Call *polymultiplicity vector (PMV)* the $(p+1)$-tuple of MVs $\Lambda^n := (\Lambda_1^n, \ldots, \Lambda_{p+1}^n)$.

Remark 10. In the particular case of diagonalizable matrices M_j or A_j the mapping Ψ is defined by the following rule (to be checked directly):

The MV $\Lambda_j^{n_1}$ defining the JNF $J_j^{n_1}$ equals $(m_{1,j} - n + n_1, m_{2,j}, m_{3,j}, \ldots, m_{i_j,j})$.

3. The case of rigid $(p+1)$-tuples.

Definition 11. *The case when $d_1 + \ldots + d_{p+1} = 2n^2 - 2$ is called* rigid. *Such $(p+1)$-tuples of matrices A_j satisfying* (1) *or of matrices M_j satisfying* (2) *or of JNFs or of PMVs are also called* rigid.

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[2]The result of the theorem does not depend on the choice of eigenvalue in the definition of Ψ

A priori, if in the rigid case for a certain $(p+1)$-tuple of conjugacy classes the DSP is solvable, then up to conjugacy it has only finitely many solutions.

Proposition 12 (see [6] and [3]). *If for a given $(p+1)$-tuple of conjugacy classes $C_j \subset GL(n, \mathbf{C})$ with generic eigenvalues and with $d_1+\ldots+d_{p+1} = 2n^2-2$ the DSP is solvable for matrices M_j, then its solution is unique up to conjugacy.*

Proposition 13. *Suppose that for a given $(p+1)$-tuple of conjugacy classes $c_j \subset gl(n, \mathbf{C})$ with generic eigenvalues and with $d_1+\ldots+d_{p+1} = 2n^2-2$ the DSP is solvable for matrices A_j. Then its solution is unique up to conjugacy.*

The proposition is proved at the end of the section.

Remark 14. Rigid representations in the multiplicative case are studied in [3] where an algorithm is given which tells whether the DSP is solvable for given conjugacy classes C_j and the construction of rigid $(p+1)$-tuples of matrices $M_j \in C_j$ is explained. The algorithm of Katz is based on a middle convolution functor in the category of pervers sheaves. The same functor is defined in a purely algebraic way in [1]. The algorithm in [3] also results in the construction $\Psi : \{J_j^n\} \mapsto \{J_j^{n_1}\}$ but in the case of rigid representations one never encounters $(p+1)$-tuples $\{J_j^{n'}\}$ satisfying condition $(\omega_{n'})$. In fact, there holds the following lemma (see [4] and [5]):

Lemma 15. *The quantity $2n^2 - \sum_{j=1}^{p+1} d_j$ is invariant for the construction $\Psi : \{J_j^n\} \mapsto \{J_j^{n_1}\}$.*

The lemma implies that it is sufficient to check that condition (α_n) holds not for $\{J_j^n\}$ (see Theorem 8) but for $\{J_j^{n'}\}$. If $n' > 1$ and condition $(\omega_{n'})$ holds, then for generic eigenvalues the DSP is solvable for the JNFs $J_j^{n'}$, see [4], hence, the necessary condition $(\alpha_{n'})$ holds – it is a strict inequality. If $n' = 1$, then condition $(\alpha_{n'})$ is an equality (this is the rigid case). Hence, in both cases condition $(\alpha_{n'})$ holds and a posteriori one knows that in fact it is not necessary to check it.

Proof of Proposition 13. 1^0. One can assume that for every j there is no non-zero integer difference between two eigenvalues of the matrix A_j (otherwise this can be achieved by a multiplication of the matrices by $c \in \mathbf{C}^*$). Hence,

1) the monodromy operators M_j of a fuchsian system with residua A_j equal up to conjugacy $\exp(2\pi i A_j)$, see Remark 1;

2) the eigenvalues of the matrices M_j are generic.

By Proposition 12, the $(p+1)$-tuple of matrices M_j is unique up to conjugacy. Indeed, denote by C_j the conjugacy class of M_j. Then $d(C_j) = d_j$ (see 1)) and $d(C_1) + \ldots + d(C_{p+1}) = 2n^2 - 2$.

2^0. Suppose that there are at least two $(p+1)$-tuples of matrices $A_j \in c_j$ (denoted by A_j^1, A_j^2) non conjugate to one another which are solutions to the DSP. Denote by (F_1), (F_2) two fuchsian systems with residua equal respectively to A_j^1, A_j^2 and with one and the same poles. Then these systems have one and the same monodromy group, see 1^0. Hence, there exists a meromorphic change $X \mapsto W(t)X$ bringing (F_1) to (F_2). (The fact that W can have a priori at most poles as singularities follows from the regularity of (F_1) and (F_2)).

3^0. For $t \neq a_j$, $j = 1, \ldots, p+1$, the matrix W is holomorphic and holomorphically invertible. Indeed, it equals $X_2(X_1)^{-1}$ where X_i is some fundamental solution to (F_i). Prove that W has no pole at a_j.

Suppose it has. Set $W = \sum_{k=-l}^{\infty} W_k(t-a_j)^k$, $l \in \mathbf{N}^*$. In a neighbourhood of a_j one has

$$(F_i) : \mathrm{d}X/\mathrm{d}t = (A_j^i/(t-a_j) + O(1))X, \;\; i = 1, 2.$$

Then one has

$$-W^{-1}\mathrm{d}W/\mathrm{d}t + W^{-1}(A_j^1/(t-a_j) + O(1))W = (A_j^2/(t-a_j) + O(1)), \;\text{ i.e.}$$

$$-\mathrm{d}W/\mathrm{d}t + (A_j^1/(t-a_j) + O(1))W = W(A_j^2/(t-a_j) + O(1)) \;\; \text{and, hence,}$$

$$-lW_{-l} + A_j^1 W_{-l} - W_{-l}A_j^2 = 0\;.$$

This implies that $W_{-l} = 0$, i.e. W has no pole at a_j. (Indeed, the eigenvalues of the linear operator $(.) \mapsto -l(.) + A_j^1(.) - (.)A_j^2$ acting on $gl(n, \mathbf{C})$ are of the form $\eta = -l + \lambda_1 - \lambda_2$ where λ_i is eigenvalue of A_j^i. Their set is one and the same for $i = 1, 2$ and by 1^0, one has $\eta \neq 0$.)

But then W is holomorphic on $\mathbf{C}P^1$, hence, constant, i.e. $W \in GL(n, \mathbf{C})$ which means that the two $(p+1)$-tuples (of matrices A_j^1 and A_j^2) are conjugate.

The proposition is proved. □

4. Some series of rigid representations. In this section we list several series of rigid representations with diagonalizable matrices M_j or A_j by means of their PMVs. Their existence follows from Theorem 8 and Remark 10. (The eigenvalues are presumed generic.) In Section 6 we explain how to deduce from their existence the one of other rigid series with generic eigenvalues in which at least one of the matrices M_j or A_j is not diagonalizable.

