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ASPLUND FUNCTIONS AND PROJECTIONAL RESOLUTIONS OF THE IDENTITY

Martin Zemek*

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Abstract. We further develop the theory of the so called Asplund functions, recently introduced and studied by W. K. Tang. Let f be an Asplund function on a Banach space X. We prove that (i) the subspace $Y := \overline{\mathrm{sp}}\, \partial f(X)$ has a projectional resolution of the identity, and that (ii) if X is weakly Lindelöf determined, then X admits a projectional resolution of the identity such that the adjoint projections restricted to Y form a projectional resolution of the identity on Y, and the dual X^* admits an equivalent dual norm such that its restriction to Y is locally uniformly rotund.

1. Asplund functions. It is well-known that any of the following conditions for a Banach space X is equivalent to "X is an Asplund space" [3, 8, 13, 19]:

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(i) *Every continuous convex function on X is Fréchet differentiable at all points of a dense G_δ subset of X.*

(ii) *Every bounded subset of X^* is w^*-fragmentable.*

(iii) *Every bounded subset of X^* is w^*-dentable.*

(iv) *Every bounded subset of X^* is dentable.*

(v) *For every separable subspace X_0 of X, the dual X_0^* is separable.*

W. K. Tang in [17] found a class of continuous convex functions on Banach spaces that have properties similar to those of continuous convex functions on Asplund spaces. He calls the functions from this class Asplund functions and establishes a number of conditions equivalent to say that a function is Asplund. Some of such conditions are listed in the following theorem.

Theorem 1.1. *Let f be a continuous convex function on a Banach space X. Then the following conditions are equivalent:*

(i) *If h is a continuous convex function on X such that $h \leq f$ then h is Fréchet differentiable at all points of a dense G_δ subset of X.*

(ii) *For each $n \in \mathbb{N}$, every bounded subset of the set $\{x^* \in X^* \ : \ f^*(x^*) \leq n\}$ is w^*-fragmentable.*

(iii) *Every w^*-compact subset of* $\operatorname{dom} f^*$ *is w^*-fragmentable.*

(iv) *Every w^*-compact subset of* $\operatorname{dom} f^*$ *is w^*-dentable.*

(v) *Every w^*-compact subset of* $\operatorname{dom} f^*$ *is dentable.*

(vi) *For every separable subspace X_0 of X, the space $\overline{\mathrm{sp}}\, \partial(x\restriction_{X_0})(X_0)$ is separable.*

Definition 1.2. *A continuous convex function f on a Banach space X is called an* Asplund function *if any of the conditions from Theorem* 1.1 *holds.*

Of course, every continuous convex function f on an Asplund space X is an Asplund function and a Banach space is Asplund if and only if its norm is an Asplund function. We present several further properties of Asplund functions. Namely, we prove that (i) *the subspace $Y := \overline{\mathrm{sp}}\, \partial f(X)$ has a projectional resolution of the identity (PRI) if f is an Asplund function* (it is well-known that duals of Asplund spaces admit a PRI [7]), and that (ii) *if f is an Asplund function on a* *weakly Lindelöf determined (WLD) Banach space X, then X admits a PRI such that the adjoint projections restricted to Y form a PRI on Y and that X^* admits an equivalent dual norm such that its restriction to Y is LUR.* (It is well-known that every WLD Asplund space admits a PRI such that the adjoint projections form a PRI on X^* and that X^* admits a dual LUR norm [5].)

Notation and preliminaries. X always denotes a Banach space with norm $\|\cdot\|$, X^* its dual space, B_{X^*} the unit ball of X^*. "*lsc*" means lower semi-continuous. For a function f on X and a subset $A \subset X$, $f\restriction_A$ denotes the restriction of f to A. We denote by $\operatorname{co} A$, $\overline{\operatorname{co}}\ A$, $\operatorname{sp} A$, $\overline{\operatorname{sp}}\ A$, $\overline{A}$, $\overline{A}^*$, and $\operatorname{card} A$, the convex hull, closed convex hull, linear span, closed linear span, norm closure, weak* closure, and cardinality of a set A, respectively. The *density* of a set A is the smallest cardinal $\aleph$ such that there exists a dense subset $M \subset A$ with $\operatorname{card} M = \aleph$. It is denoted by $\operatorname{dens} A$. $|\alpha|$ is the cardinality of an ordinal number α. By a *subspace* of a Banach space we always mean a closed linear subspace. X is always considered as a subspace of the second dual X^{**}. For $A \subset X$ and $B \subset X^*$, we put $A^\perp := \{x^* \in X^* : \langle x^*, A\rangle = \{0\}\}$ and $B_\perp := \{x \in X : \langle B, x\rangle = \{0\}\}$, and we say that "*$A$ norms B*" if $\|x^*\| = \sup\{\langle x^*, x\rangle : x \in A, \|x\| \leq 1\}$ for every $x^* \in B$. The meaning of "*B norms A*" is analogous. If (T, τ) is a topological space and Δ is a metric on T we say that T is *fragmented by the metric* Δ if every subset of T has a relatively τ-open subset of Δ-diameter as small as we wish. If T is a subset of a dual Banach space X^*, by saying "*T is w^*-fragmentable*" we mean that (T, w^*) is fragmented by the metric generated by the dual norm. A set-valued mapping $\Gamma : (T_1, \tau_1) \to (T_2, \tau_2)$ is said to be *upper semi-continuous* (*usc*) if the set $\{t_1 \in T_1 : \Gamma(t_1) \subset U\}$ is τ-open for every τ_2-open set $U \subset T_2$. For a continuous convex function $f : X \to \mathbb{R} \cup \{\infty\}$ we denote

$$\partial f(x) = \{x^* \in X^* : \langle x^*, y - x\rangle \leq f(y) - f(x) \text{ for all } y \in X\}, \quad x \in X.$$

f^* denotes the (Fenchel) conjugate function to f defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in X\} \quad \text{for} \quad x^* \in X^*.$$

By $\operatorname{dom} f^*$ we denote the set $\{x^* \in X^* : f^*(x^*) < \infty\}$. The infimal convolution of f and of another function g on X is defined by

$$(f \,\square\, g)(x) = \inf\{f(y) + g(x - y) : y \in X\}.$$

We will use the following well-known (and easy to prove) facts:

- f^* is w^*-lower semi-continuous convex,
- $\langle x^*, x\rangle \le f(x) + f^*(x^*)$ for all $x \in X$, $x^* \in X^*$ with equality holding if and only if $x^* \in \partial f(x)$ (and this holds if and only if $x \in \partial f^*(x^*)$),
- $\partial f(X) \subset \operatorname{dom} f^* \subset \overline{\partial f(X)}$ (for the latter inclusion, see Proposition 1.3),
- $f \,\square\, g$ is convex and $(f \,\square\, g)^* = f^* + g^*$.

In the following propositions, f and g are always continuous convex functions on a Banach space X.

Proposition 1.3. $\operatorname{dom} f^* \subset \overline{\partial f(X)}$.

