Global regularity of nonlinear dispersive equations and Strichartz estimates.

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Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Evgeni Y Ovcharov*)

To my parents

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Abstract

Our thesis is concerned with Strichartz estimates both in terms of their method of proof and application to nonlinear partial differential equations (PDEs). We make an overview of the method in the abstract setting following the style of exposition introduced by Keel and Tao [30] (1998) and the subsequent work by Foschi [20] (2005). Into that we incorporate generalizations to many special advances that have appeared recently. In a separate chapter we make explicit derivations to concrete equations like the wave, the Schrödinger , the Dirac, the Klein-Gordon and other similar equations. Thus, we present the most complete sets of estimates for these equations, improving significantly what has appeared in the literature. We also uncover the fact that certain endpoint inhomogeneous Strichartz estimates contain all homogeneous Strichartz estimates as a subclass in terms of an equivalence between the two types of estimates, see Theorem 1.3.2. Thus to us it is the inhomogeneous Strichartz estimates that are of prime interest. This aforementioned equivalence helps us also to study the class of homogeneous Strichartz estimates for data classes outside the energy class. Such estimates were originally investigated by T. Kato [29] in the special context of the Schrödinger equation which we generalize and improve substantially. Our thesis contains also two separate new developments which we summarize in the paragraphs below.

The first main subject in the thesis of special interest are the Strichartz estimates for the kinetic transport equation. These estimates were studied before us by Castella and Perthame [11] (1996) and Keel and Tao [30] (1998). Our work vastly improves the family of inhomogeneous Strichartz estimates for that equation. This fact allows us to prove that the Othmer-Dunbar-Alt kinetic model of bacterial chemotaxis is globally well-posed for small enough initial data in 3d. Moreover, this is the first application of Strichartz estimates to global well-posedness in the context of a nonlinear kinetic model and therefore it validates them as a useful tool in such contexts.

Another related question is the validity of the endpoint homogeneous Strichartz estimate in the context of the kinetic transport equation in spatial dimensions bigger than one. This problem was considered by Keel and Tao in their famous work [30] but proved too difficult to be resolved. In 1d we give a new geometric counterexample to the failure of the endpoint estimate which is shown to occur on characteristic functions to Besicovitch (Kakeya) sets. We too cannot resolve the higher dimensional endpoints but instead prove weaker substitutes that are valid in the setting of finite velocity spaces. The loss of integrability compared to the original estimate can be chosen to be arbitrary small. The original estimate cannot be resolved by the techniques of Keel and Tao [30] and Foschi [20] alone, in view of a counterexample that we give. The difficulty here is that there does not exist a family of perturbed estimates in a "full neighborhood" around it in a sense to be made precise later in the text.

The other main subject in the thesis that is of special interest is related to the question of the global regularity of the Dirac-Klein-Gordon system in space dimensions above one for large initial data. That question was originally considered in the 1970's by Chadam and Glassey [12, 13, 23] but proved too difficult for the time. In the past decade there is a renewed interest that brought many advances on the front of low regularity local existence theory. We present the first proof in 2d of global existence for large initial data under the assumption that the data is spherically symmetric. Our result relies on new inhomogeneous Strichartz estimates for spherically symmetric functions that we prove in the abstract setting and in particular for the wave equation. We propose a definition of spherical symmetry for spinors and show that under it the class of spherically symmetric initial data is preserved by the time evolution of the system. Independently of us, a new preprint appeared by Grünrock and Pecher [24] that

claims to solve that problem in a higher regularity class without the assumption of spherical symmetry.

We make a number of other lesser improvements and generalizations in relation to the Strichartz estimates that shall be presented in the main body of this text.

Contents

Chapter 1

Introduction

1.1 Preliminaries

Strichartz estimates are a type of a-priori estimates for the solutions of a large class of linear partial differential equations whose common property is that their solutions tend to disperse over time. Originally, such estimates were proved by R. Strichartz [45] in the late 1970's for the wave equation but later researchers extended them to other dispersive equations. The original method of proof relied on the recently discovered by Stein and Tomas fundamental results on the restriction properties of the multidimensional Fourier transform. However, the techniques were based on heavy harmonic analysis and the estimates were limited to special cases. In his article [38], Pecher showed that the time and space exponents need not be equal and thus provided most of the Strichartz estimates for the homogeneous equation in the special context of the Klein-Gordon equation. The next major advancement in the method came out in Ginibre and Velo [21] who invented a simpler and more flexible proof that relied only on the duality principle in Functional Analysis. In the late 1980's, Yajima extended the method to equations with inhomogeneous terms to cover different time and space exponents. These ideas were finalized in the mid 1990's in the papers by Lindblad and Sogge [31] and Ginibre and Velo [22]. Today, the core of these techniques is known as the TT^*-method.

By the mid 1990's Strichartz estimates became a standard tool in the analysis of the Schrödinger and the wave equations and gradually became familiar to researches working outsides these two equations. For example, in 1996 came out Castella and Perthame's short article [11], where they prove some homogeneous Strichartz estimates for the kinetic transport equation.

The next breakthrough came in 1997 when Keel and Tao [30] brought a much awaited unification in the theory. The authors elucidated the fundamental property of scaling in the estimates, presented the method in the abstract level, and gave some new tools based on bilinear-form interpolation and scaling invariant decompositions which are today the core of studying the end-point estimates and the inhomogeneous estimates.

In a paper of 2005, Foschi [20] gave a further refinement of the method by introducing a dyadic Whitney decomposition which is more effective than the original one of [30] in the inhomogeneous setting.

Please bear in mind that in our historical remarks we have selected only these advances that have helped in a larger degree the subject to obtain its modern outlook. We are interested only in the most basic form of the subject that treats the question of the Strichartz estimates as consequences of a given dispersive (decay) estimate and an energy estimate. The latter two estimates can be regarded as axioms which form the fundamental level in the body of Strichartz estimates and the entire structure is raised only by deduction from them. There are many other special circumstances of interest which we regard as special topics and extensions of the main theory and shall not be considered here.

After these introductory remarks let us now present the subject from a mathematical perspective. We denote by $U(t)$ the continuous linear evolution group of a linear homogeneous differential equation. The two most important properties of $U(t)$ are

- the dispersive estimate:

$$\|U(t)f\|_{L^\infty_x} \lesssim \frac{1}{|t|^\sigma} \|f\|_{L^1_x}, \quad t \in \mathbb{R}, \ \forall f \in L^1(X; d\mu) \tag{1.1}$$

- the energy estimate

$$\|U(t)f\|_{L^2_x} \lesssim \|f\|_{L^2_x}, \quad t \in \mathbb{R}, \ \forall f \in L^2(X; d\mu) \tag{1.2}$$

where $\sigma > 0$ is the rate of decay, f is the initial profile of the wave, and by $L^p = L^p(X; d\mu)$ we denote the Lebesgue space L^p over some measure space $(X, d\mu)$. The two inequalities above reflect the physical phenomenon that the amplitude of the wave decays over time (equation (1.1)), while its total energy remains constant (in the case of equality in equation (1.2)).

The homogeneous Strichartz estimates have the form

$$\|U(t)f\|_{L^q_t L^r_x} \lesssim \|f\|_{L^2_x}, \qquad \forall f \in L^2_x.$$

To the inhomogeneous equation we associate the following operator

$$W(t)F = \int_{-\infty}^{t} U(t-s)F(s)ds. \tag{1.3}$$

Under the assumption that supp $F \subseteq [0, \infty) \times \mathbb{R}^n$, (1.3) gives the Duhamel's formula of the fundamental solution to the inhomogeneous PDE. The inhomogeneous Strichartz estimates have the form

$$\|W(t)F\|_{L^q_t L^r_x} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x}, \tag{1.4}$$

where by $L^q_t L^r_x$ we denote the Lebesgue space $L^q(\mathbb{R}; L^r(X; d\mu))$. We show in the sequel that the homogeneous Strichartz estimates can be identified as a special subclass of the inhomogeneous ones, see Theorem 1.3.2. From this point of view, the study of the inhomogeneous Strichartz estimates shall be our prime goal. The Lebesgue norms in the dispersive and energy inequalities shall be suitably generalized to vector-valued Lebesgue norms and abstract Banach space norms in the subsequent chapters. We shall study the explicit form of the Strichartz estimates for concrete equations and shall prove new Strichartz estimates that shall help us prove existence of solutions to nonlinear PDE's.

The thesis is organized as follows. This introductory chapter continues with an example of a typical application of the Strichartz estimates to the global well-posedness of a nonlinear dispersive equation. We also highlight a universal equivalence relation between various types of homogeneous and inhomogeneous Strichartz estimates.

Chapter 2 is the place where we begin studying in detail the methods of proof of Strichartz estimates. This is done in a concrete situation, that of the kinetic transport equation. This chapter is completely self-contained and the exposition follows closely our preprint [37]. The treatment of this setting resembles to a large degree that of the abstract setting at least on the level of how interpolation works. This is the case because in both situations we cannot use the special interpolation properties of the (isotropic) Lebesgue spaces, and in particular the fact that in the latter context the real method gives rise to a finer family of well-known spaces, that of the Lorentz spaces, for which we have extensive embedding relations in the cruder family of Lebesgue spaces.

In Chapter 3 we make an application of the Strichartz estimates for the KT equation to the global well-posedness of the nonlinear kinetic model of chemotaxis proposed by Othmer-Dunbar-Alt.

In Chapter 4 we prove some new inhomogeneous Strichartz estimates with spherical symmetry in the abstract setting. The most important concrete case, that of the wave equation is also considered explicitly. In Chapter 5 we make an application of the estimates from Chapter 4 to the global well-posedness of the Dirac-Klein-Gordon (DKG) system in two spatial dimensions when the data is spherically symmetric.

Chapter 6 is intended to be a reference containing the complete set of the Strichartz estimates in the context of the most frequently used partial differential equations. The formal proofs are given in Chapter 7.

1.2 Working example

This section provides a working example for one of the most typical applications of the Strichartz estimates in the analysis of nonlinear PDE's. First, we fix the notation of some of the most frequently used spaces and operators throughout the entire body of this text.

By D^s we denote the operator of fractional differentiation of symbol $|\xi|^s$, and analogously, by Λ^s we denote the inhomogeneous operator of symbol $(1+|\xi|^2)^{s/2}$. The homogeneous Sobolev space $\dot{H}^s_r(\mathbb{R}^n)$ on $\mathbb{R}^n$ is defined by

$$\dot{H}^s_r(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) \setminus \mathcal{P}(\mathbb{R}^n) : \|D^s u\|_{L^r(\mathbb{R}^n)} < \infty\}$$

for $1 < r < \infty$, $s \in \mathbb{R}$. In other words that is the set of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ on $\mathbb{R}^n$, factorized by all polynomials $\mathcal{P}(\mathbb{R}^n)$ on $\mathbb{R}^n$, whose Sobolev norms $\|D^s \cdot\|_{L^r(\mathbb{R}^n)}$ are finite. When $r = 2$, instead of $\dot{H}^s_2(\mathbb{R}^n)$, we simply write $\dot{H}^s(\mathbb{R}^n)$. Analogously, the inhomogeneous Sobolev space $H^s_r(\mathbb{R}^n)$ on $\mathbb{R}^n$ is defined by

$$H^s_r(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \|\Lambda^s u\|_{L^r(\mathbb{R}^n)} < \infty\}.$$

The homogeneous Besov space $\dot{B}^s_{r,q}(\mathbb{R}^n)$ on $\mathbb{R}^n$ is defined by

$$\dot{B}^s_{r,q}(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n) \setminus \mathcal{P}(\mathbb{R}^n) : \left\|\{2^{sj}\, \|\phi_j * u\|_{L^r(\mathbb{R}^n)}\}_{j\in\mathbb{Z}}\right\|_{l^q} < \infty\right\},$$

where $\{\phi_j\}_{j\in\mathbb{Z}}$ is a homogeneous Littlewood-Paley dyadic decomposition on $\mathbb{R}^n$. Analogously, the inhomogeneous Besov space $B^s_{r,q}(\mathbb{R}^n)$ on $\mathbb{R}^n$ is defined by

$$B^s_{r,q}(\mathbb{R}^n) = \left\{u \in \mathcal{S}'(\mathbb{R}^n) : \left\|\{2^{sj}\, \|\phi_j * u\|_{L^r(\mathbb{R}^n)}\}_{j=0}^{\infty}\right\|_{l^q} < \infty\right\},$$

where $\{\phi_j\}_{j=0}^{\infty}$ is a inhomogeneous Littlewood-Paley dyadic decomposition on $\mathbb{R}^n$.

Recall the well-known continuous embeddings between the Besov spaces $\dot{B}^s_{r,2}$, and the Sobolev spaces $\dot{H}^s_r$

$$\dot{H}^s_r \hookrightarrow \dot{B}^s_{r,2}, \quad 1 < r \leq 2, \qquad \dot{B}^s_{r,2} \hookrightarrow \dot{H}^s_r, \quad 2 \leq r < \infty, \tag{1.5}$$

see [3, p. 152]. Analogous embeddings are valid for the inhomogeneous Besov and Sobolev spaces too.

Now let us consider the following estimate

$$\begin{aligned} \|u(t)\|_{L^4_t L^4_x} + \left\|D^{1/2}u(t)\right\|_{L^\infty_t L^2_x} + \left\|D^{-1/2}\partial_t u(t)\right\|_{L^\infty_t L^2_x} \lesssim \\ \|f\|_{\dot{H}^{1/2}} + \|g\|_{\dot{H}^{-1/2}} + \|F\|_{L^{4/3}_t L^{4/3}_x}. \end{aligned} \tag{1.6}$$

for the solution of

$$\begin{aligned} \Box u(t,x) = F(t,x,u) \quad (t,x) \in \mathbb{R}^{1+3} \\ (u, \partial_t u)_{t=0} = (f,g), \end{aligned}$$

proved by Strichartz in his original paper.

Example 1.2.1. [41, p. 110] Let us apply the above inequality to prove global existence and uniqueness for $\Box u = u^3$ on $\mathbb{R}^{1+3}$ with data $(u, \partial_t u)_{t=0} = (f,g) \in \dot{H}^{1/2} \times \dot{H}^{-1/2}$, provided that

$$E_0 = \|f\|_{\dot{H}^{1/2}} + \|g\|_{\dot{H}^{-1/2}}$$

is sufficiently small. To see this, denote by $X(u)$ the left hand side of (1.6). We now iterate in this norm. The iterates are defined inductively by $u_0 = 0$ and

$$\Box u_j = u_{j-1}^3$$

with data (f, g), for $j \in \mathbb{N}$. Then by (1.6), using the fact that

$$\|uvw\|_{L^{4/3}} \leq \|u\|_{L^4} \|v\|_{L^4} \|w\|_{L^4}$$

we have

$$X(u_j) \leq CE_0 + CX(u_{j-1})^3.$$

So if $X(u_{j-1}) \leq 2CE_0$, then so is $X(u_j)$, provided $C(2CE_0)^2 \leq 1/2$. Then, since

$$\Box(u_{j+1} - u_j) = u_j^3 - u_{j-1}^3 = (u_j - u_{j-1})u_j^2 + u_{j-1}(u_j + u_{j-1})(u_j - u_{j-1})$$

with vanishing initial data, we have

$$X(u_{j+1} - u_j) \leq C'[X(u_j) + X(u_{j-1})]^2 X(u_j - u_{j-1}) \leq C'(4CE_0)^2 X(u_j - u_{j-1}),$$

so $\{u_j\}$ is Cauchy provided $C'16C^2E_0^2 \leq 1/2$.

1.3 Equivalent estimates

In this section we present some instances of equivalence between two given Strichartz estimates. To do so, let us first introduce the setting. Consider two abstract Banach spaces $\mathcal{B}_1$, $\mathcal{B}_2$. Suppose that the duality pairing $\langle \cdot, \cdot \rangle$ for these two spaces is the same and that $\mathcal{B}_1$ and ${\mathcal{B}_2}^*$ have a common dense subset $\mathcal{S}$. We define the adjoint $U^*(t) : \mathcal{S} \to {\mathcal{B}_1}^*$ to $U(t) : \mathcal{S} \to \mathcal{B}_2$ by

$$\langle U(t)f, g \rangle = \langle f, U^*(t)g \rangle \quad \forall f, g \in \mathcal{S}.$$

A typical example is $\mathcal{B}_1 = L^p$, $\mathcal{B}_2 = L^q$, which have the same duality pairing $\langle f, g \rangle = \int fg dx$, and $\mathcal{S}$ being taken as the Schwartz class on $\mathbb{R}^n$.