For $p = 2$ we define several series of PMVs. We avoid the letters A and M which denote already matrices and the notation should not be mixed up with similar notation for singularities or Lie algebras:

$W_k :$	$(k, k, k+1),$	$(k, k, k+1),$	$(k, k, k+1)$
$B_k :$	$(k, k, k-1),$	$(k, k, k-1),$	$(k, k, k-1)$
$C_k :$	$(k, k, k),$	$(k, k, k),$	$(k, k+1, k-1)$
$D_k :$	$(k, k, k, k+1),$	$(k, k, k, k+1),$	$(2k, 2k+1)$
$E_k :$	$(k, k, k, k-1),$	$(k, k, k, k-1),$	$(2k, 2k-1)$
$F_k :$	$(k, k, k, k),$	$(k, k, k, k),$	$(2k+1, 2k-1)$
$\Phi_k :$	$(k, k, k+1, k-1),$	$(k, k, k, k),$	$(2k, 2k)$
$G_k :$	$(k, k, k+1, k+1),$	$(k, k, k+1, k+1),$	$(2k+1, 2k+1)$
$H_k :$	$(k, k, k, k, k, k+1),$	$(3k, 3k+1),$	$(2k, 2k, 2k+1)$
$I_k :$	$(k, k, k, k, k, k-1),$	$(3k, 3k-1),$	$(2k, 2k, 2k-1)$
$J_k :$	$(k, k, k, k, k, k),$	$(3k+1, 3k-1),$	$(2k, 2k, 2k)$
$K_k :$	$(k, k, k, k, k, k),$	$(3k, 3k),$	$(2k, 2k+1, 2k-1)$
$L_k :$	$(k, k, k, k, k+1, k-1),$	$(3k, 3k),$	$(2k, 2k, 2k)$
$V_k :$	$(k, k, k, k, k+1, k+1),$	$(3k+1, 3k+1),$	$(2k, 2k+1, 2k+1)$
$N_k :$	$(k, k, k, k+1, k+1, k+1),$	$(3k+1, 3k+2),$	$(2k+1, 2k+1, 2k+1)$
$P_k :$	$(k, k, k, k, k-1, k-1),$	$(3k-1, 3k-1),$	$(2k, 2k-1, 2k-1)$

Here $k \in \mathbf{N}$ or $k \in \mathbf{N}^+$ according to the case. Each of these PMVs satisfies Conditions (α_n) and (β_n) (to be checked directly). Moreover, (α_n) is equality everywhere.

The series W_k, B_k and C_k were discovered by O. Gleizer (see [2]). We don't use his result but deduce their existence from Theorem 8 and Remark 10 (partly because we need to prove the existence of rigid triples from other series as well and partly because he claims in [2] the non-existence of the rigid series

$$OG_k : \ (2, \ldots, 2, 1, 1, 1), \ (2, \ldots, 2, 1), \ (2k-1, 1, 1)$$

which contradicts Theorem 8; we deduce the existence of this series at the end of the section).

To prove the existence of these rigid series it suffices to explicit the sequence of PMVs Λ^n, Λ^{n_1}, ..., Λ^{n_s} occurring when the construction Ψ from Section 2 is iterated, see Theorem 8 and Remark 10; we set $n_s = n'$. For $\Lambda^n = W_k$

this sequence equals W_k, B_k, W_{k-1}, B_{k-1}, ..., W_0. Write it symbolically in the form

$$W_k \to B_k \to W_{k-1} \to B_{k-1} \to \ldots \to W_0.$$

All requirements of Theorem 8 and Remark 10 are met which implies the existence of irreducible triples with PMV W_k (and B_k as well if one deletes the first term of the sequence).

One finds by analogy (for $\Lambda^n = C_k$) the sequence

$$C_k \to B_k \to W_{k-1} \to B_{k-1} \to \ldots \to W_0$$

which differs from the previous one only in its first term. Hence, there exist irreducible triples with $\Lambda^n = C_k$. In the same way one obtains the sequences

$$D_k \text{ or } F_k \text{ or } \Phi_k \to E_k \to G_{k-1} \to D_{k-1} \to E_{k-1} \to G_{k-2} \to \ldots \to G_0 \to D_0$$

$$H_k \text{ or } J_k \text{ or } K_k \text{ or } L_k \to I_k \to P_k \to N_{k-1} \to V_{k-1} \to H_{k-1} \to I_{k-1} \to \ldots \to H_0$$

from which one deduces the existence of irreducible triples with Λ^n equal to any of the other PMVs listed above.

For $p = 3$ we define two series:
R_k: (k, k), (k, k), (k, k), $(k + 1, k - 1)$;
S_k: $(k + 1, k)$, $(k + 1, k)$, $(k + 1, k)$, $(k + 1, k)$.
The corresponding sequence equals

$$S_k \text{ or } R_k \to S_{k-1} \to S_{k-2} \to \ldots \to S_0$$

For $p = 4$ we define the series

$$T_k : \ (2k + 1, 2k - 1),\ (3k, k),\ (3k, k),\ (3k, k),\ (3k, k).$$

The PMV Λ^{n_1} equals $(2k - 1)$, $(k, k - 1)$, $(k, k - 1)$, $(k, k - 1)$, $(k, k - 1)$. This means that the matrix A_1 must be scalar and the PMV of the other four matrices equals S_{k-1}. Thus, the existence of irreducible quintuples follows from the existence of irreducible quadruples with PMV S_{k-1}.

Finally, we recall the existence of other four series discovered by C. Simpson (the first three, see [6]) and by O. Gleizer (see [2]):

HG_n :	$(n-1, 1)$	$(1, \ldots, 1)$	$(1, \ldots, 1)$	hypergeometric
OF_n :	$((n+1)/2, (n-1)/2)$	$((n-1)/2, (n-1)/2, 1)$	$(1, \ldots, 1)$	odd family
EF_n :	$(n/2, n/2)$	$(n/2, (n-2)/2, 1)$	$(1, \ldots, 1)$	even family
FF_n :	$(2, 1, \ldots, 1)$	$(2, \ldots, 2, 1, \ldots, 1)$ ($n-4$ times 2)	$(n-2, 2)$	finite family, n=5, 6, 7, 8

For the series OG_k defined above one obtains the sequence

$$OG_k \to OG_{k-1} \to \ldots \to OG_1 \to HG_2 \to HG_1.$$

Note that $OG_1 = HG_3$.

The existence of the series

$$[n-1,1] \ : (n-1,1), \ldots, \ (n-1,1) \ \ (n+1 \text{ times})$$

follows from $[n-1,1] \to (1), \ldots, (1)$.

Remark 16 In the series $W_k - P_k$ the multiplicities of the eigenvalues are equal to (or differ by no more than 2 from) n/s_1, n/s_2, n/s_3 where $(s_1, s_2, s_3) \in (\mathbf{N}^*)^3$ is a solution to the equation

$$1/s_1 + 1/s_2 + 1/s_3 = 1$$

(these solutions are (3,3,3), (4,4,2) and (6,3,2) up to permutation). One can consider the series OF_n and EF_n (resp. HG_n) as corresponding to the "generalized" solution $(2, 2, \infty)$ (resp. $(1, \infty, \infty)$) of the above equation.

Remark 17. C. Simpson has shown in [6] that the three series OF_n, EF_n and HG_n include all rigid triples of diagonalizable matrices M_j in which one of them has distinct eigenvalues. Hence, this is the case of matrices A_j as well because the criterium for existence of irreducible $(p+1)$-tuples (i.e. Theorem 8) is the same in the additive and in the multiplicative situation.