P r o o f. Let $x^* \in \operatorname{dom} f^*$ be given. Then $f - x^*$ is bounded below. The rest of the proof follows from the

Claim. *If f is a continuous convex function on a Banach space X and $x^* \in X^*$ is such that $f - x^*$ is bounded below then $x^* \in \overline{\partial f(X)}$.*

P r o o f o f t h e c l a i m. Let $\varepsilon > 0$ be given. Take $z \in X$ with

$$(f - x^*)(z) \le \inf_X (f - x^*) + \varepsilon^2.$$

This means $\langle x^*, x - z\rangle \le f(x) - f(z) + \varepsilon^2$ for every $x \in X$. By the Brønsted-Rockafellar theorem ([13], [9, p. 173]), there exist $y \in X$ and $y^* \in X^*$ such that $(\|y - z\| < \varepsilon)$, $\|y^* - x^*\| < \varepsilon$, $y^* \in \partial f(y)$. $\square$

Proposition 1.4. *If $f \le g$, then $\partial f(X) \subset \overline{\partial g(X)}$.*

P r o o f. Let $x^* \in \partial f(X)$ be given. Then $f - x^*$ is bounded below and so $g - x^*$ is bounded below, too. The conclusion follows from the claim in the proof of Proposition 1.3. $\square$

Proposition 1.5. *If X_0 is a subspace of X, $f_0 = f\restriction_{X_0}$ and $Q : X^* \to X_0^*$ is the canonical restriction mapping, then $Q(\partial f(X)) \subset \overline{\partial f_0(X_0)}$.*

P r o o f. Let $x^* \in \partial f(X)$ be given. $f - x^*$ is bounded below on X and so $f_0 - Q(x^*)$ is bounded below on X_0. The conclusion $Q(x^*) \in \overline{\partial f_0(X_0)}$ then follows from the claim in the proof of Proposition 1.3. $\square$

Proposition 1.6. *Under the settings of Proposition* 1.5*, $Q(\partial f(x_0)) = \partial f_0(x_0)$ for every $x_0 \in X_0$.*

P r o o f. "$x_0^* \in \partial f_0(x_0)$" means that $\langle x_0^*, x - x_0\rangle \le f_0(x) - f_0(x_0)$ for every $x \in X_0$ while "$x_0^* \in Q(\partial f(x_0))$" means that x_0^* is the restriction to X_0

of some $x^* \in X^*$ for which $\langle x^*, x - x_0\rangle \leq f(x) - f(x_0)$ for every $x \in X$. So, the inclusion "$\subset$" is obvious. To show that "$\supset$" holds true, given $x_0^* \in \partial f_0(x_0)$, denote $g := f + \langle x_0^*, x_0\rangle - f(x_0)$. g is a continuous convex function on X and $x_0^* \leq g\restriction_{X_0}$. By the Hahn-Banach dominated extension theorem (in the version of [9, p. 68]) there is a continuous extension $x^* \in X^*$ of x_0^* (i.e. $x^*\restriction_{X_0} = x_0^*$) such that $x^* \leq g$ (on X), that is, $x^* \in \partial f(x_0)$. $\square$

2. Jayne-Rogers mappings for Asplund functions. Our methods of constructing projectional resolutions of the identity are elaborations on known methods ([6, 3]). Namely, we use a generalization of Jayne-Rogers mappings constructed originally for Asplund spaces. It is well-known that if X is an Asplund space then the duality mapping has a selector (the Jayne-Rogers selector) that is the pointwise limit of a sequence of norm-to-norm continuous mappings from X into X^* ([10], [3, Theorem I.5.2]). In other words, there exists a countable-valued mapping $\mathcal{D} : X \to 2^{X^*}$ that can be written in the form

$$\mathcal{D}(x) = \{\mathcal{D}_n(x) : n \in \mathbb{N}\}, \qquad x \in X,$$

where $\mathcal{D}_n$, $n \in \mathbb{N}$, are (single-valued) mappings $X \to X^*$ satisfying

(i) $\mathcal{D}_n$ is norm-to-norm continuous for every $n \in \mathbb{N}$,

(ii) $\lim \mathcal{D}_n(x) \in J(x)$ for every $x \in X$,

where J is the duality mapping on X defined by $J = \partial\frac{\|\cdot\|^2}{2}$ or, equivalently, by

$$J(x) = \{x^* \in X^* : \langle x^*, x\rangle = \|x\|^2 = \|x^*\|^2\}, \qquad x \in X.$$

The claim of the existence of a Jayne-Rogers mapping on an arbitrary Asplund space is contained in the following theorem that is due to Jayne and Rogers [10]. (We present it in the form of [3, Theorem I.4.7]). We will use this theorem for the construction of Jayne-Rogers mappings for Asplund functions in Theorem 2.3.

Theorem 2.1. *Let T be a topological space fragmented by a metric Δ and let $\mathcal{K}(T)$ denote the family of nonempty compact subsets of T. Then there exists a selector $s : \mathcal{K}(T) \to T$ satisfying the following property. If (Z, d) is a metric space and Γ is a set-valued upper semi-continuous mapping with compact values from Z into T, then $\varphi : z \mapsto s(\Gamma(z))$ is a Baire 1 mapping of (Z, d) into (T, Δ). If T is moreover a convex subset of a vector space V and Δ is induced* *by a topological vector space topology on V, then φ is the pointwise Δ-limit of a sequence (φ_n) of continuous mappings from (Z, d) into (T, Δ).*

Definition 2.2. *Let f be a continuous convex function on a Banach space X. Let $\mathcal{D} : X \to 2^{\overline{\partial f(X)}}$ be an at most countable-valued mapping. We call $\mathcal{D}$ a* Jayne-Rogers mapping for f *if*

(i) $\partial f(x) \cap \overline{\mathcal{D}(x)} \neq \emptyset$ *for every $x \in X$,*

and if there exist single-valued mappings $\mathcal{D}_n : X \to \overline{\partial f(X)}$, $n \in \mathbb{N}$, such that

(ii) $\mathcal{D}_n$ *is norm-to-norm continuous for every $n \in \mathbb{N}$,*

(iii) $\mathcal{D}(x) = \{\mathcal{D}_n(x) : n \in \mathbb{N}\}$ *for every $x \in X$.*

Theorem 2.3. *Let f be an Asplund function on a Banach space X. Then there exists a Jayne-Rogers mapping for f.*

Proof. By Theorem 1.1 (ii), $\operatorname{dom} f^*$ is the countable union of w^*-fragmentable w^*-closed convex sets: $\operatorname{dom} f^* = \cup\{K_m : m \in \mathbb{N}\}$, where

$$K_m = \{x^* \in X^* : f^*(x^*) \leq m\} \cap mB_{X^*}.$$

Put $M = \{m \in \mathbb{N} : \partial f(X) \cap K_m \neq \emptyset\}$. For every $m \in M$ put

$$X_m = \{x \in X : \partial f(x) \cap K_m \neq \emptyset\}$$

and define a (set-valued) mapping $g_m : X_m \to 2^{K_m}$ by

$$g_m(x) = \partial f(x) \cap K_m, \qquad x \in X_m.$$

Clearly, $\bigcup_{m \in M} X_m = X$. It is well-known that $\partial f(: X \to 2^{X^*})$ is norm-to-weak* usc compact valued (see e.g. [13, Propositions 1.11 and 2.5]) and using this it is easy to check that so are the mappings g_m and that X_m are closed sets. So, for every $m \in M$ we obtain, by Theorem 2.1, a (countable-valued) mapping $G^m : X_m \to 2^{K_m}$ of the form

$$G^m(x) = \{G^m_n(x) : n \in \mathbb{N} \cup \{\infty\}\}, \quad x \in X_m,$$

where the mappings G^m_n satisfy

(i) $G^m_n : X_m \to K_m$ is norm-to-norm continuous for every $n \in \mathbb{N}$,

(ii) $\lim_n G^m_n(x) = G^m_\infty(x)$ for every $x \in X_m$,

(iii) $G^m_\infty(x) \in g_m(x)$ for every $x \in X_m$.

(To get these mappings from the "moreover part" of Theorem 2.1, put $Z := X_m$, $T := (K_m, w^*)$, let d be the metric induced by the norm of X, Δ be the metric induced by the norm of X^*, $\Gamma := g_m$, $\varphi := G^m_\infty$, and $\varphi_n := G^m_n$.) Since the sets X^m are closed and the sets K_m are convex, by [4, Theorem 4.1], for every $m \in M$ and $n \in \mathbb{N}$ there is a norm-to-norm continuous mapping $\mathcal{D}^m_n : X \to K_m$ such that $\mathcal{D}^m_n\restriction_{X_m} = G^m_n$. Define $\mathcal{D}$ by $\mathcal{D}(x) = \{\mathcal{D}^m_n(x) : m \in M, n \in \mathbb{N}\}$. Since $\cup X_m = X$, from (ii) and (iii) it follows that $\partial f(x) \cap \overline{\mathcal{D}(x)} \neq \emptyset$ for any $x \in X$. Thus, $\mathcal{D}$ is a Jayne-Rogers mapping for f. $\square$

Lemma 2.4. *Let f be an Asplund function on a Banach space X and let $\mathcal{D}$ be a Jayne-Rogers mapping for f. Then $\mathcal{D}(\overline{A}) \subset \overline{\mathcal{D}(A)}$ for every subset $A \subset X$. Consequently, if X_0 is a (non-trivial) subspace of X then* $\operatorname{dens} \overline{\operatorname{sp}}\ \mathcal{D}(X_0) \leq \operatorname{dens} X_0$.