Lemma 1.3.1 (The Duality lemma)**.** *The following two estimates for $W(t)$ are equivalent*

$$\|W(t)F\|_{L_t^q(\mathbb{R};\mathcal{B}_2)} \lesssim \|F\|_{L_t^p(\mathbb{R};\mathcal{B}_1)},$$
$$\|W(t)F\|_{L^{p'}(\mathbb{R};{\mathcal{B}_1}^*)} \lesssim \|F\|_{L^{q'}(\mathbb{R};{\mathcal{B}_2}^*)},$$

for $1 \leq p, q \leq \infty$, whenever they are both invariant to the transformation $U(t) \leftrightarrow U(-t)$.

Proof. The proof is a straightforward generalization of the proof of Lemma 2.5.3 □

Theorem 1.3.2 (The Equivalence theorem)**. A.** *The following three estimates are equivalent*

$$\|U(t)f\|_{L_t^q(\mathbb{R};\mathcal{B}_2)} \lesssim \|f\|_{\mathcal{B}_1}, \qquad \forall f \in \mathcal{B}_1,$$
$$\|W(t)F\|_{L_t^q(\mathbb{R};\mathcal{B}_2)} \lesssim \|F\|_{L_t^1(\mathbb{R};\mathcal{B}_1)}, \qquad \forall F \in L^1(\mathbb{R};\mathcal{B}_1),$$
$$\|W(t)F\|_{L^\infty(\mathbb{R};{\mathcal{B}_1}^*)} \lesssim \|F\|_{L^{q'}(\mathbb{R};{\mathcal{B}_2}^*)}, \qquad \forall F \in L^{q'}(\mathbb{R};{\mathcal{B}_2}^*).$$

B. *If $\mathcal{B}_1$ is a Hilbert space, the homogeneous estimate above is equivalent to*

$$\|W(t)F\|_{L_t^q(\mathbb{R};\mathcal{B}_2)} \lesssim \|F\|_{L^{q'}(\mathbb{R};{\mathcal{B}_2}^*)}, \qquad \forall F \in L^{q'}(\mathbb{R};{\mathcal{B}_2}^*).$$

whenever $q > 2$. In the case when $q = 2$ we can only claim that the homogeneous estimate is implied from the latter inhomogeneous estimate.

Proof. The proof is a straightforward generalization of the proof of Theorem 2.3.5. For part B, bear in mind that in the abstract setting we use the Christ-Kiselev lemma 7.1.5 instead of Lemma 2.5.1 and thus now for the endpoint case the implication is only one-way. □

Chapter 2

Strichartz Estimates for the Kinetic Transport Equation

2.1 Introduction

The main goal of this chapter is to study the range of validity of the Strichartz estimates for the kinetic transport (KT) equation

$$\partial_t u(t,x,v) + v \cdot \nabla_x u(t,x,v) = F(t,x,v), \quad (t,x,v) \in (0,\infty) \times \mathbb{R}^n \times \mathbb{R}^n, \tag{2.1}$$

$$u(0,x,v) = f(x,v). \tag{2.2}$$

The first estimates for the KT equation appeared in the paper of Castella and Perthame [11] (1996). The authors prove a large part of the homogeneous Strichartz estimates for that equation plus some inhomogeneous estimates of special type. The next work treating that subject is the paper of Keel and Tao [30] (1998), where in a small paragraph the authors prove the full range of homogeneous Strichartz estimates, save for the endpoint ones which proved too difficult to be resolved and remained open. Following up the discussion in [30], Guo and Peng [25] (2007) resolved in the negative the endpoint homogeneous Strichartz estimate

$$\|U(t)f\|_{L^2_t L^\infty_x L^1_v} \lesssim \|f\|_{L^2_{x,v}} \tag{2.3}$$

in one spatial dimension (1d), where $U(t)f = f(x - tv, v)$ is the KT propagator, by means of two separate counterexamples. Interestingly enough, they also showed that this estimate holds if one replaces the spatial L^∞-norm by a BMO-norm, a situation that is unique for the KT equation.

Then in 2008 appeared the first application of Strichartz estimates to a kinetic model, see Bournaveas et al. [8], where it is proved that the Othmer-Dunbar-Alt (ODA) kinetic model of chemotaxis (3.1)-(3.3), (3.5), has global weak solutions in three spatial dimensions (3d). However, the authors are unable to show uniqueness and continuous dependence on the initial data, and to work in data classes that are preserved by the evolution of the system. Thus their application of Strichartz estimates leaves doubt whetether these estimates can be a very useful tool in the context of nonlinear kinetic systems when it comes to proving well-posedness.

We shall prove that the ODA model is globally well-posed in 3d for small data by use of Strichartz estimates. Our proof benefits from the larger range of inhomogeneous estimates that we prove here which is also the reason for our improvement to the result of [8]. Thus our work presents the first application of Strichartz estimates in the context of a nonlinear kinetic system that gives global well-posedness, and in the light of the above, it also validates the Strichartz estimates as a viable tool in the analysis of a kinetic equation.

One of the more interesting results of this chapter is the fact that the endpoint estimate (2.3) fails on functions f that are characteristic of Besicovitch sets on the plane. Moreover, our approach is flexible enough to resolve in the negative all estimates containing L^∞-norms in the

x-variables of the form

$$\|U(t)f\|_{L^q_t L^\infty_x L^p_v} \lesssim \|f\|_{L^b_x L^c_v}$$

in all dimensions, where $0 < q, p, b, c \leq \infty$, $c \neq \infty$.

Another intersting observation that we make here concerns the inhomogeneous operator

$$W(t)F = \int_0^t U(t-s)F(s)ds$$

that solves the inhomogeneous KT equation (2.1). As it is well-known the proof of the Strichartz estimates for $W(t)$ usually goes through proving the analogous estimates for the TT^*-operator and then using the celebrated Christ-Kiselev lemma that implies the former form the latter. However, we show directly that in the context of the KT equation the Strichartz estimates for both operators are equivalent.

We study other possible relations of equivalence between different Strichartz estimates and one particular example in this direction is the equivalence of these two estimates

$$\|U(t)f\|_{L^q_t L^r_x L^p_v} \lesssim \|f\|_{L^b_x L^c_v}, \qquad \forall f \in L^b_x L^c_v, \tag{2.4}$$

$$\|W(t)F\|_{L^q_t L^r_x L^p_v} \lesssim \|F\|_{L^1_t L^b_x L^c_v} \qquad \forall F \in L^1_t L^b_x L^c_v. \tag{2.5}$$

Note that the inhomogeneous estimate above is endpoint with respect to the temporal exponent that is equal to 1. This connection helps us study the range of validity of the homogeneous estimates of the form (2.4) which are given for data outside the "transport" class. Such estimates were investigated by T. Kato [29] in the context of the Schrödinger equation where the homogeneous estimates are of the form

$$\|U_s(t)f\|_{L^q_t L^r_x} \lesssim \|f\|_{L^p_x}, \qquad \forall f \in L^p_x \tag{2.6}$$

($U_s(t)$ here is the Schrödinger propagator) and are given for data outside the "energy" class (L^2), i.e. for $f \in L^p$ and $1 < p \leq 2$. The equivalence between the estimates in (2.4) and (2.5) helps us improve the method of Kato, and we prove a larger range of estimates that we would have obtained by just adapting his method to the present context. In this connection we should mention the fact that the inhomogeneous Strichartz estimates that we prove for the operator $W(t)F$ are based on adaptation of the methods of Foschi [20] and Keel and Tao [30] and currently present the most powerful method for proving inhomogeneous Strichartz estimates.

We also consider the endpoint homogeneous and inhomogeneous Strichartz estimates in higher dimensions. We give a counterexample showing that there does not exist a family of perturbed (local) estimates in a "full neighborhood" around each of these estimates which explains why the methods of Keel and Tao [30] and Foschi [20] cannot be applied to this context. However, we prove weaker substitutes of these estimates that can be useful to applications under the assumption of a finite velocity space. The loss of integrability of these estimates compared to the original ones can be chosen to be arbitrary small. The proof of the endpoint Strichartz estimates for the KT equation remains one of the most difficult open questions with regard to Strichartz estimates.

The estimates that we prove do not follow from the estimates proved before us in the abstract setting for two reasons. One is that the decay estimate

$$\|U(t)f\|_{L^\infty_x L^1_v} \lesssim \frac{1}{|t|^n} \|f\|_{L^1_x L^\infty_v} \tag{2.7}$$

for the KT equation has vector-valued Lebesgue norms and in order for the Strichartz estimates that follow to be also given in Lebesgue norms one needs to use the complex method of interpolation and not the real one which is used in Taggart [46]. Another, even more fundamental reason, is the fact that the KT equation enjoys a whole family of spaces that it preserves in its evolution and not just the Lebesgue space L^2. We shall call the following estimate

$$\|U(t)f\|_{L^a_{x,v}} = \|f\|_{L^a_{x,v}}, \quad \forall t \in \mathbb{R},\ 0 < a \leq \infty, \tag{2.8}$$

the transport estimate, and the class $L^a_{x,v}$ - a transport class. Initially, we shall prove the homogeneous estimates for data in $L^2_{x,v}$ and shall extend this result to all other transport classes by the following invariance

$$f \to f^{\alpha}, \quad U(t)f \to (U(t)f)^{\alpha}. \tag{2.9}$$

The exponents in the Strichartz estimates (in (2.12), analogously for (2.13)) transform according to the rule

$$(q, r, p, a) \to (\alpha q, \alpha r, \alpha p, \alpha a), \quad 0 < \alpha < \infty \tag{2.10}$$

and any two estimates whose exponents are related in such a way are equivalent.

2.2 Notation and basic facts

All estimates that we prove in the sequel involve the following two basic operators

$$U(t)f = f(x - tv, v), \quad W(t)F = \int_0^t U(t-s)F(s)ds, \tag{2.11}$$

that decompose the solution u to the Cauchy problem for the linear KT equation (2.1), (2.2) into a homogeneous and inhomogeneous part

$$u(t) = U(t)f + W(t)F.$$

The homogeneous Strichartz estimates have the form

$$\|U(t)f\|_{L^q_t L^r_x L^p_v} \lesssim \|f\|_{L^a_{x,v}}, \tag{2.12}$$

where by $L^q_t L^r_x L^p_v$ we mean $L^q([0,\infty); L^r(\mathbb{R}^n; L^p(\mathbb{R}^n)))$. Whenever we consider Lebesgue spaces over other domains, the domain shall be shown explicitly. For example, by $L^q_t L^r_x L^p_v(V)$ we mean $L^q([0,\infty); L^r(\mathbb{R}^n; L^p(V)))$.

We shall also study generalized homogeneous estimates for data outside the transport class

$$\|U(t)f\|_{L^q_t L^r_x L^p_v} \lesssim \|f\|_{L^b_x L^c_v}. \tag{2.13}$$

The inhomogeneous estimates have the form

$$\|W(t)F\|_{L^q_t L^r_x L^p_v} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x L^{\tilde{p}'}_v}. \tag{2.14}$$

As we already mentioned in the Introduction, one of the new result of the present chapter is the fact that all homogeneous estimates (2.13) (and therefore (2.12)) are contained in the inhomogeneous Strichartz estimates (2.14) of the special form with either $q = \infty$ or $\tilde{q} = \infty$ in a sense of both estimates being equivalent.

Let us now describe the range of validity of the homogeneous estimates. Following Keel and Tao [30], we shall call the Lebesgue exponents for which estimate (2.12) holds for every $f \in L^a_{x,v}$ *admissible.*

Definition 2.2.1. We say that the exponent triplet (q, r, p) is *KT-admissible* if

$$\frac{1}{q} = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{r}\right), \quad a \stackrel{def}{=} \mathrm{HM}(p, r), \tag{2.15}$$

$$1 \le p, q, r \le \infty, \quad p^*(a) \le p \le a, \quad a \le r \le r^*(a), \tag{2.16}$$

except in the case $n = 1$, $(q, r, p) = (a, \infty, a/2)$.

In the above definition we use the abbreviation $\mathrm{HM}(p, r)$ to denote the harmonic mean of p and r, i.e. $a = \mathrm{HM}(p, r)$ whenever

$$\frac{1}{a} = \frac{1}{2}\left(\frac{1}{r} + \frac{1}{p}\right).$$

For convenience we have also computed the exact lower boundary p^* to p and the exact upper boundary r^* to r which are given in

Definition 2.2.2. Set

$$\begin{cases} p^*(a) = \frac{na}{n+1}, & r^*(a) = \frac{na}{n-1}, \quad \text{if} \quad \frac{n+1}{n} \le a \le \infty, \\ p^*(a) = 1, & r^*(a) = \frac{a}{2-a}, \quad \text{if} \quad 1 \le a \le \frac{n+1}{n}. \end{cases} \tag{2.17}$$

We have used the convention that $1/0 = \infty$, i.e. for $n = 1$, $r^*(a) = \infty$. Furthermore, throughout this text we shall always use the convention $1/\infty = 0$ and $1/0 = \infty$ in the context of Lebesgue exponents.

Note that the second line in (2.17) is a cutoff that keeps the Lebesgue exponents in the range $[1, \infty]$. In view of the power invariance (2.9) this is not needed for the proof of the homogeneous estimates but in the proof of the inhomogeneous estimates one needs to exploit the fact that the Lebesgue spaces with exponents in the cited range are Banach spaces and have duals in the same range. Without this cutoff, the bounds will be $p^*(a) = \frac{na}{n+1}$ and $r^*(a) = \frac{na}{n-1}$ for any $a > 0$.

A consequence of the above definition is the fact that if the triplet (q, r, p) is KT-admissible, then $a \le q \le \infty$ and $p \le r$. Triplets of the form $(q, r, p) = (a, r^*(a), p^*(a))$, for $(n+1)/n \le a < \infty$, shall be called endpoint. When $a = 1$ the only admissible triplet is $(\infty, 1, 1)$, and similarly, when $a = \infty$ the only admissible triplet is (∞, ∞, ∞).

To describe the range of the inhomogeneous estimates we shall need the next two definitions. Following Foschi [20], we shall call the exponent triplet (q, r, p) *KT-acceptable* if it satisfies a certain condition that is necessary for the validity of the inhomogeneous estimates of the form (2.14) for any right hand side $F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}$.

Definition 2.2.3. We say that the exponent triplet (q, r, p) is *KT-acceptable* if

$$\frac{1}{q} < n\left(\frac{1}{p} - \frac{1}{r}\right), \quad 1 \le q \le \infty, \quad 1 \le p < r \le \infty, \tag{2.18}$$

or if $q = \infty$, $1 \le p = r \le \infty$.

Note that a KT-acceptable triplet is always KT-admissible. We shall later see that this condition is necessary both for the validity of the generalized homogeneous estimates and the inhomogeneous estimates. To further describe the range of validity of the inhomogeneous estimates we give the following

Definition 2.2.4. We say that the two KT-acceptable exponent triplets (q, r, p) and $(\tilde{q}, \tilde{r}, \tilde{p})$ are *jointly KT-acceptable* if

$$\frac{1}{q} + \frac{1}{\tilde{q}} = n\left(1 - \frac{1}{r} - \frac{1}{\tilde{r}}\right), \quad \frac{1}{q} + \frac{1}{\tilde{q}} \le 1, \tag{2.19}$$

$$\mathrm{HM}(p, r) = \mathrm{HM}(\tilde{p}', \tilde{r}'), \tag{2.20}$$

and if the exponents satisfy further the additional restrictions

(i)
$$\frac{n-1}{p'} < \frac{n}{\tilde{r}}, \quad \frac{n-1}{\tilde{p}'} < \frac{n}{r}, \tag{2.21}$$
for $r, \tilde{r} \ne \infty$.

(ii) if $r = \infty$ then the point $(1/q, 1/r, 1/p, 1/\tilde{q}, 1/\tilde{r}, 1/\tilde{p}) \in \Sigma_1 \cup B$,

$$\begin{aligned} &\Sigma_1 = \{(\mu, 0, \kappa, \nu, 1-\kappa, 1) \;:\; 0 < \mu, \nu < 1,\; 0 < \mu + \nu < 1,\; \kappa = (\mu+\nu)/n\}, \\ &B = (0, 0, 0, 0, 1, 1). \end{aligned} \tag{2.22}$$

(iii) if $\tilde{r} = \infty$ then the point $(1/q, 1/r, 1/p, 1/\tilde{q}, 1/\tilde{r}, 1/\tilde{p}) \in \Sigma_2 \cup C$,

$$\begin{aligned} &\Sigma_2 = \{(\mu, 1-\kappa, 1, \nu, 0, \kappa) \,:\, 0 < \mu, \nu < 1,\ 0 < \mu + \nu < 1,\ \kappa = (\mu+\nu)/n\}\,, \\ &C = (0, 1, 1, 0, 0, 0). \end{aligned} \tag{2.23}$$

We remark that only the restrictions (2.19) and (2.20) in this definition are shown to be necessary for the validity of the inhomogeneous estimates. We are being slightly inconsistent but for the sake of simplicity we have incorporated the remaining conditions in this definition which we need in our proof but for which we are not able to show their necessity. Note that the restriction (2.21) is analogous to a similar condition in the context of the Schrödinger equation whose necessity is also currently not known, see Foschi [20]. This remains an obstacle to understanding the full range of the inhomogeneous Strichartz estimates in higher dimensions and this is where the current boundary of the subject lies. The remaining conditions treat the case when the exponent r, or $\tilde{r}$, to the spatial norm is equal to ∞. Again we do not know whether these conditions are necessary due to the lack of suitable counterexamples for the inhomogeneous estimates.