5. Rigid representations with an upper bound on the multiplicities of the eigenvalues of the first matrix.

5.1. Formulation of the problem. In the present section we consider the problem:

Give the complete list of PMVs for which there exist rigid irreducible $(p+1)$-tuples of diagonalizable matrices M_j satisfying (2) (*resp. of diagonalizable matrices A_j satisfying* (1)), *with generic eigenvalues, in which the multiplicities of all eigenvalues of M_1 (resp. of A_1) are $\leq u$ for some $u \in \mathbf{N}^*$.*

We solve the problem for $u = 2$. In what follows we set $m_{1,1} = u = 2$. (If $m_{1,1} = 1$, then $u = 1$ and in this case the answer to the problem is given by Remark 17.) The techniques can be used to solve the problem for any given u. We assume that no MV equals (n) in which case the corresponding matrix A_j

or M_j must be scalar. We also assume that no MV is of the form $(1,\ldots,1)$ (see Remark 17).

Remark 18. The cases $u=1$ and $u=2$ are exceptional in the following sense – whenever one finds a rigid PMV satisfying condition (β_j), there exist rigid $(p+1)$-tuples of diagonalizable matrices with this PMV. (For $u=3$ this is not true, see Example 20.) More generally, there holds

Theorem 19. *If $u\leq 2$, then conditions (α_n) and (β_n) are necessary and sufficient for the existence for generic eigenvalues of irreducible $(p+1)$-tuples of matrices M_j satisfying* (2) *or of matrices A_j satisfying* (1).

The theorem is proved in Section 7. It generalizes Simpson's result from [6]: *if one of the matrices M_j has distinct eigenvalues, then for generic eigenvalues conditions (α_n) and (β_n) are necessary and sufficient for the existence of irreducible $(p+1)$-tuples of matrices M_j satisfying* (2). In the above theorem condition (α_n) is not presumed to be an equality, i.e. the theorem does not consider only the rigid case.

Example 20. For $p=2$, $u=3$, $n=6m+3$, $m\in\mathbf{N}^*$ the PMV $(3,\ldots,3,2,1,\ldots,1)$ (m times 3, $3m+1$ units), $(3m+1,3m+1,1)$, $(3m+1,3m+1,1)$ is rigid and satisfies condition (β_n) but the PMV obtained from it after applying the construction Ψ (one has $n_1=n-2$) does not satisfy condition (β_{n-2}).

5.2. The results. The basic result is contained in Theorems 21, 22 and 23. In the next subsection we explain the method of proof.

Theorem 21. *If $u=2$, $p=3$ and $\Lambda_4^n=(n-1,1)$, then*

1) *one has $d_1+\ldots+d_4\geq 2n^2-2$ in all cases except in*

Case Ω*: n is even, $r_2=n/2$, $r_3=n/2-1$, $\Lambda_1^n=(2,\ldots,2)$, $\Lambda_2^n=(n/2,n/2)$, $\Lambda_3^n=(n/2+1,n/2-1)$; in* **Case** Ω *one has $d_1+d_2+d_3+d_4=2n^2-4$;*

2) *the only PMVs of rigid quadruples for n even are*

Ξ_n :	$(2,\ldots,2)$	$(n/2,n/2)$	$(n/2,n/2)$	$(n-1,1)$
Θ_n :	$(2,\ldots,2,1,1)$	$(n/2,n/2)$	$(n/2+1,n/2-1)$	$(n-1,1)$
Ψ_6 :	$(2,2,2)$	$(3,3)$	$(4,1,1)$	$(5,1)$

and the only ones for n odd are

Π_n :	$(2,\ldots,2,1)$	$((n+1)/2,(n-1)/2)$	$((n+1)/2,(n-1)/2)$	$(n-1,1)$
Δ_n :	$(2,\ldots,2,1)$	$(2,\ldots,2,1)$	$(n-1,1)$	$(n-1,1)$

The theorem is proved in Subsection 5.4.

Theorem 22. *If $u = 2$, then for a rigid $(p+1)$-tuple one has $p \leq 3$. If $p = 3$, then one of the MVs of a rigid quadruple equals $(n-1, 1)$.*

The theorem is proved in Subsection 5.5.

Theorem 23. *If $u = 2$ and $p = 2$, then with the exception of finitely many cases with $n \leq 21$ the only PMVs for which there exist rigid triples are the following ones:*

For n even:

1a)	Γ_n^1	:	$(2,\ldots,2)$	$(2,\ldots,2,1,1,1,1,1,1)$	$(n-2,2)$
1b)	Γ_n^2	:	$(2,\ldots,2,1,1)$	$(2,\ldots,2,1,1,1,1)$	$(n-2,2)$
1c)	Γ_n^3	:	$(2,\ldots,2,1,1)$	$(2,\ldots,2,1,1)$	$(n-2,1,1)$
1d)	Γ_n^4	:	$(2,\ldots,2)$	$(2,\ldots,2,1,1,1,1)$	$(n-2,1,1)$
1e)	Y_n^1	:	$(2,\ldots,2,1,1,1,1)$	$(m,m,2)$	$(m+1,m+1)$
1f)	Y_n^2	:	$(2,\ldots,2,1,1)$	$(m,m,1,1)$	$(m+1,m+1)$
1g)	Y_n^3	:	$(2,\ldots,2,1,1,1,1)$	$(m+2,m,1,1)$	$(m+2,m+2)$
1h)	Y_n^4	:	$(2,\ldots,2,1,1,1,1,1,1)$	$(m+2,m,2)$	$(m+2,m+2)$
1i)	Y_n^5	:	$(2,\ldots,2,1,1)$	$(m+1,m,1)$	$(m+1,m,1)$
1j)	Y_n^6	:	$(2,\ldots,2,1,1,1,1)$	$(m,m,1,1)$	$(m+2,m)$
1k)	Y_n^7	:	$(2,\ldots,2,1,1,1,1,1,1)$	$(m,m,2)$	$(m+2,m)$

For n odd:

2a)	X_n^1	:	$(2,\ldots,2,1,1,1,1,1)$	$(2,\ldots,2,1)$	$(n-2,2)$
2b)	X_n^2	:	$(2,\ldots,2,1,1,1)$	$(2,\ldots,2,1,1,1)$	$(n-2,2)$
2c)	OG_n	:	$(2,\ldots,2,1)$	$(2,\ldots,2,1,1,1)$	$(n-2,1,1)$
2d)	Z_n^1	:	$(2,\ldots,2,1)$	$(m,m,1)$	$(m,m,1)$
2e)	Z_n^2	:	$(2,\ldots,2,1,1,1,1,1)$	$(m,m-1,2)$	$(m+1,m)$
2f)	Z_n^3	:	$(2,\ldots,2,1,1,1)$	$(m,m-1,1,1)$	$(m+1,m)$
2g)	Z_n^4	:	$(2,\ldots,2,1,1,1)$	$(m,m,1)$	$(m+1,m-1,1)$

where $m \in \mathbf{N}$ or $m \in \mathbf{N}^*$.