P r o o f. Follows from the continuity of the mappings $\mathcal{D}_n$, see Definition 2.2. $\square$

By [17], the restriction $f\restriction_{X_0}$ of an Asplund function f to a subspace X_0 is an Asplund function, too. The following proposition derives a Jayne-Rogers mapping $\widetilde{\mathcal{D}}$ for $f\restriction_{X_0}$ from a given Jayne-Rogers mapping $\mathcal{D}$ for f.

Proposition 2.5. *Let f be an Asplund function on a Banach space X and let $\mathcal{D}$ be a Jayne-Rogers mapping for f. Let X_0 be a subspace of X and let $Q : X^* \to X_0^*$ be the canonical restriction mapping. Then the mapping $\widetilde{\mathcal{D}} : X_0 \to 2^{X_0^*}$ defined by $\widetilde{\mathcal{D}} = Q \circ \mathcal{D}\restriction_{X_0}$ is a Jayne-Rogers mapping for $f\restriction_{X_0}$.*

P r o o f. Let $\mathcal{D}_n$ be the mappings from Definition 2.2. Put $\widetilde{\mathcal{D}}(x) := Q(\mathcal{D}_n(x))$ for $n \in \mathbb{N}$, $x \in X$. Obviously, $\widetilde{\mathcal{D}}(x) = \{\widetilde{\mathcal{D}}_n(x) : n \in \mathbb{N}\}$, $x \in X_0$, and $\widetilde{\mathcal{D}}_n$ are continuous, which shows that (ii) and (iii) of Definition 2.2 for $\widetilde{\mathcal{D}}$ and $\widetilde{\mathcal{D}}_n$ hold. In order to verify (i), we use Proposition 1.6 and the continuity of Q:

$$\begin{aligned}\partial f_0(x_0) \cap \overline{\widetilde{\mathcal{D}}(x_0)} &= Q(\partial f(x_0)) \cap \overline{Q(\mathcal{D}(x_0))}\\ &\supset Q(\partial f(x_0)) \cap Q(\overline{\mathcal{D}(x_0)}) \supset Q(\partial f(x_0) \cap \overline{\mathcal{D}(x_0)}\,) \neq \emptyset\end{aligned}$$

for every $x_0 \in X_0$. $\square$

In the proof of Theorem 2.6 we will use Simons' lemma ([14, Lemma 2], see also [3, Lemma I.3.7]).

Simons' Lemma. *Let B be a set and C be a set of functions defined on B such that*

(i) $\sup\limits_{h\in C} \sup\limits_{x\in B} h(x) < \infty$,

(ii) *C is stable with respect to taking countable convex combinations,*

(iii) *for every $h \in C$ there exists $x_0 \in B$ such that $h(x_0) = \sup_{x\in B} h(x)$.*

Then, whenever (h_n) is a sequence in C, we have

$$\sup_{x\in B} \limsup_{n\to\infty} h_n(x) \geq \inf_{h\in C} \sup_{x\in B} h(x).$$

(The stronger assumption of $\sup_{h\in C} \sup_{x\in B} |h(x)| < \infty$ is among the assumptions of [14, Lemma 2] instead of (i), but the proof uses only our (i).)

Theorem 2.6. *Let f be an Asplund function on a Banach space X. Then there exists a Jayne-Rogers mapping $\mathcal{D}$ for f having the following property: If X_0 is a subspace of X then $\overline{\mathrm{sp}}\ \partial(f\restriction_{X_0})(X_0) = \overline{\mathrm{sp}}\ Q(\mathcal{D}(X_0))$, where $Q : X^* \to X_0^*$ is the canonical restriction mapping.*

The proof will be divided into three steps.

Lemma 2.7. *Let f be an Asplund function on a separable Banach space X that is bounded on bounded sets and let $s : X \to X^*$ be a selector of the subdifferential mapping ∂f, that is, $s(x) \in \partial f(x)$ for every $x \in X$. Then $\partial f(X) \subset \overline{\mathrm{co}}\ s(X)$.*

P r o o f. We use an idea from [16]. Put $B = s(X)$ and $\gamma = \inf_B f^*$. Clearly $\gamma < \infty$ as B is nonempty. Since $\partial f(X) \subset \operatorname{dom} f^*$ it is sufficient to show that $\operatorname{dom} f^* \subset \overline{\mathrm{co}}\ B$. If this is not so, pick $y_0^* \in \operatorname{dom} f^* \setminus \overline{\mathrm{co}}\ B$. By the separation theorem, there is $z \in X^{**}$ and $\alpha, \beta \in \mathbb{R}$ such that $\langle z, y_0^*\rangle > \beta > \alpha > \sup\langle z, \overline{\mathrm{co}}\ B\rangle$. By scaling the functional z, we may assume that $\beta - \alpha > f^*(y_0^*) - \gamma$. B is separable since f is Asplund. (This is the only place in the proof where the Asplund property of f is used.) It implies that there exists a sequence (x_n) in the set $\{x \in X : \|x\| < \|z\|, \langle x, y_0^*\rangle \geq \beta\}$ that converges to z in the topology of pointwise convergence on B. For every $x \in X$ define a function h_x by $h_x(x^*) := \langle x, x^*\rangle - f^*(x^*)$ for $x^* \in \operatorname{dom} f^*$. Set

$$A = \left\{\sum_{k=1}^{\infty} \lambda_k x_k : \lambda_k \geq 0, \sum_{k=1}^{\infty} \lambda_k = 1\right\}, \quad C = \left\{\sum_{k=1}^{\infty} \lambda_k h_{x_k} : \lambda_k \geq 0, \sum_{k=1}^{\infty} \lambda_k = 1\right\}.$$

Let $\{\lambda_k\}$ be any sequence with $\lambda_k \geq 0$, $\sum_{k=1}^{\infty} \lambda_k = 1$. Put $x = \sum_{k=1}^{\infty} \lambda_k x_k$. We have

$x \in A$ and

$$\begin{aligned}
\sum_{k=1}^{\infty} \lambda_k h_{x_k}(s(x)) &= \left\langle \sum_{k=1}^{\infty} \lambda_k x_k, s(x) \right\rangle - f^*(s(x)) \\
&= \langle x, s(x) \rangle - f^*(s(x)) = f(x) \\
&= \sup \{ \langle x, x^* \rangle - f^*(x^*) : x^* \in \operatorname{dom} f^* \} \\
&\geq \sup \{ \langle x, x^* \rangle - f^*(x^*) : x^* \in B \} \\
&= \sup \left\{ \left\langle \sum_{k=1}^{\infty} \lambda_k x_k, x^* \right\rangle - f^*(x^*) : x^* \in B \right\} \\
&= \sup \left\{ \sum_{k=1}^{\infty} \lambda_k h_{x_k}(x^*) : x^* \in B \right\}.
\end{aligned}$$

Thus, the sets B and C satisfy (iii) from Simons' lemma if we regard functions $h \in C$ as functions defined only on $B \subset \operatorname{dom} f^*$. As regards (i), we have

$$\begin{aligned}
\sup_{h \in C} \sup_{x^* \in B} h(x^*) &= \sup_{x^* \in B} \sup \left\{ \left\langle \sum_{k=1}^{\infty} \lambda_k h_{x_k}, x^* \right\rangle : \lambda_k \geq 0, \sum_{k=1}^{\infty} \lambda_k = 1 \right\} \\
&= \sup_{x^* \in B} \sup \left\{ \sum_{k=1}^{\infty} \lambda_k (\langle x^*, x_k \rangle - f^*(x^*)) : \lambda_k \geq 0, \sum_{k=1}^{\infty} \lambda_k = 1 \right\} \\
&\leq \sup \left\{ \sum_{k=1}^{\infty} \lambda_k f(x_k) : \lambda_k \geq 0, \sum_{k=1}^{\infty} \lambda_k = 1 \right\} < +\infty
\end{aligned}$$

as f is bounded on bounded sets. Clearly, (ii) is also satisfied. So, by Simons' lemma,

$$\sup_{x^* \in B} \limsup_{n \to \infty} h_{x_n}(x^*) \geq \inf_{h \in C} \sup_{x^* \in B} h(x^*).$$