Note also that in one spatial dimension ($n = 1$) condition (2.21) is void. Thus, the full range of validity of the inhomogeneous Strichartz estimates is now known for $n = 1$.

2.3 Main results on Strichartz estimates

We begin with the Strichartz estimates for admissible exponents.

Theorem 2.3.1. *Let $u(t)$ be the solution to the Cauchy problem for* (2.1), (2.2). *Then the estimate*

$$\|u(t)\|_{L^q_t L^r_x L^p_v} \lesssim \|f\|_{L^a_{x,v}} + \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x L^{\tilde{p}'}_v}\,, \tag{2.24}$$

holds for all $f \in L^a(\mathbb{R}^{2n})$and all $F \in L^{\tilde{q}'}_t L^{\tilde{r}'}_x L^{\tilde{p}'}_v$ if and only if (q, r, p) and $(\tilde{q}, \tilde{r}, \tilde{p})$ are two KT-admissible exponent triplets and $a = \mathrm{HM}(p, r) = \mathrm{HM}(\tilde{p}', \tilde{r}')$, apart from the case when $n > 1$ and (q, r, p) is being an endpoint triplet for which the corresponding estimates in higher dimensions remain unresolved.

Note that Theorem 2.3.1 allows the second triplet $(\tilde{q}, \tilde{r}, \tilde{p})$ to be endpoint and excludes only the estimates where the first triplet (q, r, p) is endpoint.

Despite the fact that Theorem 2.3.1 is essentially optimal it does not give the full range of validity of the estimates for the operator $W(t)$. It turns out that the estimates for $W(t)$ have a bigger range of validity than these for the solution u to the inhomogeneous equation with nonzero initial data considered in Theorem 2.3.1.

Theorem 2.3.2. *Suppose that (q, r, p) and $(\tilde{q}, \tilde{r}, \tilde{p})$ are two jointly KT-acceptable exponent triplets that further satisfy the following conditions*

(i) $1 < q, \tilde{q} < \infty$, $q > \tilde{q}'$, *then the estimate*

$$\|W(t)F\|_{L^q_t L^r_x L^p_v} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x L^{\tilde{p}'}_v} \tag{2.25}$$

holds for all $F \in L^{\tilde{q}'}_t L^{\tilde{r}'}_x L^{\tilde{p}'}_v$.

(ii) $\tilde{q} = \infty$, $1 < q < \infty$, *then the estimate*

$$\|W(t)F\|_{L^{q,\infty}_t L^r_x L^p_v} \lesssim \|F\|_{L^1_t L^{\tilde{r}'}_x L^{\tilde{p}'}_v} \tag{2.26}$$

holds for all $F \in L^1_t L^{\tilde{r}'}_x L^{\tilde{p}'}_v$.

(iii) $q = \infty$, $1 < \tilde{q} < \infty$, *then the estimate*

$$\|W(t)F\|_{L^\infty_t L^r_x L^p_v} \lesssim \|F\|_{L^{\tilde{q}',1}_t L^{\tilde{r}'}_x L^{\tilde{p}'}_v}\,. \tag{2.27}$$

holds for all $F \in L^{\tilde{q}',1}_t L^{\tilde{r}'}_x L^{\tilde{p}'}_v$.

(iv) $1 < q, \tilde{q} < \infty$, $q = \tilde{q}'$, *these endpoint inhomogeneous estimates remain unresolved. Instead, we present weaker substitutes of* (2.25) *under the assumption of a finite velocity space* $V \subset \mathbb{R}^n$. *The estimate*

$$\|W(t)F\|_{L_t^q L_x^r L_v^P(V)} \lesssim_V \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{P}'}(V)}, \tag{2.28}$$

holds for all $F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}(V)$, *whenever* $P, \tilde{P}$ *are such that* $1 \le P < p$ *and* $1 \le \tilde{P} < \tilde{p}$ *and* $q \le r$, $\tilde{q} \le \tilde{r}$.

Conversely, if estimate (2.25) holds for all $F \in L_t^{q'} L_x^{r'} L_v^{p'}$, *then* (q, r, p) *and* $(\tilde{q}, \tilde{r}, \tilde{p})$ *must be two jointly KT-acceptable exponent triplets, apart from conditions* (2.21), (2.22), (2.23), *whose necessity is not fully verified.*

Remark 2.3.3. Estimate (2.26) can be strengthen by replacing the Lorentz norm $L^{q,\infty}$ by the Lebesgue norm L^q in the range $q \ge \tilde{p}'$. Analogously, the Lorentz norm $L^{\tilde{q}',1}$ in estimate (2.27) can be replaced by the Lebesgue norm $L^{\tilde{q}'}$ in the range $\tilde{q}' \le p$. In both cases see Lemma 2.8.3.

Remark 2.3.4. If we restrict ourselves to finite time intervals $[0, T]$, we have the continuous embeddings

$$\begin{aligned} L^{q,r}([0,T]) &\hookrightarrow L^p([0,T]), \quad q > p,\ 1 \le q, p, r \le \infty, \\ L^p([0,T]) &\hookrightarrow L^{q,r}([0,T]), \quad p > q,\ 1 \le q, p, r \le \infty, \end{aligned}$$

see [2, p. 217]. For example, let (∞, r, p) and $(\tilde{q}, \tilde{r}, \tilde{p})$ be such that estimate (2.27) holds and let $1 \le \widetilde{Q} < \tilde{q}$. Then we have the local inhomogeneous estimate

$$\|W(t)F\|_{L_t^\infty([0,T]; L_x^q L_v^r)} \lesssim_T \|F\|_{L_t^{\tilde{Q}'}([0,T]; L_x^{\tilde{r}'} L_v^{\tilde{p}'})}$$

for any $0 < T < \infty$ and any $F \in L_t^{\tilde{Q}'}([0,T]; L_x^{\tilde{r}'} L_v^{\tilde{p}'})$.

The different cases which Theorem 2.3.2 considers can be visualized quite easily. Let us first remember that the Lebesgue space L^p is best seen as a "function" of $1/p$ rather than p in the context of interpolation. Therefore, the range of validity of the estimate (2.25) shall be described as a region in $\mathbb{R}^6$ containing the points with coordinates $(1/q, 1/r, 1/p, 1/\tilde{q}, 1/\tilde{r}, 1/\tilde{p})$ if the corresponding exponents appear in a valid estimate (2.25). The projection of that region over the $(1/q, 1/\tilde{q})$-plane is visualized on fig. 2.1.

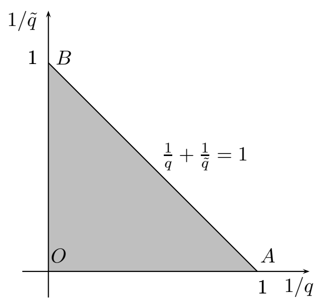

Figure 2.1: Acceptable range of $(1/q, 1/\tilde{q})$.

The inner part of ΔOAB corresponds to the non-endpoint inhomogeneous estimates, while its three sides correspond to the endpoint inhomogeneous estimates. In the context of Theorem 2.3.2, the inner part of ΔOAB corresponds to part (i), the cathetus OA - to part (ii), the cathetus OB - to part (iii), and the hypotenuse AB - to part (iv). The inhomogeneous estimates can be put into three groups in ascending order of difficulty: the inner part of ΔOAB, the two catheti OA and OB, and the hypotenuse AB.

Theorem 2.3.5 (The Equivalence theorem)**.**

A. *The following three estimates*

$$\|U(t)f\|_{L^q_t L^r_x L^p_v} \lesssim \|f\|_{L^b_x L^c_v}, \qquad \forall f \in L^b_x L^c_v, \tag{2.29}$$

$$\|W(t)F\|_{L^q_t L^r_x L^p_v} \lesssim \|F\|_{L^1_t L^b_x L^c_v}, \quad \forall F \in L^1_t L^b_x L^c_v, \tag{2.30}$$

$$\|W(t)F\|_{L^\infty_t L^{b'}_x L^{c'}_v} \lesssim \|F\|_{L^{q'}_t L^{r'}_x L^{p'}_v}, \quad \forall F \in L^{q'}_t L^{r'}_x L^{p'}_v. \tag{2.31}$$

are equivalent whenever $1 \le q, r, p \le \infty$, *and* $1 \le b, c < \infty$.

B. *Whenever* $b = c = 2$ *estimate* (2.29) *is equivalent to*

$$\|W(t)F\|_{L^q_t L^r_x L^{r'}_v} \lesssim \|F\|_{L^{q'}_t L^{r'}_x L^r_v}, \quad \forall F \in L^{q'}_t L^{r'}_x L^r_v. \tag{2.32}$$

Remark 2.3.6. In fact the only place where we use the assumption that b and c must be finite is the proof that the inhomogeneous estimate (2.30), or its equivalent (2.31), implies the homogeneous estimate (2.29). So all other implications in the theorem remain true even if $b = \infty$ or $c = \infty$.

In a direct consequence of Theorem 2.3.2 and the Equivalence theorem we obtain

Theorem 2.3.7 (Generalized homogeneous estimates)**.** *We have the estimate*

$$\|U(t)f\|_{L^{q,\infty}_t L^r_x L^p_v} \lesssim \|f\|_{L^b_x L^c_v} \tag{2.33}$$

for all $f \in L^b_x L^c_v$, *whenever the exponent vector* (q, r, p, b, c) *satisfies the following conditions*

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{b}, \quad \mathrm{HM}(p, r) = \mathrm{HM}(b, c) \stackrel{def}{=} a, \tag{2.34}$$

$$p < b \le a \le c < r, \tag{2.35}$$

$$a \le r < r^*(c), \tag{2.36}$$

in the range $1 < q < \infty$, $1 \le p, \tilde{p}, r, \tilde{r} < \infty$. *Estimate* (2.33) *also holds when* $b = c = p = r$ *and* $q = \infty$ *(the transport estimate), and whenever* $b = p$, $c = r$, *and* $1/q = n/p - n/r$ *(a consequence of the decay estimate* (2.41)*). Furthermore, if additionally* $q \ge c$, *then the sharper estimate*

$$\|U(t)f\|_{L^q_t L^r_x L^p_v} \lesssim \|f\|_{L^b_x L^c_v} \tag{2.37}$$

holds for all $f \in L^b_x L^c_v$ *in the range* $0 < q, r, p, b, c < \infty$. *Conversely, if estimate (2.37) holds for all* $f \in L^b_x L^c_v$ *then* (q, r, p, b, c) *must satisfy conditions* (2.34) *and* (2.35). *However, we do not have a counterexample showing the necessity of the upper bound* $r^*(c)$ *in dimensions* $n > 1$ *in the case when* $b \ne c$.

The Equivalence theorem, part B, together with Theorem 2.3.2, part (iv), imply the following weaker substitute for the endpoint homogeneous Strichartz estimate over finite velocity spaces.

Corollary 2.3.8. *Let* $1 \le P < p^*(a)$, $(n+1)/n \le a < \infty$, *and let* $V \subset \mathbb{R}^n$ *be bounded. Then, the following estimate*

$$\|U(t)f\|_{L^a_t L^{r^*(a)}_x L^P_v(V)} \lesssim_V \|f\|_{L^a_{x,v}}, \tag{2.38}$$

holds for all $f \in L^a_{x,v}$.

2.4 Basic properties of the kinetic transport equation

Lemma 2.4.1 (The decay estimate [40])**.** *The kinetic transport evolution group* $U(t)$ *obeys the estimate*

$$\|U(t)f\|_{L^\infty_x L^1_v} \le \frac{1}{|t|^n} \|f\|_{L^1_x L^\infty_v}, \tag{2.39}$$

for all $f \in L^1_x L^\infty_v$.

Proof.

$$\int_{\mathbb{R}^n} |U(t)f|\,dv = \int_{\mathbb{R}^n} |f(x-tv,v)|\,dv \le \int_{\mathbb{R}^n} \sup_{y\in\mathbb{R}^n} |f(x-tv,y)|\,dv$$
$$\le \frac{1}{|t|^n}\int_{\mathbb{R}^n} \sup_{y\in\mathbb{R}^n} |f(z,y)|\,dz = \frac{1}{|t|^n}\|f\|_{L^1_x L^\infty_v}\,.$$

□

Lemma 2.4.2 (The transport estimate)**.** *The kinetic transport evolution group $U(t)$ obeys the estimate*

$$\|U(t)f\|_{L^\infty_t L^a_x L^a_v} \le \|f\|_{L^a_{x,v}}\,, \quad 0 < a \le \infty, \tag{2.40}$$

for all $f \in L^a_{x,v}$.

Proof. Trivial. □

Corollary 2.4.3 (The decay estimate)**.** *The kinetic transport evolution group $U(t)$ obeys the estimate*

$$\|U(t)f\|_{L^r_x L^p_v} \le \frac{1}{|t|^{n\left(\frac{1}{p}-\frac{1}{r}\right)}}\|f\|_{L^p_x L^r_v}\,, \quad 1 \le p \le r \le \infty, \tag{2.41}$$

for all $f \in L^p_x L^r_v$.

Proof. Complex interpolation between the decay estimate (2.7) and the two transport estimates (2.8) with $a = 1$ and $a = \infty$. □

Lemma 2.4.4. *The formal adjoint to $U(t)$ is the operator $U^*(t) = U(-t)$.*

Proof. We denote by $\langle\cdot,\cdot\rangle$ the scalar product on $L^2(\mathbb{R}^{2n})$. Thus,

$$\langle U(t)f, g\rangle = \int_{-\infty}^{\infty} f(x-tv,v)\overline{g(x,v)}dxdv$$
$$= \int_{-\infty}^{\infty} f(y,v)\overline{g(y+tv,v)}dydv = \langle f, U(-t)g\rangle,$$

where we have made the substitution $y = x - tv$. □

Lemma 2.4.5 (Scaling properties of $U(t)$ and $W(t)$)**.** *The evolution operators $U(t)$ and $W(t)$ enjoy the following scaling properties*

$$U(t)f_\lambda = f\left(x/\lambda - tv/\lambda, v\right) = \{U(\cdot)f\}(t/\lambda, x/\lambda, v),$$
$$\textit{where } f_\lambda(x,v) = f\left(x/\lambda, v\right),$$
$$U(t)f_\lambda = f\left(x/\lambda - tv/\lambda, v/\lambda\right) = \{U(\cdot)f\}(t, x/\lambda, v/\lambda),$$
$$\textit{where } f_\lambda(x,v) = f\left(x/\lambda, v/\lambda\right),$$
$$W(t)F_\lambda = \lambda\int_0^{t/\lambda} F\left(s, x/\lambda - \left(t/\lambda - s\right)v, v\right)ds = \lambda\{W(\cdot)F\}\left(t/\lambda, x/\lambda, v\right),$$
$$\textit{where } F_\lambda(t,x,v) = F(t/\lambda, x/\lambda, v),$$
$$W(t)F_\lambda = \int_0^{t} F\left(s, x/\lambda - (t-s)v/\lambda, v/\lambda\right)ds = \{W(\cdot)F\}(t, x/\lambda, v/\lambda),$$
$$\textit{where } F_\lambda(t,x,v) = F\left(t, x/\lambda, v/\lambda\right).$$

Proof. Direct inspection. □

Lemma 2.4.6. *Suppose that $f \in L^r_x(\mathbb{R}^n, L^p_v(V))$, where $1 \le r, p < \infty$, and $V \subseteq \mathbb{R}^n$. We have the following two cases* *(i) If $r, p < \infty$, then*

$$U(t)f \in C(\mathbb{R}; L^r_x L^p_v).$$

Specifically, if $r = p = a < \infty$, then $U(t)$ is an isometry on $L^a_{x,v}$.

(ii) If $r = \infty$ or $p = \infty$, then

$$U(t)f \notin C(\mathbb{R}; L^r_x L^p_v).$$

If, however, f is uniformly continuous in x and V is bounded, then

$$U(t)f \in C(\mathbb{R}; L^r_x L^p_v), \quad 1 \leq r, p \leq \infty$$

and $U(t)f$ remains uniformly continuous in x for all $t \in \mathbb{R}$.