The theorem is proved in Subsection 5.6. We do not explicit the exceptional cases with $n \leq 21$. The reader can do this by iterating the construction Ψ from Section 2 backward.

5.3. The method of proof. The method of proof consists in trying to minimize the quantities d_j for r_j fixed. Denote these minimal possible values of d_j by d'_j and the PMVs realizing these minimal values by Λ'^n. (A posteriori they turn out to be unique up to permutation of the components of their MVs.)

The PMVs Λ'^n in part of the cases turn out to be rigid and then we prove the existence of the corresponding $(p+1)$-tuples of matrices by means of Theorem 8. In another part of the cases one finds out that $d'_1+\ldots+d'_{p+1} > 2n^2-2$, i.e. no rigid $(p+1)$-tuples exist for such quantities r_j. Finally, in the remaining part of the cases one has $d'_1+\ldots+d'_{p+1} < 2n^2-2$ (i.e. no irreducible $(p+1)$-tuples of matrices exist for the PMVs Λ'^n) and one finds out how to change the PMVs in order to have $d'_1+\ldots+d'_{p+1} = 2n^2-2$, without changing the quantities r_j; after this one proves the existence of rigid $(p+1)$-tuples from the new PMVs.

The following lemmas explain how this is done in more details. Recall that we denote by u the component $m_{1,1}$ of Λ_1^n and that $m_{1,1} \geq \ldots \geq m_{i_1,1}$.

Lemma 24. *If $r_j \leq n/2$ is fixed, then d_j is minimal if and only if $\Lambda_j^n = (n-r_j, r_j)$.*

P r o o f. One has $d_j = n^2 - (m_{1,j})^2 - \sum_{k=2}^{i_j}(m_{k,j})^2$ where $m_{1,j} = (n-r_j) \geq n/2$, see Remark 9. The sum $\sum_{k=2}^{i_j}(m_{k,j})^2$ is maximal if and only if $i_j = 2$, $m_{2,j} = r_j$. □

Definition 25. *Recall that $\Lambda_j^n = (m_{1,j},\ldots,m_{i_j,j})$, $m_{1,j} \geq \ldots \geq m_{i_j,j}$. If $m_{1,j} = \ldots = m_{\mu,j} > m_{\mu+1,j}$, $\mu+1 < i_j$, then the change $m_{\mu+1,j} \mapsto m_{\mu+1,j}+1$, $m_{i_j,j} \mapsto m_{i_j,j}-1$ is called a* passage. *Its inverse is called an* antipassage. *A passage preserves r_j and decreases d_j (to be checked directly). If after the change one has $m_{i_j,j} = 0$, then one deletes the last component of Λ_j^n and sets $i_j \mapsto i_j - 1$.*

Lemma 26. *If $r_j > n/2$ is fixed, then d_j is minimal if and only if $\Lambda_j^n = (m,m,\ldots,m,q)$ where $1 \leq q \leq m = n - r_j$.*

P r o o f. Perform passages as long as they are defined. No matter what the components $m_{i,j}$ are at the beginning, at the end one has $\Lambda_j^n = (m_{1,j},\ldots, m_{1,j}, q)$. □

Corollary 27. *If $u = 2$, then d_1 is minimal if and only if $\Lambda_1^n = (2,\ldots,2)$ for n even (and, hence, $d_1 = n^2-2n$) or $\Lambda_1^n = (2,\ldots,2,1)$ for n odd and $d_1 = n^2-2n+1$.*

The corollary is direct.

Remark 28. Suppose that two of the MVs equal (α,β), (v,w) with $\alpha>\beta$, $v\geq w$, $\beta\geq w$ (hence, $\alpha\leq v$) and $\beta+1\leq n/2$. Hence, their quantities d_j equal respectively $2\alpha\beta$, $2vw$. Their quantities r_j equal respectively β, w.

Change the two MVs to $(\alpha-1,\beta+1)$, $(v+1,w-1)$. Hence, their new quantities r_j are $\beta+1$, $w-1$, i.e. their sum does not change. The new quantities d_j are $\alpha\beta+\alpha-\beta-1$, $vw+w-v-1$, their sum changes by $\alpha-\beta+w-v-2=(\alpha-v)+(w-\beta)-2<0$, i.e. their sum decreases.

Lemma 29. *If $u=2$, then one has $r_2+\ldots+r_{p+1}=n$ or $n+1$.*

Indeed, if $u=2$, then $r_1=n-2$. For rigid $(p+1)$-tuples condition (β_n) holds while condition (ω_n) does not. This leaves only the two possible values (n and $n+1$) for $r_2+\ldots+r_{p+1}$.

5.4. Proof of Theorem 21. 1^0. To prove the theorem we consider all cases in which for given quantities r_j the corresponding quantities d_j are minimal. They are given by Lemmas 24, 26 and Corollary 27. We prove that among these cases **Case** Ω is the only one in which condition (α_n) does not hold. This is part 1) of the theorem. We also find all rigid cases among them (this is part 2)).

2^0. There are two possible cases: $r_2+r_3=n-1$ or n (Lemma 29).

Case 1) $r_2+r_3=n$.

Subcase 1.1) $r_2=r_3=n/2$ (i.e. n is even).

One has $\Lambda_2^n=\Lambda_3^n=(n/2,n/2)$, $d_2=d_3=n^2/2$ (Lemma 24), $d_4=2n-2$ and $d_1\geq n^2-2n$ (Corollary 27). Hence, to have rigid quadruples the last inequality must be equality and we have the series Ξ_n.

Subcase 1.2) $r_2>r_3$.

One has $r_2>n/2$, $r_3<n/2$ and by Lemmas 26 and 24 d_2, d_3 are minimal if and only if $\Lambda_2^n=(m,\ldots,m,s)$, $\Lambda_3^n=(n-m,m)$ where $n=lm+s$, $l\in\mathbf{N}$, $1\leq s\leq m$, $r_2=(l-1)m+s$, $r_3=m$.

Hence, $l\geq 2$ (otherwise $r_2\leq n/2$). One has

$$d_1\geq n^2-2n,\ d_2=l(l-1)m^2+2lms,\ d_3=2m((l-1)m+s)\text{ and }d_4=2n-2\ .$$

Set $\Delta=d_1+d_2+d_3+d_4-(2n^2-2)$. Hence,

$$\begin{aligned}\Delta&\geq -n^2-2n+2+l(l-1)m^2+2lms+2m((l-1)m+s)+2n-2=\\&=-(ml+s)^2+l(l-1)m^2+2lms+2m((l-1)m+s)=\\&=(l-2)m^2+2ms-s^2=(l-2)m^2+ms+s(m-s)>0.\end{aligned}$$

This means that rigid $(p+1)$-tuples with $r_2>n/2$, $r_3<n/2$ and $r_2+r_3=n$ do not exist.

Case 2) $r_2 + r_3 = n - 1$.

Subcase 2.1) $r_2 = r_3 = (n-1)/2$ (i.e. n is odd).

One has $d_2 = d_3 = (n^2-1)/2$, $d_4 = 2n-2$ and $d_1 \geq n^2 - 2n + 1$ with equality if and only if $\Lambda_1^n = (2, \ldots, 2, 1)$ (Corollary 27). Hence, to have a rigid quadruple the last inequality must be equality and we have the series Π_n.