Since $\lim_n h_{x_n}(x^*) = \langle z, x^* \rangle - f^*(x^*)$ for $x^* \in B$, we have

$$\alpha - \gamma > \sup_{x^* \in B} \limsup_n h_{x_n}(x^*).$$

Therefore there is $h_0 \in C$ such that $\sup\limits_{x^* \in B} h_0(x^*) < \alpha - \gamma$. But we already proved that

$$\sup_{x^* \in B} h_0(x^*) = \sup_{x^* \in \operatorname{dom} f^*} h_0(x^*)$$

and so $\alpha - \gamma > h_0(y_0^*)$. Find $\mu_k \geq 0$ such that $\sum\limits_{k=1}^{\infty} \mu_k = 1$ and $h_0 = \sum\limits_{k=1}^{\infty} \mu_k h_{x_k}$. We get

$$\alpha - \gamma > h_0(y_0^*) = \left\langle \sum_{k=1}^{\infty} \mu_k x_k, y_0^* \right\rangle - f^*(y_0^*) \geq \beta - f^*(y_0^*),$$

a contradiction. □

Proof of Theorem 2.6. *First step.* Assume that X is separable, that f is bounded on bounded sets and that $\mathcal{D}$ is an arbitrary Jayne-Rogers mapping for f. For every $x \in X$ take $s(x) \in \partial f(x) \cap \overline{\mathcal{D}(x)}$. Then from Lemma 2.7 we have $\overline{\operatorname{sp}}\, \partial f(X) = \overline{\operatorname{sp}}\, s(X)$ and so $\overline{\operatorname{sp}}\, \partial f(X) = \overline{\operatorname{sp}}\, \mathcal{D}(X)$.

Second step. Reduction to the separable case. Now, X is not required to be separable. We assume that f is bounded on bounded sets and that $\mathcal{D}$ is an arbitrary Jayne-Rogers mapping for f. We follow an idea from [3, Theorem I.5.9] to show that $\partial f(X) \subset \overline{\operatorname{sp}}\, \mathcal{D}(X)$. Let $x^* \in \partial f(X)$. We pick a separable subspace A_0 of X such that $x^* \in \partial f(A_0)$. The space $B_0 = \overline{\operatorname{sp}}\, \mathcal{D}(A_0)$ is a separable subspace of Y, by Lemma 2.4. There exists a separable subspace $A_1 \subset X$ that contains A_0 and norms B_0. Put $B_1 = \overline{\operatorname{sp}}\, \mathcal{D}(A_1)$ and find a separable subspace $A_2 \subset X$ that contains A_1 and norms B_1. Thus, by induction, we construct an increasing sequence A_n of separable subspaces of X such that A_{n+1} norms $\overline{\operatorname{sp}}\, \mathcal{D}(A_n)$. Put

$$X_0 = \overline{\bigcup_{n \in \mathbb{N}} A_n}.$$

X_0 is a separable Banach space. Let $\mathcal{D}^n$, $n \in \mathbb{N}$, be the mappings from Definition 2.2 and let Q be the canonical restriction mapping $X^* \to X_0^*$. Clearly, $x^*\restriction_{X_0} \in \partial f_0(X_0)$, where $f_0 = f\restriction_{X_0}$. Since $\widetilde{\mathcal{D}} := Q \circ \mathcal{D}\restriction_{X_0}$ is a Jayne-Rogers mapping for f_0 (Proposition 2.5), by the first step we have

$$x^*\restriction_{X_0} \in \overline{\operatorname{sp}}\, \widetilde{\mathcal{D}}(X_0).$$

Hence for every $\varepsilon > 0$ we can find $m \in \mathbb{N}$, $x_i \in X_0$, $k_i \in \mathbb{N}$ and $\lambda_i \in \mathbb{R}$, $i = 1, \dots, m$ such that

$$\left\| x^*\restriction_{X_0} - \sum_{i=1}^{m} \lambda_i \widetilde{\mathcal{D}}_{k_i}(x_i) \right\| < \varepsilon,$$

where $\widetilde{\mathcal{D}}_{k_i} = Q \circ \mathcal{D}_{k_i} \restriction_{X_0}$, $i = 1, \dots, m$. Since $X_0 = \overline{\bigcup_{n \in \mathbb{N}} A_n}$ and $\widetilde{\mathcal{D}}_{k_i}$, are continuous, there are $n_0 \in \mathbb{N}$ and $y_i \in A_{n_0}$, $i = 1, \dots, m$, such that

$$\left\| x^* \restriction_{X_0} - \sum_{i=1}^{m} \lambda_i \widetilde{\mathcal{D}}_{k_i}(y_i) \right\| < \varepsilon.$$

We have

$$y^* := x^* - \sum_{i=1}^{m} \lambda_i \mathcal{D}_{k_i}(y_i) \in \operatorname{sp} \mathcal{D}(A_{n_0})$$

and thus

$$\|y^*\| = \|y^* \restriction_{A_{n_0+1}}\| \le \|y^* \restriction_{X_0}\| < \varepsilon.$$

From $x^* - y^* = \sum_{i=1}^{m} \lambda_i \mathcal{D}_{k_i}(y_i) \in \operatorname{sp} \mathcal{D}(X)$ we conclude that $x^* \in \overline{\operatorname{sp}}\, \mathcal{D}(X)$. Therefore, $\overline{\operatorname{sp}}\, \partial f(X) = \overline{\operatorname{sp}}\, \mathcal{D}(X)$.

Third step. We do not assume that f is bounded on bounded sets. By [3, Lemma 1.4.10], there exists an increasing sequence of Lipschitz convex functions $f^m : X \to \mathbb{R}$, $m \in \mathbb{N}$ such that

(i) $f^1 \le f^2 \le \cdots \le f$ and

(ii) for every $x \in X$ there exist $m_0 \in \mathbb{N}$ and a neighbourhood V of x such that $f^m \restriction_V = f \restriction_V$ for all $m \ge m_0$.

Let $\mathcal{D}_m$ be a Jayne-Rogers mapping for f^m for $m \in \mathbb{N}$, and define $\mathcal{D}$ by $\mathcal{D}(x) := \bigcup_{m \in \mathbb{N}} \mathcal{D}^m(x)$ for $x \in X$. To show that $\mathcal{D}$ is a Jayne-Rogers mapping for f we only need to verify that $\overline{\mathcal{D}(x)} \cap \partial f(x) \neq \emptyset$ for every x and $\mathcal{D}(X) \subset \overline{\partial f(X)}$. But, given x, (ii) implies that $\partial f(x) = \partial f^{m_0}(x)$ for some m_0 and so $\overline{\mathcal{D}(x)} \cap \partial f(x) \supset \overline{\mathcal{D}^{m_0}(x)} \cap \partial f^{m_0}(x) \neq \emptyset$. Further, from Proposition 1.4 and (i) we deduce that $\overline{\partial f^1(X)} \subset \overline{\partial f^2(X)} \subset \cdots \subset \overline{\partial f(X)}$, hence that $\mathcal{D}^m(X) \subset \overline{\partial f^m(X)} \subset \overline{\partial f(X)}$ for every $m \in \mathbb{N}$, and finally that $\mathcal{D}(X) \subset \overline{\partial f(X)}$.