Proof. The first part follows from the following standard argument. Suppose that $\chi_Q(x, v)$ is the characteristic function of a square Q in $\mathbb{R}^{2n}$. The claim holds for χ_Q. Then it holds for the class of simple functions on $\mathbb{R}^{2n}$ and by density for all $f \in L^a_{x,v}$. The counterexample needed for the second part can be taken on characteristic functions of squares in $\mathbb{R}^{2n}$. The remaining claims are trivial. □

Corollary 2.4.7. *In particular if $f \in C^1_0(\mathbb{R}^{2n})$, the space of continuously differentiable functions of compact support on $\mathbb{R}^{2n}$, we have that*

$$U(t)f \in C(\mathbb{R}; L^r_x L^p_v), \tag{2.42}$$

in view of the fact that in such case both hypotheses (i) and (ii) of Lemma 2.4.6 are satisfied.

2.5 Duality and the TT^*-principle

2.5.1 Basics.

Let us consider the operator

$$T : L^2_{x,v} \to L^q_t L^r_x L^{r'}_v, \quad \{Tf\}(t, x, v) = f(x - tv, v).$$

Its formal adjoint is the L^2-valued integral

$$T^* : L^{q'}_t L^{r'}_x L^r_v \to L^2_{x,v}, \quad \{T^*F\}(x, v) = \int_{-\infty}^{\infty} F(s, x + sv, v) ds.$$

The composition of the two has the form

$$TT^* : L^{q'}_t L^{r'}_x L^r_v \to L^q_t L^r_x L^{r'}_v, \quad \{TT^*F\}(t, x, v) = \int_{-\infty}^{\infty} F(s, x - (t - s)v, v) ds.$$

In view of the TT^*-principle, see e.g. [41, p. 113], T and TT^* are equally bounded with $\|T\|^2 = \|TT^*\|$. Thus, the following two estimates are equivalent

$$\|Tf\|_{L^q_t L^r_x L^{r'}_v} \leq C \|f\|_{L^2_{x,v}}, \qquad \forall f \in L^2_{x,v}, \tag{2.43}$$

$$\|TT^*F\|_{L^q_t L^r_x L^{r'}_v} \leq C^2 \|F\|_{L^{q'}_t L^{r'}_x L^r_v}, \quad \forall F \in L^{q'}_t L^{r'}_x L^{p'}_v, \tag{2.44}$$

where $C = \|T\|$.

By duality, (2.44) is equivalent to

$$\left| \int_{-\infty}^{\infty} \langle TT^*[F](t), G(t) \rangle dt \right| \leq C^2 \|F\|_{L^{q'}_t L^{r'}_x L^r_v} \|G\|_{L^{q'}_t L^{r'}_x L^r_v},$$

and to

$$\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\langle U(s)^*F, U(t)^*G\rangle dsdt\right| \le C^2 \|F\|_{L_t^{q'}L_x^{r'}L_v^{r}} \|G\|_{L_t^{q'}L_x^{r'}L_v^{r}}.$$

$\forall F,\ \forall G \in L_t^{q'}L_x^{r'}L_v^{r}$, By symmetry, the last inequality is implied by

$$|B(F,G)| \le C^2 \|F\|_{L_t^{q'}L_x^{r'}L_v^{r}} \|G\|_{L_t^{q'}L_x^{r'}L_v^{r}}, \quad \forall F,\ \forall G \in L_t^{q'}L_x^{r'}L_v^{r}, \tag{2.45}$$

where $B(F,G)$ is the bilinear form

$$B(F,G) = \iint_{s<t} \langle U(s)^*F, U(t)^*G\rangle dsdt.$$

Here, $\langle\cdot,\cdot\rangle$ is the bilinear pairing on $\mathbb{R}^{2n}$, i.e.

$$\langle f, g\rangle = \int_{R^{2n}} f(x,v)g(x,v)dxdv.$$

Instead of proving (2.43) directly, we usually prove the equivalent estimate (2.44) for the TT^*-operator. The reason for that lies in the fact that the TT^*-operator is a convolution operator. The endpoint estimates are usually proved via the equivalent bilinear estimate (2.45).

We now turn to the inhomogeneous estimates. Suppose that (q,r,p) and $(\tilde{q},\tilde{r},\tilde{p})$ are two exponent triplets such that

$$\|Tf\|_{L_t^qL_x^rL_v^p} \le C\|f\|_{L_{x,v}^a}, \qquad \forall f \in L_{x,v}^a,$$
$$\|Tf\|_{L_t^{\tilde{q}}L_x^{\tilde{r}}L_v^{\tilde{p}}} \le C\|f\|_{L_{x,v}^{a'}}, \qquad \forall f \in L_{x,v}^{a'},$$

for some $1 \le a \le \infty$. By considering the composition

$$L_t^{\tilde{q}'}L_x^{\tilde{r}'}L_v^{\tilde{p}'} \xrightarrow{T^*} L_{x,v}^a \xrightarrow{T} L_t^qL_x^rL_v^p,$$

we obtain the consequence

$$\|TT^*F\|_{L_t^qL_x^rL_v^p} \le C^2\|F\|_{L_t^{\tilde{q}'}L_x^{\tilde{r}'}L_v^{\tilde{p}'}}, \quad \forall F \in L_t^{\tilde{q}'}L_x^{\tilde{r}'}L_v^{\tilde{p}'}. \tag{2.46}$$

Note now that the exponents in the two sides of (2.46) are not the same. This estimate does not any longer imply boundedness for the operator T. However, it implies boundedness for the inhomogeneous operator $W(t)$ due to the following

Lemma 2.5.1. *The following two estimates are equivalent*

$$\|TT^*F\|_{L_t^qL_x^rL_v^p} \lesssim \|F\|_{L_t^{\tilde{q}'}L_x^{\tilde{r}'}L_v^{\tilde{p}'}}, \quad \forall F \in L_t^{\tilde{q}'}L_x^{\tilde{r}'}L_v^{\tilde{p}'}, \tag{2.47}$$
$$\|W(t)F\|_{L_t^qL_x^rL_v^p} \lesssim \|F\|_{L_t^{\tilde{q}'}L_x^{\tilde{r}'}L_v^{\tilde{p}'}}, \quad \forall F \in L_t^{\tilde{q}'}L_x^{\tilde{r}'}L_v^{\tilde{p}'}. \tag{2.48}$$

Proof. Estimate (2.47) implies (2.48) since we can decompose the TT^*-operator into

$$\int_{\mathbb{R}} = \int_{-\infty}^{t} + \int_{t}^{\infty}$$

and then make a change of variables in the third integral to transform it to an integral like the second one. The details are left to the interested reader. The converse follows by the estimate

$$|W(t)F| \le TT^*|F|. \qquad \square$$

Furthermore, the inhomogeneous estimate

$$\|W(t)F\|_{L_t^q L_x^r L_v^p} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}}, \quad \forall F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'},$$

analogously to the proof of (2.45), is equivalent to the estimate

$$|B(F,G)| \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}} \|G\|_{L_t^{q'} L_x^{r'} L_v^{p'}}, \qquad \forall F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'},\ \forall G \in L_t^{q'} L_x^{r'} L_v^{p'}. \tag{2.49}$$

These equivalence relations between various types of Strichartz estimates shall be exploited in the sequel and for convenience we summarize in the following

Lemma 2.5.2.

(i) The boundedness of the operator $T : L^2_{x,v} \to L_t^q L_x^r L_v^{r'}$ *of the form* $Tf = U(t)f$ *is equivalent to the boundedness of the bilinear mapping* $B : L_t^{q'} L_x^{r'} L_v^r \times L_t^{q'} L_x^{r'} L_v^r \to \mathbb{C}$.

(ii) The boundedness of the operator $W(t) : L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'} \to L_t^q L_x^r L_v^p$ *is equivalent to that of the bilinear mapping* $B : L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'} \times L_t^{q'} L_x^{r'} L_v^{p'} \to \mathbb{C}$.

The bilinear formulation of the TT^*-principle of Lemma 2.5.2 was first introduced by Keel and Tao [30] in a slightly different context.

2.5.2 Equivalent estimates.

Lemma 2.5.3 (The Duality lemma)**.** *The following two estimates for* $W(t)$ *are equivalent*

$$\|W(t)F\|_{L_t^q L_x^r L_v^p} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}}, \quad \forall F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'},$$
$$\|W(t)F\|_{L_t^{\tilde{q}} L_x^{\tilde{r}} L_v^{\tilde{p}}} \lesssim \|F\|_{L_t^{q'} L_x^{r'} L_v^{p'}}, \quad \forall F \in L_t^{q'} L_x^{r'} L_v^{p'},$$

for $1 \leq p, q \leq \infty$.

Proof. By duality, the first estimate is equivalent to

$$|B(F,G)| \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}} \|G\|_{L_t^{q'} L_x^{r'} L_v^{p'}},$$

for all $F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}$, $G \in L_t^{q'} L_x^{r'} L_v^{p'}$. We have that

$$B(F,G) = \iint_{\tau<\sigma} \langle U(-\sigma)^* F, U(-\tau)^* G \rangle d\tau d\sigma,$$

by making the substitution $\sigma = -s$, $\tau = -t$ in the definition of $B(F,G)$. The integral in the line above can be written as

$$(-1)^n \iint_{\tau<\sigma} \langle U(\sigma)^* F', U(\tau)^* G' \rangle d\tau d\sigma,$$

by making the substitution $x \to -x$ and setting $F'(t,x,v) = F(-t,-x,v)$, $G'(t,x,v) = G(-t,-x,v)$. Hence the second estimate follows. The converse follows by the same argument. □

Theorem 2.5.4 (The Equivalence theorem)**.**

A. *The following three estimates*

$$\|U(t)f\|_{L_t^q L_x^r L_v^p} \lesssim \|f\|_{L_x^b L_v^c}, \qquad \forall f \in L_x^b L_v^c, \tag{2.50}$$
$$\|W(t)F\|_{L_t^q L_x^r L_v^p} \lesssim \|F\|_{L_t^1 L_x^b L_v^c}, \qquad \forall F \in L_t^1 L_x^b L_v^c, \tag{2.51}$$
$$\|W(t)F\|_{L_t^\infty L_x^{b'} L_v^{c'}} \lesssim \|F\|_{L_t^{q'} L_x^{r'} L_v^{p'}}, \qquad \forall F \in L_t^{q'} L_x^{r'} L_v^{p'}. \tag{2.52}$$

are equivalent whenever $1 \le q, r, p \le \infty$, *and* $1 \le b, c < \infty$.

B. *Whenever* $b = c = 2$ *estimate* (2.50) *is equivalent to*

$$\|W(t)F\|_{L_t^q L_x^r L_v^{r'}} \lesssim \|F\|_{L_t^{q'} L_x^{r'} L_v^r}, \quad \forall F \in L_t^{q'} L_x^{r'} L_v^r. \tag{2.53}$$

Proof. Part A. The homogeneous estimate (2.50) trivially implies the first inhomogeneous estimate (2.51). By the Duality lemma 2.5.3, the two inhomogeneous estimates (2.51) and (2.52) are equivalent. All it remains is to show that (2.51) implies (2.50).

Formally, the proof follows if we choose an inhomogeneous term $F(t) = \delta(t)f$, where $\delta(t)$ is the delta function on $\mathbb{R}$ and $f \in L_x^b L_v^c$. Indeed, we have

$$W(t)[\delta(\cdot)f] = U(t)f, \quad \|\delta(t)f\|_{L_t^1 L_x^b L_v^c} = \|f\|_{L_x^b L_v^c},$$

which furnishes the argument. To give a rigorous proof instead of $\delta(t)$ we consider a smooth approximation of the identity $\delta_\epsilon(t)$, $\epsilon > 0$. Suppose that $f \in L_x^b L_v^c$, for $1 \le b, c < \infty$, and thus by Lemma 2.4.7 $U(t)f \in C(\mathbb{R}; L_x^b L_v^c)$. In view of Lemma 2.5.6

$$\|W(t)[\delta_\epsilon f]\|_{L_x^b L_v^c} = \|\delta_\epsilon * U(t)f\|_{L_x^b L_v^c} \to \|U(t)f\|_{L_x^b L_v^c}$$

on $\mathbb{R}$ as $\epsilon \to 0$. Finally, by Fatou's Lemma 2.5.7

$$\|U(t)f\|_{L_t^q(\mathbb{R}; L_x^b L_v^c)} \lesssim \liminf_{\epsilon \to 0} \|W(t)\delta_\epsilon f\|_{L_t^q(\mathbb{R}; L_x^b L_v^c)} \lesssim \liminf_{\epsilon \to 0} \|\delta_\epsilon f\|_{L_t^1(\mathbb{R}; L_x^b L_v^c)} \lesssim \|f\|_{L_x^b L_v^c}.$$

The general case of $f \in L_x^b L_v^c$ follows by density since $C_0^1(\mathbb{R}^{2n})$ is dense in $L_x^b L_v^c$ and $U(t)$ is linear.

Part B. This follows directly from Lemma 2.5.2. □

Remark 2.5.5. The Equivalence theorem still holds if instead of the Lebesgue L^q-norm in time we have the more general Lorentz $L^{q,\tilde{q}}$-norm in time. The formulation and proof in this case are straightforward and shall be omitted. Let us note, however, that we shall use once in the sequel this more general formulation of the theorem with the $L^{q,\infty}$-norm in time.

We finish this paragraph by presenting several technical facts we needed in the proof of the Equivalence theorem. To that end, we need to generalize to the abstract setting some basic facts about approximations of the identity. Denote by $L(\mathcal{B}, \mathcal{B})$ the space of all linear continuous operators on the Banach space $\mathcal{B}$ and let $K(t) : \mathbb{R} \to L(\mathcal{B}, \mathcal{B})$ be an operator-valued function that maps $\mathbb{R}$ into $L(\mathcal{B}, \mathcal{B})$. For each $t \in \mathbb{R}$, let $m(t) : \mathbb{R} \to [0, \infty]$ denote the operator norm of $K(t)$. Suppose that $m(t) \in L^\infty \cap L^1$ and

$$\int_{-\infty}^{\infty} m(t)dt = 1.$$

We set

$$K_\epsilon(t) = \frac{1}{\epsilon} K\left(\frac{t}{\epsilon}\right), \quad F_\epsilon = F * K_\epsilon = \int_{-\infty}^{\infty} K_\epsilon(t-s)F(s)ds,$$

where $F : \mathbb{R} \to \mathcal{B}$ is a vector-valued function. In the next lemma we specify conditions on F and K under which $F_\epsilon \to F$, in the sense that $\|F(t) - F_\epsilon(t)\|_{\mathcal{B}} \to 0$ as $\epsilon \to 0$. We shall call any such family of kernels K_ϵ, for $\epsilon > 0$, an approximation of the identity.

Lemma 2.5.6. *Suppose that* $F : \mathbb{R} \to \mathcal{B}$ *belongs to* $L^1_{loc}(\mathbb{R}; \mathcal{B})$*, the space of all locally integrable* $\mathcal{B}$*-valued functions on* $\mathbb{R}$*, and* $m(t) = O(|t|^{-1})$ *as* $|t| \to +\infty$*. Then* $F_\epsilon \to F$ *at each point of continuity of* F*.*

In the classical setting of $\mathcal{B} = \mathbb{R}$ (or $\mathbb{C}$) the proof of this theorem can be found in a standard course of Real Analysis like e.g. [51, p. 152]. The generalization to the vector-valued setting is straightforward. The lemma shall be used under the same assumptions on the kernel to show that $\|F_\epsilon(t)\|_{\mathcal{B}} \to \|F(t)\|_{\mathcal{B}}$ on $\mathbb{R}$, as $\epsilon \to 0$, whenever $F \in C(\mathbb{R}; \mathcal{B})$

In the same spirit we generalize

Lemma 2.5.7 (Fatou's lemma)**.** *Suppose that $F_k \to F$ a.e. on $\mathbb{R}$, then*

$$\|F\|_{L_t^{p,q}(\mathbb{R};\mathcal{B})} \le \liminf_{k\to\infty} \|F_k\|_{L_t^{p,q}(\mathbb{R};\mathcal{B})}\,,$$

where $p=q=1$, $p=q=\infty$, or $1<p<\infty$ and $1\le q\le\infty$.

Remark 2.5.8. The classical Fatou's lemma is originally stated in the case of $L^{p,q}(\mathbb{R};\mathcal{B}) = L^1(\mathbb{R})$, i.e. for $p=q=1$ and $\mathcal{B}=\mathbb{R}$.