Subcase 2.2) n is odd and $r_2 > r_3$.

One has $r_2 > n/2$, $r_3 < n/2$ and by Lemmas 26 and 24 d_2, d_3 are minimal if and only if $\Lambda_2^n = (m, \ldots, m, s)$, $\Lambda_3^n = (n-m+1, m-1)$ where $n = lm + s$, $l \in \mathbf{N}$, $1 \leq s \leq m$, $r_2 = (l-1)m + s$, $r_3 = m - 1$.

Hence, $l \geq 2$, (otherwise $r_2 < n/2$; note that $l = 1$, $s = m$ is impossible because n is odd) and $m > 1$ (otherwise A_3 or M_3 must be scalar). One has

$$\Delta \geq -n^2 - 2n + 2 + l(l-1)m^2 + 2lms + 2(m-1)((l-1)m + s + 1) + 2n - 2 =$$
$$= -(ml+s)^2 + l(l-1)m^2 + 2lms + 2(m-1)((l-1)m + s + 1) =$$
$$= (l-2)m^2 + 2ms - s^2 - 2(l-1)m - 2s - 2 + 2m =$$
$$= (l-2)m^2 + ms + s(m-s) - 2(l-2)m - 2s - 2 =$$
$$= (l-2)m(m-2) + s(m-s) + (m-2)s - 2 > 0$$

for $m > 2$ because either $l > 2$ or $l = 2$ and $m \geq s \geq 1$. Hence, there are no rigid quadruples in this case. If $m = 2$, then $\Lambda_3 = \Lambda_4 = (n-1, 1)$ – this gives the series Δ_n.

Subcase 2.3) n is even, $r_2 > r_3$, $r_2 > n/2$ and $r_3 < n/2$.

Like in **Subcase 2.2)** we show that no rigid quadruples exist (it is impossible to have $l = 1$, $s = m$ because in this case $r_2 = n/2$).

Subcase 2.4) n is even, $r_2 = n/2$, $r_3 = n/2 - 1$ and $\Lambda_1^n = (2, \ldots, 2)$, $\Lambda_2^n = (n/2, n/2)$, $\Lambda_3^n = (n/2 + 1, n/2 - 1)$.

One has $d_1 + d_2 + d_3 + d_4 = 2n^2 - 4$. This is precisely **Case** Ω. In this case to have an irreducible representation one cannot choose for all three matrices A_1, A_2, A_3 (or M_1, M_2, M_3) the Jordan normal forms defined by the MVs Λ_1^n, Λ_2^n, Λ_3^n.

All conjugacy classes are even-dimensional. To have a rigid quadruple one has to choose only for one of the indices $j = 1, 2, 3$ a conjugacy class of dimension ρ next after the minimal one $\rho_{\min} = d_j$ and one must have $\rho = \rho_{\min} + 2$ (because $d_1 + \ldots + d_4$ has to increase by 2).

For $n \geq 8$ this can be done only for $j = 1$ and this gives the series Θ_n. For $n = 6$ one can choose $j = 3$ as well (but not $j = 2$) and this gives the case Ψ_6. For $n = 4$ the only possibility is $(2, 1, 1)$, $(2, 2)$, $(3, 1)$, $(3, 1)$ which is the case Θ_4.

3^0. Prove that rigid quadruples from the five cases Ξ_n, Θ_n, Ψ_6, Π_n and Δ_n really exist. Use the notation from Section 4. One has

$$\Xi_{n+1} \to \Pi_n \text{ and } \Pi_n \to \Pi_{n-2} \to \ldots \to \Pi_3 = [2,1],$$

this proves the existence of the rigid series Ξ_n and Π_n. One also has

$$\Theta_n \to \Theta_{n-2} \to \ldots \to \Theta_4 \to HG_2 \text{ and } \Psi_6 \to \Theta_4 \to HG_2.$$

This proves the existence of the series Θ_n and of Ψ_6. The one of the series Δ_n follows from $\Delta_n \to \Delta_{n-2}$ and $\Delta_3 = [2,1]$.

The theorem is proved.

5.5. Proof of Theorem 22. 1^0. Recall that the change of two MVs $(n-r_j, r_j)$, $(n-r_i, r_i)$ to $(n-r_j-1, r_j+1)$, $(n-r_i+1, r_i-1)$ (provided that $r_i \leq r_j \leq n/2$ and $r_j+1 \leq n/2$) does not change the sum r_j+r_i and decreases the sum d_j+d_i, see Remark 28. In what follows when such a change is performed and after it a MV becomes equal to (n) we delete it because the corresponding matrix A_j or M_j must be scalar.

Remind that MVs like the above ones give the minimal value of d_j when r_j is fixed and $r_j \leq n/2$, see Lemma 24.

2^0. Consider only these $(p+1)$-tuples ($p \geq 4$) in which the MVs provide minimal possible values for d_j when r_j is fixed (see Lemmas 24 and 26 and Corollary 27). For all of them we show that condition (α_n) holds and is a strong inequality. Hence, it is strong for all other possible MVs with these values of r_j, i.e. no rigid $(p+1)$-tuples exist for $p \geq 4$.

As a result of suitably chosen changes of MVs like in 1^0 one comes to the case $p=3$, $\Lambda_4^n = (n-1,1)$. In this case one has $d_1+\ldots+d_4 < 2n^2-2$ only in **Case** Ω, see Theorem 21, when one has $d_1+\ldots+d_4 = 2n^2-4$.

Hence, if starting with a $(p+1)$-tuple one comes as a result of such changes of MVs to the case $p=3$, $\Lambda_4^n = (n-1,1)$, but not to **Case** Ω, then the $(p+1)$-tuple is not rigid, see Remark 28.

3^0. So consider only the $(p+1)$-tuples which after a change like in 1^0 become the quadruple from **Case** Ω. This means that either $p=4$ or $p=3$ (as a result of a change of MVs no more than one MV of the form (n) can appear).

We show in 4^0 why the case $p=3$ needs not to be considered. If $p=4$, then there are only two possibilities:

1) $\Lambda_1^n = (2,\ldots,2)$ $\Lambda_2^n = \Lambda_3^n = (n/2+1, n/2-1)$ $\Lambda_4^n = \Lambda_5^n = (n-1,1)$

2) $\Lambda_1^n = (2,\ldots,2)$ $\Lambda_2^n = (n/2, n/2)$, $\Lambda_3^n = (n/2+2, n/2-2)$ $\Lambda_4^n = \Lambda_5^n = (n-1,1)$.

One has respectively $d_1 + \ldots + d_5 = 2n^2 + 2n - 8$ and $2n^2 + 2n - 12$. Hence, the first possibility never gives a rigid quintuple (one has $n \geq 4$). The second can give a rigid quintuple only for $n = 5$, but n must be even. Note that for $n = 4$ the MV Λ_3^n from 2) equals (4), so this is in fact a quadruple, not a quintuple.