Finally, let X_0 be a subspace of X. Put $f_0 := f \restriction_{X_0}$ and $f_0^m := f^m \restriction_{X_0}$ for every $m \in \mathbb{N}$. Obviously, f_0^m are Lipschitz and

(i_0) $f_0^1 \le f_0^2 \le \cdots \le f_0$,

(ii_0) for every $x \in X_0$ there exist $m_0 \in \mathbb{N}$ and a neighbourhood V_0 of x in X_0 such that $f_0^m \restriction_{V_0} = f_0 \restriction_{V_0}$ for all $m \ge m_0$.

Again, by Proposition 1.4, from (i_0) it follows that $\overline{\partial f_0^1(X_0)} \subset \overline{\partial f_0^2(X_0)} \subset \cdots \subset \overline{\partial f_0(X_0)}$. From ($\mathrm{ii}_0$) it follows that $\partial f_0(X_0) \subset \bigcup_{m\in\mathbb{N}} \partial f_0^m(X_0)$. So,

$$\overline{\bigcup_m \partial f_0^m(X_0)} = \overline{\partial f(X_0)}.$$

Put $\widetilde{\mathcal{D}} := Q\circ\mathcal{D}\restriction_{X_0}$ and $\widetilde{\mathcal{D}}^m := Q\circ\mathcal{D}^m\restriction_{X_0}$, for $m \in \mathbb{N}$. Obviously, $\widetilde{\mathcal{D}}(x) = \bigcup_m \widetilde{\mathcal{D}}^m(x)$ for every $x \in X_0$. By Proposition 2.5, $\widetilde{\mathcal{D}}^m$ is a Jayne-Rogers mapping for f_0^m, $m \in \mathbb{N}$. So, using the second step we get

$$\begin{aligned}\overline{\mathrm{sp}}\ \widetilde{\mathcal{D}}(X_0) &= \overline{\mathrm{sp}} \bigcup_m \widetilde{\mathcal{D}}^m(X_0) = \overline{\bigcup_m \overline{\mathrm{sp}}\ \widetilde{\mathcal{D}}^m(X_0)} \\ &= \overline{\bigcup_m \overline{\mathrm{sp}}\ \partial f_0^m(X_0)} = \overline{\mathrm{sp}} \bigcup_m \partial f_0^m(X_0) = \overline{\mathrm{sp}}\ \partial f_0(X_0). \qquad \square\end{aligned}$$

Lemma 2.8. *Let f be an Asplund function on a Banach space X and let $\mathcal{D}$ be a Jayne-Rogers mapping for f satisfying the conclusion of Theorem 2.6. Let X_0 be a subspace of X, put $Y = \overline{\mathrm{sp}}\ \partial f(X)$, $Y_0 = \overline{\mathrm{sp}}\ \mathcal{D}(X_0)$ and let $Q : X^* \to X_0^*$ be the canonical restriction mapping. Suppose that X_0 norms Y_0. Then*

(i) *Y_0 is isometrical to $\overline{\mathrm{sp}}\ \partial(f\restriction_{X_0})(X_0)$,*

(ii) *there exists a linear projection P of Y onto Y_0 of norm one that assigns to each $y \in Y$ the unique $y_0 \in Y_0$ satisfying $y\restriction_{X_0} = y_0\restriction_{X_0}$.*

P r o o f. Put $f_0 = f\restriction_{X_0}$. Since X_0 norms Y_0, $Q\restriction_{Y_0}$ is an isometry. By Proposition 1.5 and Theorem 2.6 we get

$$\begin{aligned}Q(Y) &= Q(\overline{\mathrm{sp}}\ \partial f(X)) \subset \overline{\mathrm{sp}}\ Q(\partial f(x)) \\ &\subset \overline{\mathrm{sp}}\ \overline{\partial f_0(X_0)} = \overline{\mathrm{sp}}\ \partial f_0(X_0) = \overline{\mathrm{sp}}\ Q(\mathcal{D}(X_0)) \\ &= Q(\overline{\mathrm{sp}}\ \mathcal{D}(X_0)) = Q(Y_0).\end{aligned}$$

This proves (i) and shows that $Q\restriction_{Y_0}$ is an isometry between Y_0 and $Q(Y)$. Hence, there is an inverse mapping $(Q\restriction_{Y_0})^{-1}$ of norm one from $Q(Y)$ onto Y_0 and so

$$P := (Q\restriction_{Y_0})^{-1} \circ (Q\restriction_{Y})$$

is a projection of norm one from Y onto Y_0. Finally, suppose that for some $y \in Y$, $y_0 \in Y_0$ we have $y\restriction_{X_0} = y_0\restriction_{X_0}$. It means that $Q(y) = Q(y_0)$. Applying $(Q\restriction_{Y_0})^{-1}$

to both sides of this equality yields $Py = y_0$. This shows that for every $y \in Y$ there is just one $y_0 \in Y_0$ satisfying $y\restriction_{X_0} = y_0\restriction_{X_0}$ and that $y_0 = Py$. This proves (ii). $\square$

3. PRI for the subspace $\overline{\mathrm{sp}}\ \partial f(X)$.

Definition 3.1. *Given a Banach space X, denote by μ the smallest ordinal such that $|\mu| = \operatorname{dens} X$. A* projectional resolution of the identity (PRI) *for X is a collection $\{P_\alpha : \omega_0 \le \alpha \le \mu\}$ of linear projections of X into X that satisfy, for every $\alpha \in [\omega_0, \mu]$, the following conditions:*

(i) $\|P_\alpha\| = 1$,

(ii) $P_\alpha P_\beta = P_\beta P_\alpha = P_\beta$ *if* $\omega_0 \le \beta \le \alpha$,

(iii) $\operatorname{dens} P_\alpha X \le |\alpha|$,

(iv) $\cup\{P_\beta X : \omega_0 \le \beta < \alpha\}$ *is dense in* $P_\alpha X$ *if* α *is a limit ordinal,*

(v) $P_\mu = \mathrm{Id}_X$.

Lemma 3.2. *Let X be a Banach space. Let $\mathcal{D} : X \to 2^{X^*}$ be an at most countable valued mapping. Let $\aleph$ be an infinite cardinal number and let $A_0 \subset X$ be a subset with* $\operatorname{card} A_0 \le \aleph$. *Then there exists a set $A_0 \subset A \subset X$ such that $\overline{A}$ is linear,* $\operatorname{card} A \le \aleph$ *and A norms* $\operatorname{sp} \mathcal{D}(A)$.

Proof. Follows from Lemma 4.2 below. $\square$

Theorem 3.3. *Let f be an Asplund function on a Banach space X. Then the space $\overline{\mathrm{sp}}\ \partial f(X)$ admits a PRI $\{P_\alpha : \omega_0 \le \alpha \le \mu\}$ such that for every $\alpha \in [\omega_0, \mu]$ there is a subspace X_α of X such that $P_\alpha(\overline{\mathrm{sp}}\ \partial f(X))$ is isometrical to $\overline{\mathrm{sp}}\ \partial(f\restriction_{X_\alpha})(X_\alpha)$.*

Proof. Denote $Y = \overline{\mathrm{sp}}\ \partial f(X)$. Let $\mathcal{D} : X \to 2^Y$ be a Jayne-Rogers mapping for f satisfying the conclusion of Theorem 2.6. Let μ be the smallest ordinal with $\operatorname{dens} X = |\mu|$ and let $\{x_\alpha : \omega_0 \le \alpha < \mu\}$ be a dense set in X. By transfinite induction, we construct a "long sequence" $\{A_\alpha : \omega_0 \le \alpha \le \mu\}$ of subsets of X satisfying, for every $\alpha \in [\omega_0, \mu]$,

(i) $\overline{A_\alpha}$ is linear,

(ii) $\operatorname{card} A_\alpha \le |\alpha|$,

(iii) $A_\beta \subset A_\alpha$ if $\omega_0 \le \beta \le \alpha$,

(iv) $A_\alpha = \cup\{A_\beta : \omega_0 \le \beta < \alpha\}$ if α is a limit ordinal,

(v) A_α norms $\operatorname{sp}\mathcal{D}(A_\alpha)$.