Proof. The limit $F_k \to F$ a.e. on $\mathbb{R}$ means that $\|F(t)-F_k(t)\|_{\mathcal{B}} \to 0$ for almost all $t\in\mathbb{R}$. This implies the limit $\|F_k\|_{\mathcal{B}} \to \|F\|_{\mathcal{B}}$ a.e. on $\mathbb{R}$. By considering $f_k(t):\mathbb{R}\to[0,\infty)$, $f_k(t)=\|F_k(t)\|_{\mathcal{B}}$, and $f(t):\mathbb{R}\to[0,\infty)$, $f(t)=\|F(t)\|_{\mathcal{B}}$ it will be enough to show the claim only in the scalar case. However, the latter is a direct consequence of the Monotone convergence theorem for the Lorentz space $L^{p,q}(\mathbb{R})$ stated below. Indeed, let $g_k=\inf\{f_k,f_{k+1},\dots\}$. Then $g_k \nearrow f$ and $0\le g_k\le f_k$. Thus,

$$\|f\|_{L^{p,q}} = \lim \|g_k\|_{L^{p,q}} \le \liminf \|f_k\|_{L^{p,q}}$$

and the claim follows. □

Theorem 2.5.9 (Monotone convergence theorem for Lorentz spaces)**.** *Let (X,Σ,μ) be a measure space and let $\{f_k\}$ be a sequence of measurable functions on X. If $0\le f_k\nearrow f$ μ-a.e. on X, then*

$$\|f_k\|_{L^{p,q}(X;d\mu)} \to \|f\|_{L^{p,q}(X;d\mu)}\,,$$

where $p=q=1$, $p=q=\infty$, or $1<p<\infty$ and $1\le q\le\infty$.

Proof. In the case when $p=q=1$ the proof can be found in [51, p. 172]. The case when $p=q=\infty$ is trivial. The rest follows from the special representation of the Lorentz norm

$$\|f\|^q_{L^{p,q}(X;d\mu)} = \int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t}, \quad q<\infty,$$
$$\|f\|_{L^{p,q}(X;d\mu)} = \sup_{0<t<\infty} \{t^{1/p} f^*(t)\}, \quad q=\infty,$$

see [2, p. 216]. Indeed, the claim follows from the property

$$|f_k| \nearrow |f| \ \mu\text{-a.e.} \quad \Rightarrow \quad f_k^* \nearrow f^*,$$

see [2, p. 41], where by f^* we have denoted the decreasing rearrangement of f, for a definition see [2, p. 39]. □

2.5.3 Local in time decompositions and scaling

The full power of the bilinear formulation of the TT^* method only comes to light when one decomposes the bilinear operator $B(F,G)$ into local in time dyadic pieces that are scaling invariant with regard to Strichartz estimates. In order to present that idea in more detail, we need several definitions in advance.

Denote by Ω the region $\{(t,s)|s<t\}$ on the (t,s)-coordinate plane.

Definition 2.5.10. We call any positive integer that is a power of two a dyadic number. Furthermore, we call a square Q in $\mathbb{R}^2$ dyadic if its side length is a dyadic number and the coordinates of its vertices are integer multiples of dyadic numbers.

We apply Whitney's dyadic decomposition on Ω and obtain the family $\mathcal{O}$ of essentially disjoint dyadic squares Q (overlapping on the sides is not excluded) such that the distance between any square $Q\in\mathcal{O}$ and the boundary of Ω ($\{(t,s)|t=s\}$) is approximately proportional to the diameter of Q, see figure 2.2. By $\mathcal{O}_\lambda$ we denote the collection of all squares in $\mathcal{O}$ whose side length is λ.

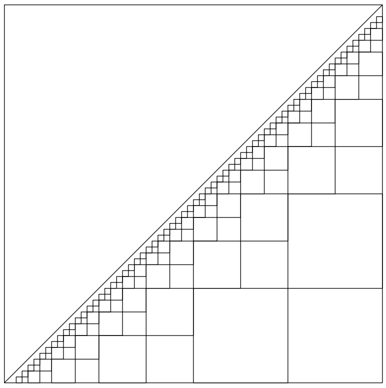

Figure 2.2: Whitney's decomposition for the region $s < t$

Thus we obtain the representations

$$\Omega = \bigcup_{\lambda} \bigcup_{Q\in\mathcal{O}_\lambda}, \quad B(F,G) = \sum_{\lambda} \sum_{Q\in\mathcal{O}_\lambda} B_Q(F,G),$$

where

$$B_Q(F,G) = \iint_Q \langle U^*(s)F(s), U^*(t)G(t)\rangle dsdt. \tag{2.54}$$

The advantage of the above decomposition lies in the fact that whenever $Q = J \times I$ and $Q \in \mathcal{O}_\lambda$, we have

$$\lambda = |I| = |J| \sim \operatorname{dist}(\Omega, \partial\Omega) \sim \operatorname{dist}(I, J). \tag{2.55}$$

This special property (2.55) of Whitney's decomposition plays a role in the proof of the following scaling invariance

$$|B_Q(F,G)| \lesssim \lambda^{\beta(q,r,\tilde{q},\tilde{r})} \|F\|_{L_t^{\tilde{q}'}(J;L_x^{\tilde{r}'}L_v^{\tilde{p}'})} \|G\|_{L_t^{q'}(I;L_x^{r'}L_v^{p'})} \tag{2.56}$$

of the localized bilinear operator B_Q. Here we have used the notation

$$\beta(q,r,\tilde{q},\tilde{r}) = \frac{1}{q} + \frac{1}{\tilde{q}} - n\left(1 - \frac{1}{r} - \frac{1}{\tilde{r}}\right). \tag{2.57}$$

The range of the Lebesgue exponents $(1/q, 1/r, 1/p, 1/\tilde{q}, 1/\tilde{r}, 1/\tilde{p})$ for which we can prove that (2.56) holds for every $Q \in \mathcal{O}_\lambda$ shall be presented in Lemma 2.7.4. Let us assume for the present that we have chosen Lebesgue exponents for which this lemma holds. Now, it is even more important to know whether the sum of all dyadic pieces B_Q with $Q \in \mathcal{O}_\lambda$ obeys the same scaling invariance, which is addressed in

Lemma 2.5.11. *If* $\frac{1}{q} + \frac{1}{\tilde{q}} \leq 1$, *then*

$$\sum_{Q\in\mathcal{O}_\lambda} |B_Q(F,G)| \lesssim \lambda^{\beta(q,r,\tilde{q},\tilde{r})} \|F\|_{L_t^{\tilde{q}'}(\mathbb{R};L_x^{\tilde{r}'}L_v^{\tilde{p}'})} \|G\|_{L_t^{q'}(\mathbb{R};L_x^{r'}L_v^{p'})}. \tag{2.58}$$

Proof. In view of (2.56)

$$\sum_{Q\in\mathcal{O}_\lambda} |B_Q(F,G)| \lesssim \lambda^{\beta(q,r,\tilde{q},\tilde{r})} \sum_{Q\in\mathcal{O}_\lambda,\, Q=J\times I} \|F\|_{L_t^{\tilde{q}'}(J;L_x^{\tilde{r}'}L_v^{\tilde{p}'})} \|G\|_{L_t^{q'}(I;L_x^{r'}L_v^{p'})}.$$

An application of Lemma 2.5.12 below concludes the proof. □

Lemma 2.5.12. *Suppose* $\frac{1}{p} + \frac{1}{\tilde{p}} \geq 1$. *Then*

$$\sum_{Q\in\mathcal{O}_\lambda,\ Q=J\times I} \|f\|_{L^{\tilde{p}}(J)} \|g\|_{L^p(I)} \leq \|f\|_{L^{\tilde{p}}(\mathbb{R})} \|g\|_{L^p(\mathbb{R})}.$$

Proof. The lemma follows directly from the inequality

$$\sum_j |a_j b_j| \leq \left(\sum_j |a_j|^{\tilde{p}}\right)^{\frac{1}{\tilde{p}}} \left(\sum_j |b_j|^p\right)^{\frac{1}{p}},$$

which holds in the range $\frac{1}{p} + \frac{1}{\tilde{p}} \geq 1$, and the fact that for each dyadic interval I there are at most two dyadic squares in $\mathcal{O}_\lambda$ with side I. □

Consider now the bilinear operator $A : L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'} \times L_t^{q'} L_x^{r'} L_v^{p'} \to l_s^\infty$, (for a definition of l_s^∞ see below), defined by the formula

$$A(F, G) = \{b_\lambda\}_{\lambda \in 2^{\mathbb{Z}}} = \left\{ \sum_{Q \in \mathcal{O}_\lambda} |B_Q(F, G)| \right\}_{\lambda \in 2^{\mathbb{Z}}}.$$

This definition is well-made whenever (q, r, p) and $(\tilde{q}, \tilde{r}, \tilde{p})$ satisfy the assumptions of Lemma 2.7.4, and if $s = -\beta(q, r, \tilde{q}, \tilde{r})$. The boundedness of $A : L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'} \times L_t^{q'} L_x^{r'} L_v^{p'} \to l^1$ implies the boundedness of $B : L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'} \times L_t^{q'} L_x^{r'} L_v^{p'} \to \mathbb{C}$. Thus, in view of Lemma 2.5.2, the estimate

$$\|\{b_\lambda\}\|_{l^1} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}} \|G\|_{L_t^{q'} L_x^{r'} L_v^{p'}}, \quad \forall F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}, \ \forall G \in L_t^q L_x^r L_v^p,$$

implies the boundedness of $W(t) : L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'} \to L_t^q L_x^r L_v^p$. We summarize this argument in

Lemma 2.5.13. *The boundedness of the bilinear operator* $A : L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'} \times L_t^{q'} L_x^{r'} L_v^{p'} \to l^1$ *implies the inhomogeneous Strichartz estimate*

$$\|W(t)F\|_{L_t^q L_x^r L_v^p} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}}, \quad \forall F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}.$$

To highlight the importance of Lemma 2.5.13, we mention that it presents the furthest reduction of the problem of proving the inhomogeneous Strichartz estimates for the operator $W(t)$. This lemma shall also be used in the proof of the homogeneous endpoint estimates (2.38).

In the remaining part of this paragraph we collect several facts from Real Interpolation that shall be used throughout this work. By $L^p = L^p(X; \mathcal{B})$ and $L^{p,q} = L^{p,q}(X; \mathcal{B})$ we denote the Lebesgue space and the Lorentz space, respectively, of vector-valued functions that map a fixed measure space $(X, d\mu)$ to a fixed Banach space $\mathcal{B}$.

Lemma 2.5.14 (see [3, p. 113])**.** *Suppose that* $0 < p_0, p_1, q_0, q_1 \leq \infty$, $0 < \theta < 1$, *and* $p_0 \neq p_1$. *Then*

$$(L^{p_0, q_0}, L^{p_1, q_1})_{\theta, q} = L^{p, q},$$

where $1/p = (1 - \theta)/p_0 + \theta/p_1$.

Suppose that $\mathcal{B}_0$ and $\mathcal{B}_1$ are two Banach spaces that are compatible for interpolation.

Lemma 2.5.15 (see the Appendix of [15])**.** *For every* $1 \leq p_0, p_1 < \infty$, $0 < \theta < 1$, $1/p = (1 - \theta)/p_0 + \theta/p_1$ *and* $p \leq q$ *we have*

$$L^p(X; (\mathcal{B}_0, \mathcal{B}_1)_{\theta, q}) \hookrightarrow (L^{p_0}(X; \mathcal{B}_0), L^{p_1}(X; \mathcal{B}_1))_{\theta, q}.$$

Denote by l_s^p the space of number sequences with a norm

$$\|\{a\}_{j \in \mathbb{Z}}\|_{l_s^p} = \left(2^{js} |a_j|^p\right)^{1/p}, \quad 1 \leq p < \infty,$$
$$\|\{a\}_{j \in \mathbb{Z}}\|_{l_s^\infty} = \sup_{j \in \mathbb{Z}} 2^{js} |a_j|, \quad p = \infty.$$

Lemma 2.5.16 (See Theorem 5.6.1 in [3])**.** *We have the identity*

$$\left(l_{s_0}^\infty, l_{s_1}^\infty\right)_{\theta, 1} = l_s^1,$$

where $s_0, s_1 \in \mathbb{R}$, $s_0 \neq s_1$ *and* $s = (1-\theta)s_0 + \theta s_1$.

Lemma 2.5.17 (See pp. 76-77 in [3])**.** *Suppose that* $(\mathcal{A}_0, \mathcal{A}_1)$, $(\mathcal{B}_0, \mathcal{B}_1)$, $(\mathcal{C}_0, \mathcal{C}_1)$ *are interpolation couples and that the bilinear operator* T *acts as a bounded transformation as indicated below:*

$$T : \mathcal{A}_0 \times \mathcal{B}_0 \to \mathcal{C}_0,$$
$$T : \mathcal{A}_0 \times \mathcal{B}_1 \to \mathcal{C}_1,$$
$$T : \mathcal{A}_1 \times \mathcal{B}_0 \to \mathcal{C}_1.$$

If $\theta_0, \theta_1 \in (0,1)$ *and* $p, q, r \in [1, \infty]$ *such that* $1/p + 1/q \geq 1$, *then* T *also acts as a bounded transformation in the following way:*

$$T : (\mathcal{A}_0, \mathcal{A}_1)_{\theta_0, pr} \times (\mathcal{B}_0, \mathcal{B}_1)_{\theta_1, qr} \to (\mathcal{C}_0, \mathcal{C}_1)_{\theta_0+\theta_1, r}.$$

2.6 Strichartz estimates for the Cauchy problem

In this paragraph we prove the claim of Theorem 2.3.1 in the direction of its sufficiency condition. The proof of the claim in the part of its necessity shall be proved in Section 2.9 by means of counterexamples. Let us first consider the case of non-endpoint exponent triplets. We recall that a simple condition for an exponent triplet (q, r, p) to be non-endpoint is $q > a$, where $a = \mathrm{HM}(p, r)$.

The main characteristic of the present setting is the fact that the homogeneous and the inhomogeneous estimate

$$\|U(t)f\|_{L^q_t L^r_x L^p_v} \lesssim \|f\|_{L^a_x}, \quad 1 \leq a < \infty, \tag{2.24a}$$
$$\|W(t)F\|_{L^q_t L^r_x L^p_v} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x L^{\tilde{p}'}_v}, \tag{2.24b}$$

are equivalent in view of the TT^*-principle and Lemma 2.5.1. Here, we have assumed that (q, r, p) and $(\tilde{q}, \tilde{r}, \tilde{p})$ are two non-endpoint KT-admissible exponent triplets subject to the scaling condition $a = \mathrm{HM}(p, r) = \mathrm{HM}(\tilde{p}', \tilde{r}')$. A further reductions will be to consider only the estimate

$$\|U(t)f\|_{L^q_t L^r_x L^{r'}_v} \lesssim \|f\|_{L^2_x},$$

since, in view of the power invariance (2.9), we can generalize it to any transport norm.

Proof. In view of the decay estimate

$$\|U(t)f\|_{L^r_x L^{r'}_v} \lesssim \frac{1}{|t|^{\beta(r)}} \|f\|_{L^{r'}_x L^r_v}, \quad 2 \leq r \leq \infty,$$

where $\beta(r) = n(1 - 2/r)$, cf. Corollary 2.4.3, we obtain the following estimate for TT^*F

$$\|TT^*F\|_{L^r_x L^{r'}_v} \lesssim \int_{-\infty}^{\infty} \|U(t-s)F(s)\|_{L^r_x L^{r'}_v}\, ds \lesssim \int_{-\infty}^{\infty} \frac{\|F(s)\|_{L^{r'}_x L^r_v}}{|t-s|^{\beta(r)}} ds.$$

We take the L^q-norm in t and in view of the Hardy-Littlewood-Sobolev (HLS) theorem of fractional integration, see [2, pp. 228-229], [41], we obtain

$$\|TT^*F\|_{L^q_t L^r_x L^{r'}_v} \lesssim \|F\|_{L^{q'}_t L^{r'}_x L^r_v},$$

whenever $0 < \beta(r) < 1$, $1 + 1/q = 1/q' + \beta(r)$. The latter conditions are equivalent to $2 < r < r^*(2)$, $1/q + n/r = n/2$. The left endpoint $r = 2$ follows trivially from the transport estimate (2.8). □

The right endpoint $r = r^*(2)$ remains unresolved in the context of the KT equation, unlike that of the wave and the Schrödinger equations, where it has been resolved (in the positive) by Keel and Tao [30] (1997).