4^0. If as a result of changes of MVs a $(p+1)$-tuple with $p \geq 4$ becomes first a quadruple different from the one of **Case** Ω and then the one from **Case** Ω, then it cannot be rigid – each change decreases $d_1 + \ldots + d_{p+1}$ by at least 2 and in **Case** Ω this sum equals $2n^2 - 4$.

The theorem is proved.

5.6. Proof of Theorem 23. Set $\Delta = d_1 + d_2 + d_3 - (2n^2 - 2)$. Irreducible (resp. rigid) triples can exist only for $\Delta \geq 0$ (resp. $\Delta = 0$), see condition (α_n). Like in the proof of Theorem 21 we consider all cases in which for given quantities r_j the corresponding quantities d_j are minimal, see Lemmas 24, 26 and Corollary 27. We assume that no MV equals $(1, \ldots, 1)$, see Remark 17.

Case 1) $r_2 + r_3 = n$.

Subcase 1.1) $r_2 > n/2$, $r_3 < n/2$.

1^0. The quantities d_2 and d_3 are minimal if and only if one has $\Lambda_2^n = (m, \ldots, m, s)$, $\Lambda_3^n = (n-m, m)$, $n = lm + s$, $l \in \mathbf{N}$, $1 \leq s \leq m$, $r_2 = (l-1)m + s$, $r_3 = m$, see Lemmas 24 and 26. For such Λ_2^n, Λ_3^n one has $d_2 = l(l-1)m^2 + 2lms$, $d_3 = 2m((l-1)m + s)$. Hence,

$$\begin{aligned}\Delta &\geq -n^2 - 2n + 2 + l(l-1)m^2 + 2lms + 2m((l-1)m + s) = \\ &= -(ml + s)^2 - 2ml - 2s + 2 + l(l-1)m^2 + 2lms + 2m((l-1)m + s) = \\ &= (l-2)m^2 + 2ms - s^2 - 2(lm + s - 1) = \\ &= (l-2)m(m-2) + s(2m - s - 2) - 4m + 2.\end{aligned}$$

One has $m \geq 2$, otherwise $\Lambda_2^n = (1, \ldots, 1)$. Hence, $s(2m - s - 2) \geq 0$.

2^0. If $l \geq 3$, $m \geq 6$ or $l \geq 4$, $m \geq 4$, then $(l-2)m(m-2) - 4m > 0$ and the triple cannot be rigid. On the other hand $l \geq 2$, otherwise $r_2 \leq n/2$. Hence, rigid triples exist only for $l = 2$ or 3 or for $m = 2$ or 3.

3^0. If $m = 2$, then $\Delta < 0$ (for $s = 1$ or 2). The PMVs for which the minimal value of Δ is attained are:

1) $(2, \ldots, 2)$, $(2, \ldots, 2)$, $(n-2, 2)$ for n even; $\Delta = -6$;

2) $(2, \ldots, 2, 1)$, $(2, \ldots, 2, 1)$, $(n-2, 2)$ for n odd; $\Delta = -4$.

Find all rigid triples with such values of r_2, r_3 (i.e. $n-2, 2$). To this end one has to replace 3 multiplicities equal to 2 for n even (resp. 2 multiplicities equal to 2 for n odd) by couples of multiplicities 1,1. (Indeed, the biggest component of Λ_1^n is ≤ 2, the ones of Λ_2^n and Λ_3^n do not change because they define r_2 and r_3.) Each change of 2 by 1,1 increases Δ by 2.

The possibilities (up to permutation of Λ_1^n and Λ_2^n) for n even are 1a) – 1d), for n odd they are 2a) – 2c). Possibility 2c) is the series OG_k introduced in the previous section.

4^0. If $m = 3$, $l \geq 5$, then $\Delta > 0$, see 1^0. Hence, for $m = 3$ rigid triples can exist only for $n \leq 15$.

5^0. If $l = 3$, then $\Delta = m(m-2) + s(2m-s-2) - 4m + 2$ and $\Delta > 0$ if $m \geq 5$ or $m = 4$, $s = 2, 3, 4$ (to be checked directly). Hence, rigid triples with $l = 3$ exist only for $n \leq 13$.

6^0. If $l = 2$, then $\Delta = s(2m-s-2) - 4m + 20$ and if $4 \leq s \leq m-2$, then $\Delta > 0$. Hence, rigid triples can exist only for $s = 1, 2, 3, m-1, m$.

If $s = 3$ and $m \geq 7$, then $\Delta > 0$, i.e. with $s = 3$ rigid triples can exist only for $n \leq 21$.

If $s = m$, then for $m \geq 6$ one has $\Delta > 0$, i.e. such rigid triples can exist only for $n \leq 20$.

If $s = m-1$, then again for $m \geq 6$ one has $\Delta > 0$, i.e. such rigid triples can exist only for $n \leq 17$.

If $s = 1$, then we have $\Lambda_1^n = (2, \ldots, 2, 1)$, $\Lambda_2^n = (m, m, 1)$, $\Lambda_3^n = (m+1, m)$. One has $\Delta = -(n-1)$, i.e. for $n > 1$ one cannot choose these MVs to have rigid triples. Give the list of the MVs with the same quantities r_j for which $\Delta = 0$. They are obtained from the given ones as a result of one or several antipassages, see Subsection 5.3.

The MV Λ_2^n after one antipassage becomes $(m, m-1, 2)$ (and d_2 increases by $(n-1)-4$) or $(m, m-1, 1, 1)$ (and d_2 increases by $(n-1)-2$). The MV Λ_3^n after one antipassage becomes $(m+1, m-1, 1)$ and d_3 increases by $(n-1)-2$. To increase d_1 by $2s$ one has to make s antipassages in which a component 2 is replaced by a couple of units. However, we avoid to have $\Lambda_1^n = (1, \ldots, 1)$ which case was considered in Section 4. Therefore for $n \geq 22$ the only PMVs which give rigid triples for $s = 1$ are Z_n^2, Z_n^3 and Z_n^4.

If $s = 2$, one gets the series $\Lambda_1^n = (2, \ldots, 2, 2)$, $\Lambda_2^n = (m, m, 2)$, $\Lambda_3^n = (m+2, m)$ with $\Delta = -6$. The only ways to increase Δ by 6 for $m \geq 10$ are to make three antipassages changing a component 2 by two components 1,1. This yields possibilities 1j) and 1k).

Subcase 1.2) $r_2 = r_3 = n/2$ (n is even).

We assume that $n \geq 22$. The PMV which minimizes the sum $d_1 + d_2 + d_3$ equals $(2, \ldots, 2)$, $(n/2, n/2)$, $(n/2, n/2)$ and one has $\Delta = -2n + 2$. Hence, to obtain irreducible triples one has to choose another PMV, in which at least one MV defines a JNF giving a greater value of the corresponding quantity d_j.

For the PMV as above one has $d_1 = n^2 - 2n$, $d_2 = d_3 = n^2/2$. By replacing consecutively components equal to 2 of Λ_1^n by couples of units one can obtain as values of d_1 all even numbers from $n^2 - 2n$ to $n^2 - n$.