This is done as follows. Put $A_{\omega_0} = \operatorname{sp}\{x_{\omega_0}\}$ and assume that, for an ordinal $\alpha \in (\omega_0, \mu]$, we have constructed A_β for all $\beta \in [\omega_0, \alpha)$. We will construct A_α. If α is a limit ordinal then we put $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$. It is easy to check that (i)–(v) are satisfied. If α is a non-limit ordinal then we put $A_\alpha := A$ where A is found by Lemma 3.2 for $A_0 := A_{\alpha-1} \cup \{x_\alpha\}$. Again, it is easy to verify (i)–(v). This completes the construction.

Now, fix $\alpha \in [\omega_0, \mu]$. By Lemma 2.4, $\overline{\mathcal{D}(A_\alpha)} \supset \mathcal{D}(\overline{A_\alpha})$ and so, by (v), $\overline{A_\alpha}$ norms $Y_\alpha := \overline{\operatorname{sp}}\ \mathcal{D}(A_\alpha) = \overline{\operatorname{sp}}\ \overline{\mathcal{D}(A_\alpha)} = \overline{\operatorname{sp}}\ \mathcal{D}(\overline{A_\alpha})$. Lemma 2.8 then yields a norm one projection $P_\alpha : Y \to Y$ such that $P_\alpha Y = Y_\alpha$, Y_α is isometrical to $\overline{\operatorname{sp}}\ \partial(f\restriction_{\overline{A_\alpha}})(\overline{A_\alpha})$ and

$$(*) \qquad \begin{array}{l} P_\alpha \text{ assigns to each } y^* \in Y \text{ the unique } y^*_\alpha \in Y_\alpha, \\ \text{satisfying } y^*\restriction_{\overline{A_\alpha}} = y^*_\alpha\restriction_{\overline{A_\alpha}}. \end{array}$$

We claim that $\{P_\alpha : \omega_0 \le \alpha \le \mu\}$ is a PRI on Y. Indeed, $\overline{A_\mu} = X$ since $A_\mu \supset \{x_\alpha : \omega_0 \le \alpha < \mu\}$ and thus $Y_\mu = Y$, $P_\mu = \mathrm{Id}_Y$. For any $\omega_0 \le \beta \le \alpha \le \mu$ we have $P_\beta Y = Y_\beta \subset P_\alpha Y = Y_\alpha$ by (iii), and so $P_\alpha P_\beta = P_\beta$. Also, by $(*)$ it is clear that $P_\beta P_\alpha = P_\beta$. If α is a limit ordinal then (iv) yields $Y_\alpha = \overline{\bigcup_{\beta<\alpha} Y_\beta}$, i.e. $P_\alpha Y = \overline{\bigcup_{\beta<\alpha} P_\beta Y}$. Finally, by Lemma 2.4, $\operatorname{dens} Y_\alpha \le \operatorname{dens} \overline{A_\alpha} \le |\alpha|$. $\square$

4. Asplund functions on weakly Lindelöf determined Banach spaces. A Banach space X is called *weakly Lindelöf determined* (WLD) if there are a nonempty set Γ and an injective continuous mapping $T : (X^*, w^*) \to \mathbb{R}^\Gamma$ such that for every $x^* \in X^*$ the set $\{\gamma \in \Gamma : Tx^*(\gamma) \neq 0\}$ is at most countable. We recall that every WCG, even every WCD space is weakly Lindelöf determined. See e.g. [6, 8] for details.

Lemma 4.1 ([6, Lemma 6.1.1])**.** *Let X be a Banach space and suppose there exist two sets $A \subset X$, $B \subset X^*$ such that $\overline{A}$, $\overline{B}$ are linear and*

(i) $\|x\| = \sup\langle B \cap B_{X^*}, x\rangle$ *for every* $x \in A$,

(ii) $A^\perp \cap \overline{B}^* = \{0\}$.

Then there exists a norm one linear projection $P : X \to X$ such that $PX = \overline{A}$, $P^{-1}(0) = B_\perp$ and $P^* X^* = \overline{B}^*$.

The following is a variant of [6, Lemma 6.1.3].

Lemma 4.2. *Let X be a Banach space. Let $\Phi : X^* \to 2^X$, $\Psi : X \to 2^{X^*}$ and $\mathcal{D} : X \to 2^{X^*}$ be three at most countable valued mappings. Let $\aleph$ be an infinite cardinal number and let $A_0 \subset X$, $B_0 \subset X^*$ be two subsets with $\operatorname{card} A_0 \le \aleph$, $\operatorname{card} B_0 \le \aleph$.*

Then there exist sets $A_0 \subset A \subset X$, $B_0 \subset B \subset X^*$ such that $\overline{A}$, $\overline{B}$ are linear, $\operatorname{card} A \le \aleph$, $\operatorname{card} B \le \aleph$, $\Phi(B) \subset A$, $\mathcal{D}(A) \cup \Psi(A) \subset B$ and A norms $\operatorname{sp} \mathcal{D}(A)$.

Proof. We will use an old glueing argument due to S. Mazur. By induction we will construct sequences of sets $A_0 \subset A_1 \subset A_2 \subset \cdots \subset X$ and $B_0 \subset B_1 \subset B_2 \subset \cdots \subset X^*$ as follows. If, for some $n \in \mathbb{N}$, A_{n-1} and B_{n-1} are already found then find a set $C_n \subset X$ with $\operatorname{card} C_n \le \aleph$ that norms $\operatorname{sp} \mathcal{D}(A_{n-1})$ and put

$$A_n = \left\{ \sum_{i=1}^{m} r_i x_i : m \in \mathbb{N}, x_i \in C_n \cup A_{n-1} \cup \Phi(B_{n-1}), r_i \text{ rational},\ i = 1, \dots, m \right\}$$

and

$$B_n = \left\{ \sum_{i=1}^{m} r_i x_i^* : m \in \mathbb{N}, x_i^* \in B_{n-1} \cup \mathcal{D}(A_{n-1}) \cup \Psi(A_{n-1}), \right. \\ \left. r_i \text{ rational},\ i = 1, \dots, m \right\}.$$

Now, put A $= \bigcup_{n=1}^{\infty} A_n$, $B = \bigcup_{n=1}^{\infty} B_n$. If $x_1, x_2 \in A$ then $x_1, x_2 \in A_n$ for some $n \in \mathbb{N}$ and so $x_1 + x_2 \in A_{n+1} \subset A$. Similarly, if $x \in A$ and $\lambda \in \mathbb{R}$ there is $n \in \mathbb{N}$ so that $\lambda x \in \lambda A_n \subset \overline{A_{n+1}} \subset \overline{A}$, which shows that $\overline{A}$ is linear. An analogous argument guarantees the linearity of $\overline{B}$. The remaining properties of the sets A and B claimed in the statement of the lemma are easy to check □

The following lemma is due to Valdivia. It claims the existence of so called projectional generator on any WLD space.

Lemma 4.3 ([6, Proposition 8.3.1]). *Let X be a weakly Lindelöf determined Banach space. Then there exists an at most countably valued mapping $\Phi : X^* \to 2^X$ such that $\Phi(B)^\perp \cap \overline{B}^* = \{0\}$ whenever $\emptyset \ne B \subset X^*$ and $\overline{B}$ is linear.*

Theorem 4.4. *Let f be an Asplund function on a weakly Lindelöf determined Banach space. Then X admits a PRI $\{P_\alpha : \omega_0 \le \alpha \le \mu\}$ such that $\{P_\alpha^*\restriction_{\overline{\mathrm{sp}}\,\partial f(X)} : \omega_0 \le \alpha \le \mu\}$ is a PRI on $\overline{\mathrm{sp}}\,\partial f(X)$ and $P_\alpha^*\overline{\mathrm{sp}}\,\partial f(X)$ is isometrical to $\overline{\mathrm{sp}}\,\partial(f\restriction_{P_\alpha X})(P_\alpha X)$.*

P r o o f. Denote $Y = \overline{\mathrm{sp}}\,\partial f(X)$. Let $\Phi : X^* \to 2^X$ be a mapping found in Lemma 4.3 and let $\mathcal{D} : X \to 2^Y$ be a Jayne-Rogers mapping for f satisfying the conclusion of Theorem 2.6. For every $x \in X$ find $\Psi(x) \in B_{X^*}$ with $\langle \psi(x), x\rangle = \|x\|$.