The equivalent inhomogeneous estimate (2.24b) to the homogeneous endpoint (2.24a) with $(q,r,p) = (2, r^*(2), r^*(2)')$ has both triplets (q,r,p) and $(\tilde{q},\tilde{r},\tilde{p})$ equal to $(2, r^*(2), r^*(2)')$. However, the case when (q,r,p) is non-endpoint and $(\tilde{q},\tilde{r},\tilde{p})$ is endpoint corresponds to a valid inhomogeneous estimate that is considered in the proof of Theorem 2.3.2 in Section 2.8.

2.7 Local inhomogeneous estimates

Following Foschi [20], we want to find the range of local estimates for $W(t)$ that are invariant to the scaling

$$\|W(t)[\chi_{\lambda J}F]\|_{L^q(\lambda I;L^r_x L^p_v)} \lesssim \lambda^{\frac{1}{q}+\frac{1}{\tilde{q}}-n\left(1-\frac{1}{r}-\frac{1}{\tilde{r}}\right)} \|F\|_{L^{\tilde{q}'}(\lambda J;L^{\tilde{r}'}_x L^{\tilde{p}'}_v)}, \quad \forall \lambda > 0, \tag{2.59}$$

where I and J are two unit intervals separated by a unit distance and $\chi_{\lambda J}$ is the characteristic of the rescaled interval λJ.

The bilinear formulation of (2.59) is

$$|B_Q(F,G)| \lesssim \lambda^{\beta(q,r,\tilde{q},\tilde{r})} \|F\|_{L^{\tilde{q}'}_t(J;L^{\tilde{r}'}_x L^{\tilde{p}'}_v)} \|G\|_{L^{q'}_t(I;L^{r'}_x L^{p'}_v)}, \tag{2.60}$$

where Q is the square $I \times J$.

Lemma 2.7.1. *Estimate* (2.59) *holds for any two (non-endpoint) KT-admissible triplets* (q,r,p) *and* $(\tilde{q},\tilde{r},\tilde{p})$ *with* $a = \tilde{a}'$.

Proof. The proof follows trivially from Theorem 2.3.1 due to the fact that $\beta(q,r,\tilde{q},\tilde{r}) = 0$ under the hypothesis of the lemma. □

Lemma 2.7.2. *Estimate* (2.59) *holds with* $(q,r,p) = (\infty,r,p)$ *and* $(\tilde{q},\tilde{r},\tilde{p}) = (\infty,p',r')$*, where* $1 \leq p \leq r \leq \infty$.

Proof. Due to the decay estimate (2.41) we have that

$$\begin{aligned}\sup_{t\in\lambda I} \|W(t)[\chi_{\lambda J}F]\|_{L^r_x L^p_v} &\lesssim \sup_{t\in\lambda I} \int_{\lambda J} \frac{\|F(\tau)\|_{L^p_x L^r_v}}{|t-\tau|^{n\left(\frac{1}{p}-\frac{1}{r}\right)}} d\tau \\ &\lesssim \lambda^{\beta(\infty,r,\infty,p')} \|F\|_{L^1(\lambda J;L^p_x L^r_v)}.\end{aligned}$$

□

Lemma 2.7.3. *Whenever* (q,r,p) *and* $(\tilde{q},\tilde{r},\tilde{p})$ *are exponent triplets for which estimate* (2.59) *holds, we have that* (2.59) *also holds with* (Q,r,p) *and* $(\tilde{Q},\tilde{r},\tilde{p})$*, where* $1 \leq Q \leq q$*,* $1 \leq \tilde{Q} \leq \tilde{q}$.

Proof. A trivial application of Hölder's inequality

$$\begin{aligned}\|W(t)[\chi_{\lambda J}F]\|_{L^Q(\lambda I;L^r_x L^p_v)} &\lesssim \lambda^{\frac{1}{Q}-\frac{1}{q}} \|W(t)[\chi_{\lambda J}F]\|_{L^q(\lambda I;L^r_x L^p_v)} \\ &\lesssim \lambda^{\beta(Q,r,\tilde{q},\tilde{r})} \|F\|_{L^{\tilde{q}'}(\lambda J;L^{\tilde{r}'}_x L^{\tilde{p}'}_v)} \lesssim \lambda^{\beta(Q,r,\tilde{Q},\tilde{r})} \|F\|_{L^{\tilde{Q}'}(\lambda J;L^{\tilde{r}'}_x L^{\tilde{p}'}_v)}.\end{aligned}$$

□

Let us define the range of validity of the local estimates (2.59) as the set $\mathcal{E}$ in $\mathbb{R}^6$. Each point in $\mathcal{E}$ corresponds to the vector $(1/q, 1/r, 1/p, 1/\tilde{q}, 1/\tilde{r}, 1/\tilde{p})$ in $\mathbb{R}^6$. Below we find the convex hull $\mathcal{E}^*$ ($\mathcal{E}^* \subseteq \mathcal{E}$) of the points in $\mathbb{R}^6$ that correspond to the estimates in the three lemmas above. We shall call any point or collection of points in $\mathcal{E}$ *acceptable.*

Lemma 2.7.4 (Local inhomogeneous estimates)**.** *Estimate* (2.59) *holds whenever the exponent* *triplets* (q,r,p), $(\tilde{q},\tilde{r},\tilde{p})$ *satisfy the following conditions*

$$0<\frac{1}{q},\frac{1}{\tilde{q}}<1,\quad 0<\frac{1}{p},\frac{1}{\tilde{p}},\frac{1}{r},\frac{1}{\tilde{r}}\le 1, \tag{2.61}$$

$$\frac{1}{r}\le\frac{1}{p},\quad \frac{1}{\tilde{r}}\le\frac{1}{\tilde{p}},\quad \mathrm{HM}(p,r)=\mathrm{HM}(\tilde{p}',\tilde{r}'), \tag{2.62}$$

$$\frac{1}{\tilde{r}}-\frac{1}{\tilde{p}}-\frac{1}{r}+\frac{1}{p}\le\frac{2}{nq},\quad \frac{1}{r}-\frac{1}{p}-\frac{1}{\tilde{r}}+\frac{1}{\tilde{p}}\le\frac{2}{n\tilde{q}}, \tag{2.63}$$

$$\frac{n-1}{p'}<\frac{n}{\tilde{r}},\quad \frac{n-1}{\tilde{p}'}<\frac{n}{r}, \tag{2.64}$$

or if the point $(1/q,1/r,1/p,1/\tilde{q},1/\tilde{r},1/\tilde{p})$ *lies inside one of the cubes in* $\mathbb{R}^6$ *below*

$$\begin{aligned}&(\kappa,0,\mu,\nu,1-\mu,1)\,, && 0\le\kappa,\mu,\nu\le 1,\\ &(\kappa,1-\mu,1,\nu,0,\mu)\,, && 0\le\kappa,\mu,\nu\le 1.\end{aligned} \tag{2.65}$$

Proof. We apply the Riesz-Thorin convexity theorem to interpolate between the already proven local estimates. In essence, we find the convex hull of the locally acceptable sets associated with Lemmas 2.7.1 and 2.7.2 and then expand that set by the rule given in Lemma 2.7.3.

The set of acceptability S_1 of the local estimates in Lemma 2.7.1 is given by the system

$$0<\frac{1}{r},\frac{1}{\tilde{r}}\le 1,\quad 0\le\frac{1}{q},\frac{1}{\tilde{q}},\frac{1}{p},\frac{1}{\tilde{p}}\le 1, \tag{2.66}$$

$$\frac{1}{q}=\frac{n}{2}\left(\frac{1}{p}-\frac{1}{r}\right),\quad \frac{1}{\tilde{q}}=\frac{n}{2}\left(\frac{1}{\tilde{p}}-\frac{1}{\tilde{r}}\right), \tag{2.67}$$

$$\frac{1}{r}+\frac{1}{p}+\frac{1}{\tilde{r}}+\frac{1}{\tilde{p}}=2, \tag{2.68}$$

$$\frac{n-1}{p}<\frac{n+1}{r},\qquad \frac{n-1}{\tilde{p}}<\frac{n+1}{\tilde{r}}, \tag{2.69}$$

or if $(1/q,1/r,1/p,1/\tilde{q},1/\tilde{r},1/\tilde{p})\in\{B=(0,0,0,0,1,1),C=(0,1,1,0,0,0)\}$.

Note that S_1 is a convex polyhedron in $\mathbb{R}^6$ and the two points B and C lie on its boundary. The set of acceptability S_2 of the local estimates in Lemma 2.7.2 is the convex hull (in fact a triangle) of the three points

$$A=(0,0,1,0,0,1),\qquad B=(0,0,0,0,1,1),\qquad C=(0,1,1,0,0,0). \tag{2.70}$$

Vertices B and C are already included in S_1, thus it would suffice to take only the vertex A. Hence, we obtain the following set

$$\begin{aligned}&\frac{1}{Q}=\frac{\theta}{q}, && \frac{1}{R}=\frac{\theta}{r}, && \frac{1}{P}=1-\theta+\frac{\theta}{p},\\ &\frac{1}{\tilde{Q}}=\frac{\theta}{\tilde{q}}, && \frac{1}{\tilde{R}}=\frac{\theta}{\tilde{r}}, && \frac{1}{\tilde{P}}=1-\theta+\frac{\theta}{\tilde{p}}, && 0\le\theta\le 1,\end{aligned}$$

where $(1/Q,1/R,1/P,1/\tilde{Q},1/\tilde{R},1/\tilde{P})$ are the coordinates of the new set S_3 written in terms of $(1/q,1/r,1/p,1/\tilde{q},1/\tilde{r},1/\tilde{p})$ and θ. Of course, to S_3 we must also add the line segments $[A,B]$ and $[A,C]$. We shall treat this case separately at the end.

Finally, we apply the rule given in Lemma 2.7.3 and thus we replace the equations for Q and $\tilde{Q}$ above with the following inequalities

$$1\ge\frac{1}{Q}\ge\frac{\theta}{q},\qquad 1\ge\frac{1}{\tilde{Q}}\ge\frac{\theta}{\tilde{q}},$$

plus the restrictions

$$\frac{1}{r} \leq \frac{1}{p}, \quad \frac{1}{\tilde{r}} \leq \frac{1}{\tilde{p}}, \tag{2.71}$$

which were implicitly assumed in (2.67).

1. We first eliminate q and $\tilde{q}$ from the system for S_1 to obtain

$$\frac{1}{Q} \geq \frac{n}{2}\left(\frac{\theta}{p} - \frac{\theta}{r}\right), \quad \Leftrightarrow \quad \frac{1}{Q} \geq \frac{n}{2}\left(\theta - 1 + \frac{1}{P} - \frac{1}{R}\right),$$
$$\Leftrightarrow \quad \theta \leq \frac{1}{P'} + \frac{1}{R} + \frac{2}{nQ}.$$

Similarly,

$$\theta \leq \frac{1}{\tilde{P}'} + \frac{1}{\tilde{R}} + \frac{2}{n\tilde{Q}}, \qquad \frac{1}{Q}, \frac{1}{\tilde{Q}} \leq 1.$$

2. As expected, condition (2.68) is invariant

$$\frac{1}{R} + \frac{1}{P} + \frac{1}{\tilde{R}} + \frac{1}{\tilde{P}} = 2.$$

3. Reworking condition (2.69), we obtain

$$\theta < \frac{n+1}{n-1}\frac{1}{R} + \frac{1}{P'}, \qquad \theta < \frac{n+1}{n-1}\frac{1}{\tilde{R}} + \frac{1}{\tilde{P}'}.$$

4. Condition (2.71) is replaced by

$$\frac{1}{P'} + \frac{1}{R} \leq \theta, \quad \frac{1}{\tilde{P}'} + \frac{1}{\tilde{R}} \leq \theta.$$

5. Finally, conditions (2.66) are transformed into

$$\frac{1}{P'}, \frac{1}{\tilde{P}'}, \frac{1}{R}, \frac{1}{\tilde{R}} \leq \theta, \quad 0 \leq \frac{1}{Q}, \frac{1}{\tilde{Q}}, \frac{1}{P}, \frac{1}{\tilde{P}}, \frac{1}{R}, \frac{1}{\tilde{R}} \leq 1.$$

6. We group all conditions obtained in the previous 5 steps according to their type

$$0, \frac{1}{P'}, \frac{1}{\tilde{P}'}, \frac{1}{R}, \frac{1}{\tilde{R}}, \frac{1}{P'} + \frac{1}{R}, \frac{1}{\tilde{P}'} + \frac{1}{\tilde{R}} \leq \theta. \tag{2.72}$$

$$\theta \leq \frac{1}{R} + \frac{1}{P'} + \frac{2}{nQ}, \frac{1}{\tilde{R}} + \frac{1}{\tilde{P}'} + \frac{2}{n\tilde{Q}}, \frac{n+1}{n-1}\frac{1}{R} + \frac{1}{P'}, \frac{n+1}{n-1}\frac{1}{\tilde{R}} + \frac{1}{\tilde{P}'}, 1. \tag{2.73}$$

$$0 \leq \frac{1}{Q}, \frac{1}{\tilde{Q}}, \frac{1}{P}, \frac{1}{\tilde{P}}, \frac{1}{R}, \frac{1}{\tilde{R}} \leq 1, \qquad \frac{1}{P} + \frac{1}{R} + \frac{1}{\tilde{R}} + \frac{1}{\tilde{P}} = 2. \tag{2.74}$$

7. We discard the redundant conditions like

$$0, \frac{1}{P'}, \frac{1}{\tilde{P}'}, \frac{1}{R}, \frac{1}{\tilde{R}} \leq \theta,$$

which are all weaker than the other two in (2.72).

There exists θ solving all inequalities in (2.72), (2.73), if and only if every quantity in (2.72) is bounded from above by any quantity in (2.73). Thus we form all possible combinations between the quantities in the two types of (reduced) inequalities to obtain the following set of conditions

$$
\begin{gathered}
\frac{1}{R}+\frac{1}{P'}\le\frac{1}{\tilde R}+\frac{1}{\tilde P'}+\frac{2}{n\tilde Q},\quad \frac{1}{\tilde R}+\frac{1}{\tilde P'}\le\frac{1}{R}+\frac{1}{P'}+\frac{2}{nQ},\\
\frac{1}{R}+\frac{1}{P'}<\frac{n+1}{n-1}\frac{1}{\tilde R}+\frac{1}{\tilde P'},\quad \Leftrightarrow\quad \frac{n-1}{P'}<\frac{n}{\tilde R},\\
\frac{1}{\tilde R}+\frac{1}{\tilde P'}<\frac{n+1}{n-1}\frac{1}{R}+\frac{1}{P'},\quad \Leftrightarrow\quad \frac{n-1}{\tilde P'}<\frac{n}{R},\\
\frac{1}{R}\le\frac{1}{P},\quad \frac{1}{\tilde R}\le\frac{1}{\tilde P},
\end{gathered}
$$

describing the region S_3.