Hence, one cannot increase enough Δ by changing only Λ_1^n. If one changes Λ_2^n and/or Λ_3^n without changing r_2 and r_3, the new choices have to be among the following MVs, otherwise Δ increases by more than $2n - 2$:

$$\begin{array}{lll} 1) & (n/2, n/2-1, 1) & d_j = n^2/2 + n - 2; \\ 2) & (n/2, n/2-2, 2) & d_j = n^2/2 + 2n - 8; \\ 3) & (n/2, n/2-2, 1, 1) & d_j = n^2/2 + 2n - 6. \end{array}$$

If one uses possibility 2) or 3), then the only cases in which $\Delta = 0$ are 1g) and 1h). If one uses possibility 1), then this leads to case 1i) or to the series EF_n, see Section 4.

Case 2) $r_2 + r_3 = n + 1$.
Subcase 2.1) $r_2 > n/2$, $r_3 \leq n/2$.
The quantities d_2 and d_3 are minimal if and only if one has

$$\Lambda_2^n = (m, \ldots, m, s),\ \Lambda_3^n = (n - m - 1, m + 1),\ n = lm + s,$$

$$l \in \mathbf{N},\ 1 \leq s \leq m,\ m \geq 2,\ r_2 = (l-1)m + s,\ r_3 = m + 1,$$

see Lemmas 24 and 26. For such Λ_2^n, Λ_3^n one has

$$d_2 = l(l-1)m^2 + 2lms,\ d_3 = 2(m+1)((l-1)m + s - 1) = 2m((l-1)m + s) + \delta$$

where $\delta = 2(l-1)m + 2s - 2 - 2m = 2(l-2)m + 2s - 2$. Like in 1^0 one finds

$$\Delta \geq (l-2)m(m-2) + s(2m - s - 2) - 4m + 2 + \delta = (l-2)m^2 + s(2m - s) - 4m$$

(the difference in the estimation of d_3 w.r.t. 1^0 equals δ). For $l > 3$ one has $\Delta > 0$. The same is true for $l = 3$ except for $m = 2$. In the latter case one has $n = 7$ or 8.

For $l = 2$ one does not have $\Delta > 0$ only if $s = 1, 2, 3$ or 4; if $s = 4$, then $m = s = 4$ and $n = 12$; if $s = 3$, then $m = 3$ or 4, resp. $n = 9$ or 11.

The case $l = 2$, $s = 1$ is impossible because then one has $r_2 = m + 1$, $r_3 = m$ and $r_2 + r_3 = n < n + 1$.

If $l = 2$, $s = 2$, then $n = 2m + 2$ is even and for $\Lambda_1^n = (2, \ldots, 2)$, $\Lambda_2^n = (m, m, 2)$, $\Lambda_3^n = (m + 1, m + 1)$ one has

$$\Delta = (n^2 - 2n) + 2(n/2 - 1)^2 + 4(n - 2) + n^2/2 - 2n^2 + 2 = -4.$$

If $m \leq 9$, then $n \leq 20$. If $m \geq 10$, i.e. $n \geq 22$, then it is possible to increase Δ by 4 (without changing r_1, r_2, r_3) only by replacing the PMV by one of the PMVs from 1e) or 1f).

For all other choices of Λ_j^n with $r_1 = n - 2$, $r_2 = m + 2$, $r_3 = m + 1$ one has $\Delta > 0$. Hence, no rigid triples exist for such PMVs.

Subcase 2.2) $r_2 > n/2$, $r_3 > n/2$.

Necessarily n is odd ($n = 2m + 1$) and the minimal possible value of $d_1 + d_2 + d_3$ is attained for and only for $\Lambda_1^n = (2, \ldots, 2, 1)$, $\Lambda_2^n = (m, m, 1)$, $\Lambda_3^n = (m, m, 1)$ (see Lemma 26). Such triples are rigid. They give possibility 2d).

Prove the existence of the listed series. With the notation from Section 4 one has

$$\Gamma_n^i \to \Gamma_{n-2}^i, \quad \Gamma_6^1 \to X_5^1 \to Y_4^1 = \Gamma_4^2 = \Gamma_4^4 \to HG_3, \quad \Gamma_4^3 \to HG_2,$$

which proves the existence of the series Γ_n^i, $i = 1, 2, 3, 4$. One also has

$$X_n^i \to X_{n-2}^i, \quad X_5^1 \to \Gamma_4^2, \quad X_3^2 = HG_3$$

which proves the existence of the series X_n^i. Next,

$$\text{for} \quad n > 4 \quad Y_n^1 \to Z_{n-1}^2 \to Z_{n-3}^2 \to \ldots \to Z_5^2 \to \Gamma_4^4,$$

hence, the series Y_n^1 and Z_n^2 also exist. From $Z_n^i \to Z_{n-2}^i$, $Z_3^i = HG_3$, $i = 3, 4$ there follows the existence of the series Z_n^3, Z_n^4. From

$$Z_n^1 \to Y_{n-1}^5 \to Y_{n-3}^5 \to \ldots \to Y_2^5 = HG_2$$

follows the existence of Z_n^1 and Y_{n-1}^5. From $Y_n^2 \to Z_{n-1}^3$ follows the one of Y_n^2. One has

$$Y_n^3 \to Y_{n-2}^6 \to Y_{n-4}^3 \to Y_{n-6}^6 \to \ldots,$$

hence, Y_n^6 and Y_n^3 also exist (we let the reader prove the existence of Y_4^3; one has $Y_4^6 = HG_4$). Finally, one has

$$Y_n^4 \to Y_{n-2}^7 \to Y_{n-4}^4 \to Y_{n-6}^7 \to \ldots$$

which proves the existence of Y_n^4 and Y_n^7 (the reader has to prove the existence of Y_6^4 and Y_6^7).

The theorem is proved.

6. The case of arbitrary (not necessarily diagonal) Jordan normal forms.

Definition 30. *For a given JNF $J^n = \{b_{i,l}\}$ define its* corresponding *diagonal JNF J'^n. (We say that J^n and J'^n are corresponding to one another.) A diagonal JNF is a partition of n defined by the multiplicities of the eigenvalues. For each l the family $\{b_{i,l}\}$ is a partition of $\sum_{i\in I_l} b_{i,l}$ and J'^n is the disjoint sum of the dual partitions.*

Example 31. If a JNF is defined by the family $B = \{b_{i,l}\}$ where $l = 1, 2$ and $B = \{4, 2, 2\}\{5, 1\}$, i.e. there are two eigenvalues, the first (resp. the second) with three Jordan blocks, of sizes 4, 2, 2 (resp. with two Jordan blocks, of sizes 5, 1), then the corresponding diagonal JNF is defined by the MV $(3, 3, 1, 1, 2, 1, 1, 1, 1)$ (or, better, by the MV with non-increasing components $(3, 3, 2, 1, 1, 1, 1, 1, 1)$). Indeed, $(3, 3, 1, 1)$ (resp. $(2, 1, 1, 1, 1)$) is the partition dual to $(4, 2, 2)$ (resp. to $(5, 1)$).

The following theorem explains why it is sufficient to know (for generic eigenvalues) the solution to the DSP only in the case of diagonalizable matrices. The theorem is announced in [4] and proved in [5].