Let μ be the smallest ordinal with $\operatorname{dens} X = |\mu|$ and let $\{x_\alpha : \omega_0 \le \alpha < \mu\}$ be a dense set in X. By transfinite induction, we construct "long sequences" $\{A_\alpha : \omega_0 \le \alpha \le \mu\}$ and $\{B_\alpha : \omega_0 \le \alpha \le \mu\}$ of subsets of X and X^*, respectively, satisfying, for every $\alpha \in [\omega_0, \mu]$,

(i) $\overline{A_\alpha}$, $\overline{B_\alpha}$ are linear,

(ii) $\operatorname{card} A_\alpha \le |\alpha|$, $\operatorname{card} B_\alpha \le |\alpha|$,

(iii) $\Phi(B_\alpha) \subset A_\alpha$, $\mathcal{D}(A_\alpha) \cup \Psi(A_\alpha) \subset B_\alpha$,

(iv) $A_\beta \subset A_\alpha$, $B_\beta \subset B_\alpha$ if $\omega_0 \le \beta \le \alpha$,

(v) $A_\alpha = \cup\{A_\beta : \omega_0 \le \beta < \alpha\}$, $B_\beta = \cup\{B_\beta : \omega_0 \le \beta < \alpha\}$ if α is a limit ordinal,

(vi) A_α norms $\mathrm{sp}\,\mathcal{D}(A_\alpha)$.

This is done as follows. Put $A_{\omega_0} = \mathrm{sp}\{x_{\omega_0}\}$, $B_{\omega_0} = \{0\}$. Assume that, for an ordinal $\alpha \in (\omega_0, \mu]$, we have constructed A_β, B_β for every $\beta \in [\omega_0, \alpha)$. We will construct A_α and B_α. If α is a limit ordinal then we put $A_\alpha = \bigcup\limits_{\beta<\alpha} A_\beta$, $B_\alpha = \bigcup\limits_{\beta<\alpha} B_\beta$. It is easy to check that (i)–(vi) are satisfied. If α is a non-limit ordinal then we put $A_\alpha = A$, $B_\alpha = B$, where A and B are found in Lemma 4.2 for $A_0 := A_{\alpha-1} \cup \{x_\alpha\}$ and $B_0 := B_{\alpha-1}$. Again, it is easy to verify (i)–(vi). This completes the construction.

Now, let us fix $\alpha \in [\omega_0, \mu]$ and verify the assumptions of Lemma 4.1. For $x \in A_\alpha$ we have

$$\|x\| = \langle \Psi(x), x\rangle \le \sup\langle B_\alpha \cap B_{X^*}, x\rangle \le \|x\|$$

by (iii). Further,

$$A_\alpha^\perp \cup \overline{B_\alpha}^* \subset \Phi(B_\alpha)^\perp \cup \overline{B_\alpha}^* = \{0\}$$

by (iii) and by Lemma 4.3. Thus, Lemma 4.1 yields a norm one projection $P_\alpha : X \to X$ such that $P_\alpha X = \overline{A_\alpha}$, $P_\alpha^{-1}(0) = B_{\alpha\perp}$ and $P_\alpha^* X^* = \overline{B_\alpha}^*$.

By Lemma 2.4, $\overline{\mathcal{D}(A_\alpha)} \supset \mathcal{D}(\overline{A_\alpha})$ and so, by (vi), $\overline{A_\alpha}$ norms $Y_\alpha := \overline{\mathrm{sp}}\,\mathcal{D}(A_\alpha) = \overline{\mathrm{sp}}\,\overline{\mathcal{D}(A_\alpha)} = \overline{\mathrm{sp}}\,\mathcal{D}(\overline{A_\alpha})$. Lemma 2.8 then yields a norm one projection $\widetilde{P}_\alpha : Y \to Y$ such that $\widetilde{P}_\alpha Y = Y_\alpha$, $\widetilde{P}_\alpha Y$ is isometrical to $\overline{\mathrm{sp}}\,\partial(f\restriction_{P_\alpha X})(P_\alpha X)$ and

$$(*)\qquad \begin{array}{l}\widetilde{P}_\alpha \text{ assigns to each } y^* \in Y \text{ the unique } y^*_\alpha \in Y_\alpha \\ \text{satisfying } \ y^*\restriction_{\overline{A_\alpha}} = y^*_\alpha\restriction_{\overline{A_\alpha}}.\end{array}$$

We have to verify $\widetilde{P}_\alpha = P_\alpha^*\restriction_Y$. To this end, consider any $y^* \in Y$ and $x \in X$. Then

$$\begin{aligned}\left\langle \widetilde{P}_\alpha y^*, x\right\rangle &= \left\langle \widetilde{P}_\alpha y^*, P_\alpha x + (x - P_\alpha x)\right\rangle \\ = \left\langle \widetilde{P}_\alpha y^*, P_\alpha x\right\rangle &= \langle y^*, P_\alpha x\rangle = \langle P_\alpha^* y^*, x\rangle\end{aligned}$$

as

$$x - P_\alpha x \in P_\alpha^{-1}(0) = B_{\alpha\perp} \subset \mathcal{D}(A_\alpha)_\perp = (\overline{\mathrm{sp}}\,\mathcal{D}(\overline{A_\alpha}))_\perp = (\widetilde{P}_\alpha Y)_\perp$$

by (iii) and Lemma 2.4. We claim that $\{P_\alpha : \omega_0 \le \alpha \le \mu\}$ is a PRI on X. Indeed, $\overline{A_\mu} = X$ since $A_\mu \supset \{x_\alpha : \omega_0 \le \alpha < \mu\}$. For any $\omega_0 \le \beta \le \alpha \le \mu$ we have $P_\beta X \subset P_\alpha X$ and $P_\beta^{-1}(0) \supset P_\alpha^{-1}(0)$ by (iv) and so $P_\beta P_\alpha = P_\alpha P_\beta = P_\beta$. For limit ordinals α, (v) implies $\overline{A_\alpha} = \overline{\overline{\bigcup_{\beta<\alpha} A_\beta}}$, i.e. $P_\alpha X = \overline{\bigcup_{\beta<\alpha} P_\beta X}$.

That $\{P_\alpha^*\restriction_Y : \omega_0 \le \alpha \le \mu\}$ forms a PRI on Y follows in the same manner as in the proof of Theorem 3.3, because the construction of $\widetilde{P}_\alpha$ in this proof is the same as that of P_α in Theorem 3.3. □

5. LUR-renorming of $\overline{\mathrm{sp}}\,\partial f(X)$. We will use the following theorem by Troyanski and Zizler ([18, 20]). We present it in the form of [3, Theorem VII.1.8 and Remark VII.1.7].

Theorem 5.1. *Let $\mathcal{P}$ be a class of Banach spaces such that every Y in $\mathcal{P}$ admits a PRI $\{P_\alpha : \omega_0 \le \alpha \le \mu\}$ such that $P_\alpha Y$ belongs to $\mathcal{P}$ for every $\alpha \in [\omega_0, \mu)$. Then every Y in $\mathcal{P}$ admits an equivalent LUR norm. Moreover, if the spaces Y in $\mathcal{P}$ are subspaces of dual Banach spaces and if the projections P_α are weak*-to-weak* continuous then every $Y \in \mathcal{P}$ admits an equivalent w^*-lower semi-continuous LUR norm.*

Theorem 5.2. *Let f be an Asplund function on a WLD Banach space X. Then there exists an equivalent dual norm on X^* such that its restriction to $\overline{\mathrm{sp}}\,\partial f(X)$ is LUR.*

P r o o f. Denote by $\mathcal{P}$ the class of all Banach spaces that are isometrical to a subspace Y of dual Banach spaces Z^* such that (i) Z is WLD and (ii) there exists an Asplund function g on Z with $Y = \overline{\mathrm{sp}}\, \partial g(Z)$. It follows from Theorem 4.4 and from the fact that subspaces of WLD spaces are WLD that this class $\mathcal{P}$ satisfies the assumptions of the "moreover part" of Theorem 5.1. Consequently, there exists an equivalent w^*-lower semi-continuous LUR norm on every $Y \in \mathcal{P}$. The conclusion follows from the following proposition. □