8. We apply the rule given in Lemma 2.7.3 to the two line segments $[A,B]$ and $[A,C]$ to obtain the following two cubes in $\mathbb{R}^6$

$$
\begin{aligned}
&(\mu,0,\kappa,\nu,1-\kappa,1)\,, &&0\le\mu,\nu,\kappa\le 1,\\
&(\mu,1-\kappa,1,\nu,0,\kappa)\,, &&0\le\mu,\nu,\kappa\le 1.
\end{aligned}
\tag{2.75}
$$

Hence, the computation of the set $\mathcal{E}^*$ is finished. □

2.8 Proof of the global inhomogeneous Strichartz estimates

2.8.1 Global inhomogeneous non-endpoint estimates

In this paragraph we prove the inhomogeneous Strichartz estimate

$$
\|W(t)F\|_{L^q_tL^r_xL^p_v}\lesssim\|F\|_{L^{\tilde q'}_tL^{\tilde r'}_xL^{\tilde p'}_v}\,,\quad \forall F\in L^{\tilde q'}_tL^{\tilde r'}_xL^{\tilde p'}_v,
$$

in the range $q>\tilde q'$. Thanks to Lemma 2.5.13, we have reduced this problem to showing the estimate

$$
\|\{b_\lambda\}\|_{l^1}\lesssim\|F\|_{L^{\tilde q'}_tL^{\tilde r'}_xL^{\tilde p'}_v}\|G\|_{L^{q'}_tL^{r'}_xL^{p'}_v}\,,\quad \forall F\in L^{\tilde q'}_tL^{\tilde r'}_xL^{\tilde p'}_v,\ \forall G\in L^{q'}_tL^{r'}_xL^{p'}_v.
$$

For convenience we recall that

$$
\{b_\lambda\}_{\lambda\in 2^{\mathbb{Z}}}=\left\{\sum_{Q\in\mathcal{O}_\lambda}|B_Q(F,G)|\right\}_{\lambda\in 2^{\mathbb{Z}}}.
$$

We shall next specify the range of validity of the above estimates in terms of the vector

$$
(1/q,1/r,1/p,1/\tilde q,1/\tilde r,1/\tilde p)
$$

that we assume to be fixed. Let us first for simplicity denote by Δ the triangle

$$
(x,1/r,1/p,y,1/\tilde r,1/\tilde p)\cap\{x>0,y>0,x+y<1\}
$$

in $\mathbb{R}^6$. We denote by $P(x,y)$ any point in Δ with running coordinates x and y. Suppose that

$$
1/q+1/\tilde q=n\,(1-1/r-1/\tilde r)\,,\tag{2.76}
$$

and that $P(1/q,1/\tilde q)\in\Delta\cap\mathcal{E}^*$ such that there exist a small enough open neighborhood $U(1/q,1/\tilde q)$ of points $P(x,y)\in\Delta\cap\mathcal{E}^*$ around $P(1/q,1/\tilde q)$. Therefore, in view of Corollary 2.5.11, we have the estimate

$$
|b_\lambda|\lesssim\lambda^{\beta(q,r,\tilde q,\tilde r)}\|F\|_{L^{\tilde q'}_tL^{\tilde r'}_xL^{\tilde p'}_v}\|G\|_{L^{q'}_tL^{r'}_xL^{p'}_v}\,,
$$

or equivalently that $\{b_\lambda\} \in l^\infty_s$ with $s = -\beta(q, r, \tilde{q}, \tilde{r})$. Let us set

$$1/q_0 = 1/q + \epsilon, \quad 1/\tilde{q}_0 = 1/\tilde{q} + \epsilon, \quad 1/q_1 = 1/q - 3\epsilon, \quad 1/\tilde{q}_1 = 1/\tilde{q} - 3\epsilon,$$

for some small enough $\epsilon > 0$, such that the corresponding exponent vectors do not leave $U(1/q, 1/\tilde{q})$. Then we have that $\beta(q_0, r, \tilde{q}_0, \tilde{r}) = -2\epsilon$, and $\beta(q_1, r, \tilde{q}_0, \tilde{r}) = \beta(q_0, r, \tilde{q}_1, \tilde{r}) = 2\epsilon$. The following bilinear maps

$$\begin{aligned} A &: L_t^{\tilde{q}_0'} L_x^{\tilde{r}'} L_v^{\tilde{p}'} \times L_t^{q_0'} L_x^{r'} L_v^{p'} \to l^\infty_{-2\epsilon}, \\ A &: L_t^{\tilde{q}_0'} L_x^{\tilde{r}'} L_v^{\tilde{p}'} \times L_t^{q_1'} L_x^{r'} L_v^{p'} \to l^\infty_{2\epsilon}, \\ A &: L_t^{\tilde{q}_1'} L_x^{\tilde{r}'} L_v^{\tilde{p}'} \times L_t^{q_0'} L_x^{r'} L_v^{p'} \to l^\infty_{2\epsilon}, \end{aligned}$$

are bounded. In virtue of Lemma 2.5.17, we have that the map

$$A : (L_t^{\tilde{q}_0'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}, L_t^{\tilde{q}_1'} L_x^{\tilde{r}'} L_v^{\tilde{p}'})_{1/4, \tilde{q}'} \times (L_t^{q_0'} L_x^{r'} L_v^{p'}, L_t^{q_1'} L_x^{r'} L_v^{p'})_{1/4, q'} \to (l^\infty_{2\epsilon}, l^\infty_{-2\epsilon})_{1/2,1}$$

is also bounded. Finally, in view of Lemma 2.5.16 and the embeddings of the Lorentz spaces, we obtain

$$A : L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'} \times L_t^{q'} L_x^{r'} L_v^{p'} \to l^1.$$

All assumption made in this paragraph are explicitly stated in Theorem 2.3.2, part (i). We remark that condition (2.63), together with (2.76), is equivalent to (q, r, p) and $(\tilde{q}, \tilde{r}, \tilde{p})$ being KT-acceptable. Let us also note that the two locally acceptable "cubic" sets in (2.75) give rise to the two globally acceptable "cubic cross-section" sets Σ_1 and Σ_2 in Definition 2.2.4.

2.8.2 Global inhomogeneous endpoint estimates with $\tilde{q} = \infty$

In this paragraph we prove the inhomogeneous Strichartz estimates with $P(1/q,\, 1/\tilde{q})$ lying on either of the two catheti of Δ. Since by duality both type of estimates are equivalent, it is enough to consider only the case $\tilde{q} = \infty$. We exclude the two endpoints $(0, 0)$ and $(1, 0)$ from our considerations. We suppose that $P(1/q, 0) \in \{0 < 1/q < 1\} \cap \mathcal{E}^*$, such that there exist a small enough open neighborhood $U(1/q, 0)$ of points $P(x, 0) \in \{0 < 1/q < 1\} \cap \mathcal{E}^* \cap \{1/\tilde{q} = 0\}$ around $P(1/q, 0)$. We also assume the scaling condition (2.76) from the previous paragraph. Then we have

$$\begin{aligned} A &: L_t^1 L_x^{\tilde{r}'} L_v^{\tilde{p}'} \times L_t^{q_0'} L_x^{r'} L_v^{p'} \to l^\infty_\epsilon, \\ A &: L_t^1 L_x^{\tilde{r}'} L_v^{\tilde{p}'} \times L_t^{q_1'} L_x^{r'} L_v^{p'} \to l^\infty_{-\epsilon}, \end{aligned}$$

where

$$\frac{1}{q_0} = \frac{1}{q} - \frac{1}{\epsilon}, \quad \frac{1}{q_1} = \frac{1}{q} + \frac{1}{\epsilon}.$$

The real method with parameters $(\theta, q) = (1/2, 1)$ gives that

$$A : L_t^1 L_x^{\tilde{r}'} L_v^{\tilde{p}'} \times L_t^{q',1} L_x^{r'} L_v^{p'} \to l^1.$$

Equivalently, in view of the TT^*-principle,

$$\|W(t)F\|_{L_t^{q,\infty} L_x^r L_v^p} \lesssim \|F\|_{L_t^1 L_x^{\tilde{r}'} L_v^{\tilde{p}'}}, \tag{2.77}$$

for all $F \in L_t^1 L_x^{\tilde{r}'} L_v^{\tilde{p}'}$. The explicit restrictions on the Lebesgue exponents (q, r, p) and $(\infty, \tilde{r}, \tilde{p})$ are stated in Theorem 2.3.2, part (ii). Analogously, the dual case is stated as part (iii) of that theorem.

There are two remaining issues that we would like to address a) the corresponding homogeneous estimates to (2.77) via the Equivalence theorem 2.3.5 in its stronger form for Lorentz spaces and b) the sharpening of (2.77) to the Lebesgue norm L_t^q.

Lemma 2.8.1. *The estimate*

$$\|U(t)f\|_{L_t^{q,\infty}L_x^rL_v^p} \lesssim \|f\|_{L_x^bL_v^c}, \tag{2.78}$$

holds for all $f \in L_x^bL_v^c$, whenever

$$\frac{1}{q}+\frac{n}{r}=\frac{n}{b}, \quad \mathrm{HM}(r,p)=\mathrm{HM}(b,c) \stackrel{def}{=} a, \quad r<\frac{nc}{n-1},$$
$$p<b\le a\le c<r, \quad 1<q<\infty,\, 1\le p,\tilde{p},r,\tilde{r}<\infty.$$

Proof. The range of validity of estimate (2.78) is the determined in the following way. We first take the restrictions on the Lebesgue exponents that define the range of the validity of the local inhomogeneous estimates of Lemma 2.7.4, i.e. the set $\mathcal{E}^*$. However, any nonstrict inequalities where q appears should be taken as strict inequalities to ensure the existence of the small open neighborhood $U(1/q,0)$ around $P(1/q,0)$. To that we add the restrictions $0<1/q<1$ and the scaling condition (2.76). Thus, we have that $1/q=n\,(1-1/r-1/\tilde{r})$, and

$$0<\frac{1}{q},\frac{1}{\tilde{q}}<1, \quad 0<\frac{1}{p},\frac{1}{\tilde{p}},\frac{1}{r},\frac{1}{\tilde{r}}\le 1,$$
$$\frac{1}{r}\le\frac{1}{p}, \quad \frac{1}{\tilde{r}}\le\frac{1}{\tilde{p}}, \quad \mathrm{HM}(p,r)=\mathrm{HM}(\tilde{p}',\tilde{r}'),$$
$$\frac{1}{q}<n\left(\frac{1}{p}-\frac{1}{r}\right), \quad 0\le n\left(\frac{1}{\tilde{p}}-\frac{1}{\tilde{r}}\right),$$
$$\frac{n-1}{p'}<\frac{n}{\tilde{r}}, \quad \frac{n-1}{\tilde{p}'}<\frac{n}{r},$$

or that the point $(1/q,1/r,1/p,0,1/\tilde{r},1/\tilde{p})$ belongs to the set

$$(\kappa,0,\mu,0,1-\mu,1)\,, \quad 0<\kappa,\mu<1,\ \kappa=n\mu.$$

The latter set of exponents does not give as anything new as it essentially expresses a special case of the decay estimate

$$\|U(t)f\|_{L_t^{q,\infty}L_x^\infty L_v^{nq}} \lesssim \|f\|_{L_x^{nq}L_v^\infty}\,.$$

Let us use the more natural notation for the exponents $b=\tilde{r}'$ and $c=\tilde{p}'$. Thus we have the conditions

$$\frac{1}{q}+\frac{n}{r}=\frac{n}{b}, \quad \mathrm{HM}(p,r)=\mathrm{HM}(\tilde{p}',\tilde{r}'),$$
$$r<\frac{nc}{n-1}, \quad p<b\le c<r,$$
$$1<q<\infty, \quad 1\le p,\tilde{p},r,\tilde{r}<\infty.$$

□

Corollary 2.8.2. *The estimate*

$$\|U(t)f\|_{L_t^qL_x^rL_v^p} \lesssim \|f\|_{L_x^bL_v^c}, \quad b\ne c, \tag{2.79}$$

holds for all $f\in L_x^bL_v^c$ whenever

$$\frac{1}{q}+\frac{n}{r}=\frac{n}{b}, \quad p<b<a<c<r, \quad r<\frac{n}{n-1}c,$$
$$\mathrm{HM}(r,p)=\mathrm{HM}(b,c)\stackrel{def}{=} a, \quad 1<q,b,c,p,r<\infty, \quad q\ge c.$$

The L_t^q-norm in (2.79) can be replaced by the $L_t^{q,c}$-norm. In such case the assumption $q\ge c$ can be removed.

Proof. Each estimate in the statement of this corollary can be proved by interpolating two estimates (2.78) with the real method. Indeed, let us perturb slightly the exponents q, b, and c, keeping r and p fixed, in such a way that they remain in the range of validity of the estimates (2.78). For example, the perturbed exponents can be taken as follows

$$\begin{aligned} 1/q_1 &= 1/q + n/\epsilon, \quad 1/b_1 = 1/b + 1/\epsilon, \quad 1/c_1 = 1/c - n/\epsilon, \\ 1/q_2 &= 1/q - n/\epsilon, \quad 1/b_2 = 1/b - 1/\epsilon, \quad 1/c_2 = 1/c + n/\epsilon. \end{aligned}$$

We then interpolate by the real method with $(\theta, q) = (1/2, c)$, and make use of Proposition 2.5.15. $\square$

Lemma 2.8.3. *Suppose that (q, r, p) and $(\infty, \tilde{r}, \tilde{p})$ are two jointly KT-acceptable exponent triplets such that $r < \frac{n}{n-1}\tilde{r}'$ ($r < \infty$ for $n = 1$), $q \geq \tilde{p}'$, and $1 < q, b, c, p, r < \infty$. Then the estimate*

$$\|W(t)F\|_{L_t^q L_x^r L_v^p} \lesssim \|F\|_{L_t^1 L_x^{\tilde{r}'} L_v^{\tilde{p}'}},$$

holds for all $F \in L_t^1 L_x^{\tilde{r}'} L_v^{\tilde{p}'}$. Similarly, if (∞, r, p) and $(\tilde{q}, \tilde{r}, \tilde{p})$ are two jointly KT-acceptable exponent triplets such that $\tilde{r} < \frac{n}{n-1} r'$ ($\tilde{r} < \infty$ for $n = 1$), $\tilde{q}' \leq p$, and $1 < q, b, c, p, r < \infty$, then the estimate

$$\|W(t)F\|_{L_t^\infty L_x^r L_v^p} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}}$$

holds for all $F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}$.

Proof. The lemma follows directly from Lemma 2.8.1, Corollary 2.8.2, and the Equivalence theorem 2.3.5. The range of validity of these estimates is identical to that of the generalized homogeneous estimates except for the usual change of notation. $\square$

2.8.3 Global inhomogeneous endpoint estimates with $q = \tilde{q}'$

In this paragraph we assume that the L_v^p-norms are given over a bounded velocity space $V \subset \mathbb{R}^n$ and prove the inhomogeneous estimates (2.28).

We suppose now that $P(1/q, 1/\tilde{q})$ lies on the hypotenuse of Δ, see fig. 2.1. Denote by $V(1/r, 1/p, 1/\tilde{r}, 1/\tilde{p})$ a small open neighborhood of $P(1/q, 1/\tilde{q})$ in the 4-dimensional affine subspace of $\mathbb{R}^6$ that is orthogonal to the plane of Δ and contains the point $P(1/q, 1/\tilde{q})$. We assume that $P(1/q, 1/\tilde{q})$ satisfies the scaling condition (2.76) and that it belongs to $\mathcal{E}^*$ together with a small open neighborhood $V(1/r, 1/p, 1/\tilde{r}, 1/\tilde{p})$. As we did before, we perturb slightly the exponents of $P(1/q, 1/\tilde{q})$

$$\begin{aligned} \frac{1}{r_0} &= \frac{1}{r} + \epsilon, \quad \frac{1}{\tilde{r}_0} = \frac{1}{\tilde{r}} + \epsilon, \quad \frac{1}{p_0} = \frac{1}{p} - \epsilon, \quad \frac{1}{\tilde{p}_0} = \frac{1}{\tilde{p}} - \epsilon, \\ \frac{1}{r_1} &= \frac{1}{r} - 3\epsilon, \quad \frac{1}{\tilde{r}_1} = \frac{1}{\tilde{r}} - 3\epsilon, \quad \frac{1}{p_1} = \frac{1}{p} + 3\epsilon, \quad \frac{1}{\tilde{p}_1} = \frac{1}{\tilde{p}} + 3\epsilon. \end{aligned}$$

We have that $\beta(q, r_0, \tilde{q}, \tilde{r}_0) = 2n\epsilon$ and $\beta(q, r_1, \tilde{q}, \tilde{r}_0) = \beta(q, r_0, \tilde{q}, \tilde{r}_1) = -2n\epsilon$. Hence the maps

$$\begin{aligned} A &: L_t^{\tilde{q}'} L_x^{\tilde{r}_0'} L_v^{\tilde{p}_0'} \times L_t^{q'} L_x^{r_0'} L_v^{p_0'} \to l_{-2\epsilon}^\infty, \\ A &: L_t^{\tilde{q}'} L_x^{\tilde{r}_0'} L_v^{\tilde{p}_0'} \times L_t^{q'} L_x^{r_1'} L_v^{p_1'} \to l_{2\epsilon}^\infty, \\ A &: L_t^{\tilde{q}'} L_x^{\tilde{r}_1'} L_v^{\tilde{p}_1'} \times L_t^{q'} L_x^{r_0'} L_v^{p_0'} \to l_{2\epsilon}^\infty, \end{aligned}$$

are bounded. In virtue of Lemma 2.5.17 and the well-known interpolation identity

$$(L^p(\mathbb{R}; \mathcal{A}_0), L^p(\mathbb{R}; \mathcal{A}_1))_{\theta,p} = L^p(\mathbb{R}; (\mathcal{A}_0, \mathcal{A}_1)_{\theta,p}), \quad 1 < p < \infty, \tag{2.80}$$

see [3], the map

$$A : (L_t^{\tilde{q}'} L_x^{\tilde{r}_0'} L_v^{\tilde{p}_0'}, L_t^{\tilde{q}'} L_x^{\tilde{r}_1'} L_v^{\tilde{p}_1'})_{1/4,\tilde{q}'} \times$$
$$(L_t^{q'} L_x^{r_0'} L_v^{p_0'}, L_t^{q'} L_x^{r_1'} L_v^{p_1'})_{1/4,q'} \to (l^\infty_{2\epsilon}, l^\infty_{-2\epsilon})_{1/2,1}$$

is also bounded. In view of the fact that V is bounded we have that $L^{\tilde{P}'}(V) \hookrightarrow L^{\tilde{p}_0'}(V)$ and $L^{\tilde{P}}(V) \hookrightarrow L^{\tilde{p}_1'}(V)$ whenever $1 \leq \tilde{P} \leq min(\tilde{p}_0, \tilde{p}_1)$. Analogously, $L^{P'}(V) \hookrightarrow L^{p_0'}(V)$ and $L^{P'}(V) \hookrightarrow L^{p_1'}(V)$ whenever $1 \leq P \leq min(p_0, p_1)$. Thus we also have that the map

$$A : (L_t^{\tilde{q}'} L_x^{\tilde{r}_0'} L_v^{\tilde{P}'}, L_t^{\tilde{q}'} L_x^{\tilde{r}_1'} L_v^{\tilde{P}'})_{1/4,\tilde{q}'} \times$$
$$(L_t^{q'} L_x^{r_0'} L_v^{P'}, L_t^{q'} L_x^{r_1'} L_v^{P'})_{1/4,q'} \to (l^\infty_{2\epsilon}, l^\infty_{-2\epsilon})_{1/2,1}$$

is bounded. Finally, in view of the interpolation identity (2.80), it follows that

$$A : L_t^{\tilde{q}'} L_x^{\tilde{r}',\tilde{q}'} L_v^{\tilde{P}'} \times L_t^{q'} L_x^{r',q'} L_v^{P'} \to l^1.$$

By the TT^*-principle, this implies the estimate

$$\|W(t)F\|_{L_t^q L_x^r L_v^P(V)} \lesssim_V \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{P}'}(V)}, \tag{2.81}$$

for any $P, \tilde{P}$, such that $1 \leq P < p$ and $1 \leq \tilde{P} < \tilde{p}$, and any two jointly KT-acceptable exponent triplets (q, r, p) and $(\tilde{q}, \tilde{r}, \tilde{p})$ whose exponents further satisfy the following conditions

$$1 < q, \tilde{q} < \infty, \quad q = \tilde{q}', \quad q \leq r, \quad \tilde{q} \leq \tilde{r}$$
$$\frac{n-1}{p'} < \frac{n}{\tilde{r}}, \quad \frac{n-1}{\tilde{p}'} < \frac{n}{r}.$$

2.9 Counterexamples

In this section we present counterexamples to the validity of the Strichartz estimates for the KT equation. Thus we give necessary conditions for the range of validity of these estimates.