Theorem 32. *If for some eigenvalues the DSP is weakly solvable for a given $\{J_j^n\}$ (resp. for $\{J'^n_j\}$), then it is solvable for $\{J'^n_j\}$ (resp. for $\{J_j^n\}$) for any generic eigenvalues.*

Thus if one knows that the DSP is solvable for a certain PMV Λ^n for generic eigenvalues, then one knows that it is solvable (for generic eigenvalues) for all $(p+1)$-tuples of JNFs $\{J_j^n\}$ such that the JNF defined by Λ_j^n corresponds to J_j^n. This allows one to construct new series of $(p+1)$-tuples of JNFs (not all of which diagonal) for which there exist rigid $(p+1)$-tuples of matrices A_j or M_j. One should know, however, that for certain $(p+1)$-tuples of JNFs one cannot have generic eigenvalues.

Example 33. Consider the series C_k from Section 4 for matrices A_j. A possible triple of JNFs corresponding to the diagonal ones defined by the PMV is the following one: J_1^n and J_2^n are the same as before, i.e. diagonalizable, with MVs of the eigenvalues equal to (k, k, k) while J_j^3 has a single eigenvalue with Jordan blocks of sizes $(1, 2, 3, \ldots, 3)$. Hence, the multiplicities of all eigenvalues are divisible by k. The sum of all eigenvalues counted with multiplicities k times smaller equals 0 and this is a non-genericity relation.

Consider the same example for matrices M_j. The product of all eigenvalues with multiplicities k times smaller is a root of unity of order k. If this root is non-primitive, then again a non-genericity relation holds and there exist no such generic eigenvalues. In this case the set of possible eigenvalues with these JNFs is a reducible variety with k connected components each of which corresponds to one of the roots of unity. The eigenvalues from the components corresponding to non-primitive roots are all non-generic.

7. Proof of Theorem 19. 1^0. Theorem 32 allows one to prove the theorem only in the case of diagonalizable matrices. For $n \leq 3$ the reader can check the theorem oneself, so suppose that $n \geq 4$.

It suffices to prove that the PMV Λ^{n_1} obtained from Λ^n after applying Ψ (see Section 2) satisfies condition (β_{n_1}). (The PMV Λ^{n_1} satisfies condition (α_{n_1}) if and only if Λ^n satisfies (α_n), see Lemma 15.)

If one of the MVs is of the form $(1, \ldots, 1)$ and conditions (α_n), (β_n) hold, then in the case of matrices M_j the answer to the DSP is positive, see [6], hence, it is positive for matrices A_j as well (for generic eigenvalues the criterium is the same in the case of matrices A_j or M_j). Therefore we assume that for all j one has $m_{1,j} \geq 2$.

Remark 34. Remind that

1) the maximal value of d_j equals $n^2 - n$ and it is attained only for a MV of the form $(1, \ldots, 1)$;

2) for the MV $(n-1, 1)$ the quantity d_j equals $2n-2$; hence, if $p = 2$ and one of the MVs equals $(n-1, 1)$, then (α_n) holds only if the other two equal $(1, \ldots, 1)$;

3) for the MVs $(n/2, n/2)$ and $(n/2, n/2 - 1, 1)$ the values of d_j equal respectively $n^2/2$ and $n^2/2 + n - 2$.

2^0. Set $\rho_j := r_1 + \ldots + \hat{r}_j + \ldots + r_{p+1}$. One has $r_1 = n-1$ or $n-2$, therefore for $j \neq 1$, $p \geq 3$ one has $\rho_j \geq n - 2 + p - 1 \geq n$; this is true for $p = 2$ as well because no MV equals $(n-1, 1)$, otherwise (α_n) does not hold, see Remark 34. Therefore we check only that after performing the construction Ψ from Section 2 one has $\rho_1 \geq n_1$.

One has $n - n_1 \leq 2$, see Remark 10. If $n - n_1 = 1$, then every quantity r_j remains the same or decreases by 1. The second possibility takes place only if $r_j \geq n/2$ and Λ_j^n has two equal greatest components. Denote by l the number of indices j for which $r_j \geq n/2$. Hence, $j = 1$ is always among them. Three cases are possible:

Case 1) $l \leq 2$.

Condition (β_n) satisfied by Λ^n implies that Λ^{n_1} satisfies condition (β_{n_1}) because for $j > 1$ either all r_j remain the same or only one decreases by 1 when Ψ is performed.

Case 2) $l \geq 3$ and $p \geq 3$.

After applying Ψ in the sum ρ_1 there are two quantities r_j which are $\geq n/2 - 1$ and one which is ≥ 1, so Λ^{n_1} satisfies condition (β_{n_1}).

Case 3) $p = 2$ and $l = 3$.

The sum ρ_1 can become $< n - 1$ after applying Ψ only if $\Lambda_2^n = \Lambda_3^n = (n/2, n/2)$ and n is even. But in this case condition (α_n) does not hold for any MV Λ_1^n (see Remark 34), hence, the case has to be excluded. In all other cases the sum ρ_1 decreases by 1 and the PMV Λ^{n_1} satisfies condition (β_{n_1}).

3^0. Let $n - n_1 = 2$. Like in the case $n - n_1 = 1$, for $j \neq 1$ the sum ρ_j is $\geq n$. Indeed, if $p > 2$, then such a sum contains $r_1 \geq n - 2$ and two more quantities r_j which are ≥ 1. If $p = 2$ and $u = 2$, then no MV is of the form $(n - 1, 1)$ because condition (α_n) would not hold, see Remark 34. Hence, except $r_1 \geq n - 2$, ρ_j contains $r_2 \geq 2$ or $r_3 \geq 2$, i.e. $\rho_j \geq n$. So there remains to check that after applying Ψ one has $\rho_1 \geq n - 2$.

4^0. Denote by s_j the difference $m_{1,j} - m_{2,j}$ and by κ the number of quantities s_j which are ≤ 1. Hence, s_1 is always one of them. Four cases are possible:

Case 4) $\kappa = 1$ or 2.

At most one quantity r_j from ρ_1 decreases by at most 2, so Λ^{n_1} satisfies condition (β_{n_1}).

Case 5) $\kappa \geq 4$.

After performing Ψ one has $r_j \geq n/2 - 2$ for three indices $j > 1$, hence, $\rho_1 \geq 3n/2 - 6 \geq n - 2$ because $n \geq 4$.

Case 6) $\kappa = 3$, $p \geq 3$.

In this case after performing Ψ one has $r_j \geq n/2 - 2$ for two indices $j > 1$ and $r_j \geq 1$ for another one, so $\rho_1 \geq n - 3$ with equality only if two MVs Λ_j^n with $j > 1$ equal $(n/2, n/2)$ and a third equals $(n-1, 1)$. But in such a case $n - n_1 = 1$, so the case has to be excluded.

Case 7) $\kappa = 3$, $p = 2$.

After performing Ψ one has $\rho_1 < n - 2$ only if n is even and either both Λ_2^n, Λ_3^n are of the form $(n/2, n/2)$ or one is of this form while the other equals $(n/2, n/2 - 1, 1)$. In the first case condition (α_n) does not hold for any Λ_1^n, see Remark 34. In the second it holds only for $\Lambda_1^n = (1, \ldots, 1)$, but in this case $n - n_1 = 1$, so both cases have to be excluded.

The theorem is proved.

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