Proposition 5.3. *Let Y be a subspace of a dual Banach space X^* and let $|\cdot|$ be an equivalent w^*-lower semi-continuous norm on Y. Then there exists an equivalent dual norm $|||\cdot|||$ on X^* such that $|||\cdot|||\restriction_Y = |\cdot|$.*

P r o o f. Let B_{X^*} be the unit ball of X^* in the original norm $\|\cdot\|$ and let $B_1 \subset Y$ be the $|\cdot|$-unit ball of Y. We may and do assume that $|\cdot| \leq \|\cdot\|\restriction_Y \leq \beta|\cdot|$ for some $\beta \in \mathbb{R}$. Since B_1 is relatively w^*-closed in Y it can be written as an intersection of relatively w^*-closed halfspaces (in Y), i.e. there exist an index set I and a family $\{p_\alpha : \alpha \in I\}$ of w^*-continuous linear functionals on Y such that

$$B_1 = \{y^* \in Y : p_\alpha(y^*) \leq 1, \alpha \in I\}.$$

Since B_1 is symmetric, it also satisfies

$$B_1 = \{y^* \in Y : |p_\alpha(y^*)| < 1, \alpha \in I\}.$$

Clearly, $p_\alpha \leq \|\cdot\|\restriction_Y$ for all $a \in I$. By the Hahn-Banach dominated extension theorem([9, p. 68]), every p_α can be extended to a w^*-continuous linear functional $\widetilde{p}_\alpha$ defined on the whole of X^* in such a way that $\widetilde{p}_\alpha \leq \|\cdot\|$ on X^*. Denote

$$B_2 = \{y^* \in X^* : |\widetilde{p}_\alpha(y^*)| \leq 1, \alpha \in I\} \cap \beta B_{X^*}.$$

From the construction it follows that

(i) B_2 is w^*-closed, symmetric and convex,

(ii) $B_{X^*} \subset B_2 \subset \beta B_{X^*}$,

(iii) $B_2 \cap Y = B_1$.

Consequently, the Minkowski's functional $|||\cdot|||$ of the set B_2 is an equivalent dual norm on X^* such that $|||\cdot|||\restriction_Y = |\cdot|$. □

Theorem 5.4. *Let f be an Asplund function on a WLD Banach space X.*

(a) *If f is bounded on bounded sets then f can be approximated by Fréchet differentiable convex functions uniformly on bounded sets.*

(b) *If f is Lipschitz then f can be approximated by Fréchet differentiable convex functions uniformly on X.*

Proof. We use the argument from [15]. Denote $Y = \overline{\mathrm{sp}}\,\partial f(X)$ and define a sequence (h_n) of functions on X^* by $h_n(x^*) = f^*(x^*) + \frac{1}{n^3}\|x^*\|^{*2}$ for $x^* \in Y$ and $h_n(x^*) = \infty$ for $x^* \in X^* \setminus Y$, where $\|\cdot\|^*$ is an equivalent dual norm on X^* whose restriction to Y is LUR (Theorem 5.2). Clearly, $\operatorname{dom} h_n = \operatorname{dom} f^* \subset Y$. Define $g_n := f \,\square\, \frac{n^3}{4}\|\cdot\|^2$, where $\square$ denotes the infimal convolution. g_n is a continuous convex function satisfying $g_n^* = h_n$ for all n. If $n \in \mathbb{N}$, $x \in X$ and $y^* \in \partial g_n(x)$ are given, then $y^* \in Y$ and h_n is rotund at y^* with respect to x in the sense of [2], i.e. for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\{x^* : h_n(y^* + x^*) - h_n(y^*) - \langle x^*, x\rangle \le \delta\} \subset \varepsilon B_{X^*}$. By [2, Proposition 4], g_n is Fréchet differentiable at x with the derivative y^*. By [11, Lemma 2.4] (resp. by its proof) $g_n \to f$ uniformly on bounded sets (resp. uniformly on X). $\square$

Remarks. In the context of the presented results the following questions arise.

Q1. *Given an Asplund space X, does its dual X^* admit a PRI with respect to any equivalent nondual norm?* We note that the result of Fabian and Godefroy from [7] reads as: The dual to every Asplund space, that is, every dual Banach space with the Radon-Nikodým property, admits a PRI with respect to any dual norm. Once a norm is not dual, the argument of [7] does not work.

Q2. *Given an Asplund space X and a non-weak*-closed subspace Y of X, does there exist a PRI on Y with respect to the restriction of the dual norm of X^* to Y?* Theorem 3.3 says that the answer is "Yes" if Y is of the form $\overline{\mathrm{sp}}\,\partial f(X)$, where f is a continuous convex function on X. The answer is also "Yes" if the subspace Y is weak*-closed. Indeed, if we put $Z = Y_\perp$ then the quotient X/Z is also Asplund and $(X/Z)^*$ is isometrical to $Z^\perp = (Y_\perp)^\perp = \overline{Y}^* = Y$.

Q3. *Given an arbitrary Banach space X and a non-weak*-closed weak*-dentable subspace Y of X^*, does there exist a PRI on Y with respect to the restriction of the dual norm of X^* to Y?* The question is stronger than Q2, because the dual to every Asplund space is weak* dentable.

Q4. *If a Banach space X is dentable, does there exist a PRI on X?* The question Q3 is stronger than Q4 because here X is a weak*-dentable subspace of X^{**}. A positive answer to Q4 would imply that spaces with Radon-Nikodým property admit an equivalent locally uniformly rotund norm, which is, however, a long standing and widely open problem.

Example of an Asplund function. Consider the Banach space

$$X = (\ell_1(\Gamma), \|\cdot\|_1),$$

where Γ is an infinite set and $\|\cdot\|_1$ is the canonical norm on $\ell_1(\Gamma)$, and consider a function $f : X \to \mathbb{R}$ defined by

$$f(x) = \sum_{\gamma\in\Gamma} x_\gamma^2, \quad x = (x_\gamma) \in \ell_1(\Gamma).$$

We can easily check that, for every $x \in X$, $\partial f(x) = \{2x\}$ ($\subset (\ell_\infty(\Gamma), \|\cdot\|_\infty) = X^*$, where $\|\cdot\|_\infty$ is the canonical norm on $\ell_\infty(\Gamma)$). Since the "identity" mapping $(\ell_1(\Gamma), \|\cdot\|_1) \to (\ell_\infty(\Gamma), \|\cdot\|_\infty)$ is norm-to-norm continuous, f is Fréchet differentiable on X and thus Asplund. (f' is continuous on X and so, for every separable subspace $X_0 \subset X$, $\overline{\mathrm{sp}}\, \partial(f\restriction_{X_0})(X_0) = \overline{\mathrm{sp}}(f'\restriction_{X_0})(X_0)$ is separable.) We can also easily check that

$$\partial f(x) = \ell_1(\Gamma), \quad \mathrm{dom}\, f^* = \ell_2(\Gamma), \quad \overline{\mathrm{dom}\, f^*} = c_0(\Gamma).$$

Thus,

$$\partial f(X) \subsetneqq \mathrm{dom}\, f^* \subsetneqq \overline{\mathrm{dom}\, f^*} \subsetneqq \ell_\infty(\Gamma) = X^*$$

and

$$Y = \overline{\mathrm{sp}}\, \partial f(X) = \overline{\mathrm{sp}}\ \mathrm{dom}\, f^* = c_0(\Gamma) \subsetneqq X^*.$$

Note that $(c_0(\Gamma), \|\cdot\|_\infty)$ is not dentable (it contains an infinite 2-tree) and so Y, as a subspace of X^*, is not weak*-dentable. But, by Theorem 4.1, $(c_0(\Gamma), \|\cdot\|_\infty)$ has a PRI.

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Nad záložnou 5
182 00 Prague 8
Czech Republic

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