2.9.1 Geometric representation

Consider the velocity averages of the kinetic transport propagator $U(t)$

$$\{Af\}(t, x) = \int_{-\infty}^{\infty} f(x - tv, v)dv$$

in one spatial dimension ($n = 1$). In fact, we are dealing with the line integral

$$Af = \frac{1}{\sqrt{1+t^2}} \int_{\gamma_{x,t}} f(l)dl$$

along the straight line $\gamma_{x,t}$ in the x-v-plane, defined by the point $X = (x, 0)$, and the gradient $-1/t$. Let χ_Q be the characteristic function of some measurable set Q in the x-v-plane. Thus, we have

$$\|U(t)\chi_Q\|_{L_x^\infty L_v^1} = \frac{1}{\sqrt{1+t^2}} \operatorname{ess}\sup_{x\in\mathbb{R}} \int_{\gamma_{x,t}} \chi_Q(l)dl.$$

The geometrical interpretation of the latter identity is that $\|U(t)\chi_Q\|_{L_x^\infty L_v^1}$ is equal to the (essential) supremum of the line measure of the intersections of Q with straight lines of gradient $-1/t$, up to a fixed factor in t. As t traverses $\mathbb{R}$, this procedure will be repeated in all possible directions. Let us recall the celebrated

Lemma 2.9.1 (Besicovitch (1919), see [4]). *There exists a sequence of polygons $\{Q_j\}_{j=0}^\infty$ on the x-v-plane that are uniformly bounded and whose area monotonically converges to zero, with* *the property that each Q_j contains a unit line segment in all directions.*

The existence of Besicovitch sets immediately implies the failure of the endpoint estimate.

$$\|U(t)f\|_{L^2_t L^\infty_x L^1_v} \lesssim \|f\|_{L^2_{x,v}} .$$

Indeed, for $f = \chi_{Q_j}$ the left handside in the line above remains bigger than a fixed positive constant, while the right handside tends to zero, as j tends to infinity. This argument recovers the result of [25], but we can say more.

Lemma 2.9.2. *The Strichartz estimate for the kinetic transport equation on* $\mathbb{R}^n$

$$\|U(t)f\|_{L^q_t L^\infty_x L^p_v} \lesssim \|f\|_{L^b_x L^c_v} \tag{2.82}$$

fails on characteristic functions of (Cartesian products of) Besicovitch sets on the plane for all $0 < q, p, b, c \leq \infty$, $c \neq \infty$.

Proof. The proof is essentially the same as the argument above. Let us consider the case when $n = 1$ in detail. Suppose that the area of the Besicovitch set Q is equal to ϵ. Then, provided that $c < \infty$, $\|\chi_Q\|_{L^b_x L^c_v} \lesssim \|\chi_Q\|_{L^c_x L^c_v} = \epsilon^{1/c}$. Thus the right handside in (2.82) will tend to zero as the area of Q does so. Note that we can always assume that $p = 1$ in (2.82) due to the power invariance (2.9). Thus as before, the left handside will remain bigger than some fixed positive constant.

For higher dimensions $n > 1$ we repeat the same argument with the product set $Q^n = Q \times Q \times \ldots Q$. $\square$

Corollary 2.9.3. *All Strichartz estimates for the kinetic transport equation on* $\mathbb{R}^n$ *of the form*

$$\|U(t)f\|_{L^q_t L^\infty_x L^p_v} \lesssim \|f\|_{L^b_x L^c_v}, \quad 0 < q, p, b, c \leq \infty,$$

fail, except for the trivial identity

$$\|U(t)f\|_{L^\infty_t L^\infty_x L^\infty_v} \lesssim \|f\|_{L^\infty_{x,v}} .$$

Proof. In view of Lemma 2.9.2, we need to consider only the case when $c = \infty$. Due to the scaling properties of $U(t)$ given in Lemma 2.4.5, we have that $p = b$. However, in such case the triplet (q, ∞, p) cannot be KT-acceptable, unless if $q = r = p = b = c = \infty$. The necessity of the latter is proved below in this paragraph, see Lemma 2.9.4. $\square$

We next propose a generalization to other Strichartz norms of the geometric representation for the $L^\infty_x L^1_v$-norm given above. The only application of it that we make here is to proving Lemma 2.9.4, which shall be reproved on an easier counterexample in what follows. Therefore the remaining of this paragraph is only informative and can be skipped.

Let us define the quantity

$$N_r(\theta, Q) = \|R(\pi/2 - \theta)\chi_Q\|_{L^r_x L^1_v} ,$$

where by $R(\pi/2 - \theta)$ we have denoted rotation around the origin by the angle $\pi/2 - \theta$, and $\cot\theta = -t$. By a simple geometric reasoning it is not hard to see that we have the identity

$$\int_{-\infty}^{\infty} \left(\int_{\gamma_{x,t}} \chi_Q(l)dl \right)^r dx = \sqrt{1+t^2} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} R(\pi/2 - \theta)\chi_Q(l, m)dl \right)^r dm,$$

where dl and dm lie on orthogonal axes. (Make a diagram!) Thus, we obtain the representation

$$\begin{aligned}
\|U(t)\chi_Q\|_{L^q_t L^r_x L^1_v} &= \left\| \left(\frac{1}{\sqrt{1+t^2}} \right)^{1/r'} N_r(\theta, Q) \right\|_{L^q(\mathbb{R})} \\
&= \left\| (\sin\theta)^{1/r' - 2/q} N_r(\theta, Q) \right\|_{L^q(0,\pi)} .
\end{aligned}$$

By considering the Cartesian product $Q^n = Q\times Q\times \dots Q \in \mathbb{R}^{2n}$, we obtain the representation of the Strichartz norms in n spatial dimensions

$$\|U(t)\chi_{Q^n}\|_{L^q_t L^r_x L^1_v} = \left\| \left(\frac{1}{\sqrt{1+t^2}}\right)^{n/r'} N_r^n(\theta, Q) \right\|_{L^q(\mathbb{R})}.$$

Lemma 2.9.4. *The following conditions*

$$r \geq p, \quad \frac{1}{q} < n\left(\frac{1}{p} - \frac{1}{r}\right), \quad or \;\; q = \infty, \; 1 \leq p = r \leq \infty, \tag{2.83}$$

are necessary for the validity of the the generalized homogeneous Strichartz estimates (2.13).

Proof. For example, let Q be the unit disk on the x_1-v_1-plane. Then obviously $N_r(\theta, Q) =$ const for all $\theta \in [0, \pi]$. Hence, the norm above is only finite if $r \geq 1$, $qn/r' > 1$, or if $q = \infty$ and $r = 1$. In view of the power invariance (2.9), we generalize this example to any exponent triplet (q, r, p), and obtain the restrictions (2.83) to the range of validity of the generalized homogeneous Strichartz estimates (2.13). □

2.9.2 Failure of the L^∞ norm in the inhomogeneous estimates

The idea to use characteristic functions of Besicovitch sets on the plane in order to show the failure of the homogeneous Strichartz estimates (2.82) cannot be applied in a straightforward way in the inhomogeneous setting. Instead, we shall apply an argument similar to that in the first counterexample of Guo and Peng [25].

Lemma 2.9.5. *All Strichartz estimates in $\mathbb{R}^n$ of the form*

$$\|W(t)F\|_{L^q_t L^\infty_x L^p_v} \lesssim \|F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x L^{\tilde{p}'}_v}, \quad 1 \leq q, p, \tilde{q}, \tilde{r}, \tilde{p} \leq \infty, \tag{2.84}$$

fail in the range $n/(n-1) \leq \tilde{r} \leq \infty$ $(1 \leq \tilde{r}' \leq n)$ for some $F \in L^{\tilde{q}'}_t L^{\tilde{r}'}_x L^{\tilde{p}'}_v$.

Proof. We shall find a function F explicitly for which (2.86) fails. Suppose that

$$F(t, x, v) = \phi(t)\psi(x)\xi(v) \geq 0, \quad \phi \in L^{\tilde{q}'}(\mathbb{R}), \; \psi \in L^{\tilde{r}'}(\mathbb{R}^n), \; \psi \in L^{\tilde{p}'}(\mathbb{R}^n).$$

We choose $\phi(t)$, $\psi(x)$, and $\xi(v)$ as follows

$$\begin{aligned} \phi(t) &= \chi(0 \leq t \leq 1), \\ \psi(x) &= \frac{1}{|x|^{n/\tilde{r}'}(-\ln|x|)^{1/\tilde{r}'+\epsilon}}\chi(|x| < 1/2), \\ \xi(v) &= \frac{1}{|v|^{n/\tilde{p}'}(-\ln|x|)^{1/\tilde{p}'+\epsilon}}\chi(|x| < 1/2). \end{aligned}$$

We take also the function $g \in L^{p'}(\mathbb{R}^n)$,

$$g(v) = \frac{1}{|v|^{n/p'}(-\ln|x|)^{1/p'+\epsilon}}\chi(|x| < 1/2).$$

Note also that due to the scaling invariances in Lemma 2.4.5, we have $1/p' + 1/\tilde{r}' + 1/\tilde{p}' = 1$. We have the following simple inequality

$$\|W(t)F\|_{L^\infty_x L^p_v} \geq \left\| \int_{\mathbb{R}^n} W(t)Fg(v)dv \right\|_{L^\infty_x}.$$

The latter integral can be rewritten as

$$\int_{\mathbb{R}^n} W(t)Fg(v)dv = \int_0^t \int_{\mathbb{R}^n} \phi(s)\psi(x-(t-s)v)\xi(v)g(v)dvds.$$

The function

$$h(t,x) = \int_{\mathbb{R}^n} \phi(s)\psi(x - tv)\xi(v)g(v)dv$$

is continuous in x, for $t > 0$. Indeed, consider the inequality

$$|h(t,x_1) - h(t,x_2)| \le \|\psi(x_1 - tv) - \psi(x_2 - tv)\|_{L_v^{\tilde{r}'}} \|\xi g\|_{L_v^{\tilde{r}}}.$$

Since $\tilde{r}' < \infty$, we have that the first norm in the line above tends to zero as $|x_2 - x_1| \to 0$. Hence,

$$\|W(t)F\|_{L_x^\infty L_v^p} \ge \int_0^t h(t-s,0)ds. \tag{2.85}$$

For $0 < t < 1/2$, we have that

$$h(t,0) \ge \frac{1}{t^{n/\tilde{r}'}} \int_{|v|\le t} \frac{1}{|v|^n} \frac{1}{(-\ln|v|)^{1+3\epsilon}} dv \gtrsim \frac{1}{t^{n/\tilde{r}'}(\ln t)^{3\epsilon}}.$$

Hence, the norm in (2.85) is infinite whenever $n/\tilde{r}' \ge 1$. ☐

Lemma 2.9.6. *All Strichartz estimates in $\mathbb{R}^n$ of the form*

$$\|W(t)F\|_{L_t^q L_x^r L_v^p} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^1 L_v^{\tilde{p}'}}, \quad 1 \le q,p,r,\tilde{q},\tilde{p} \le \infty, \tag{2.86}$$

fail in the range $n/(n-1) \le r \le \infty$ for some $F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}$.

Proof. Follows by Lemma 2.9.5, in view of the Equivalence theorem 2.3.5. ☐

In particular, the double endpoint in $\mathbb{R}$

$$\|W(t)F\|_{L_t^2 L_x^\infty L_v^1} \lesssim \|F\|_{L_t^2 L_x^1 L_v^\infty}$$

fails.

2.9.3 Homogeneous estimates

By scaling, that is Lemma 2.4.5, estimate

$$\|U(t)f\|_{L_t^q L_x^r L_v^p} \lesssim \|f\|_{L_{x,v}^a}, \quad \forall f \in L_{x,v}^a, \tag{2.12}$$

holds only if

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{a}, \quad a = \text{HM}(p,r).$$

Let us next find the upper bound $r \le r^*(a)$. It is enough to consider only the special case $a = 2$. We shall prove the equivalent condition $q \ge 2$. (In general $r \le r^*(a)$ and $q \ge a$ are equivalent.) The claim follows directly by the translation invariance in t of the TT^*-operator. Indeed, first recall that estimate (2.12) with $a = 2$ is equivalent to

$$\|TT^*F\|_{L_t^q L_x^r L_v^{r'}} \lesssim \|F\|_{L_t^{q'} L_x^{r'} L_v^r}, \quad \forall F \in L_t^{q'} L_x^r L_v^{r'}.$$

Then, in view of the famous Hörmander's lemma 2.9.7, we have that $q \ge q'$, or equivalently $q \ge 2$.

We remark in passing that by translation invariance in x we also have $r \ge r'$, or equivalently $r \ge 2$. The latter two inequalities have already been proved in Lemma 2.9.4, and in particular are equivalent to the first inequality in (2.83).

As usual, the general case (for $0 < a < \infty$) follows by the power invariance (2.9).

Lemma 2.9.7 (Hörmander [27]). *Whenever a (non-trivial) linear and bounded operator maps a vector-valued L^p-space to another vector-valued L^q-space, $1 \le p, q \le \infty$, and additionally this operator is translation invariant, then we must have that $p \le q$.*

2.9.4 Generalized homogeneous estimates

Let us consider the homogeneous Strichartz estimate

$$\|U(t)f\|_{L_t^q L_x^r L_v^p} \lesssim \|f\|_{L_x^b L_v^c}, \quad \forall f \in L_x^b L_v^c, \tag{2.13}$$

for data outside the transport class. Most of the arguments from the preceding paragraph apply to this case as well. By scaling, we have that the conditions

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{b}, \quad \mathrm{HM}(p,r) = \mathrm{HM}(b,c) \stackrel{def}{=} a, \tag{2.87}$$

are necessary. The conditions $p \le a \le r$ and $a \le r$ have already been proved in Lemma 2.9.4.

Let us now verify that $b \le c$. To that end see that estimate (2.13) is equivalent to

$$\|TT^*F\|_{L_t^q L_x^r L_v^p} \lesssim \|F\|_{L_t^1 L_x^b L_v^c}, \quad \forall F \in L_t^1 L_x^b L_v^c.$$

By duality, the latter is equivalent to

$$\|TT^*F\|_{L_t^\infty L_x^{b'} L_v^{c'}} \lesssim \|F\|_{L_t^{q'} L_x^{r'} L_v^{p'}}, \quad \forall F \in L_t^{q'} L_x^{r'} L_v^{p'}.$$

The exponent pair must be KT-acceptable (proved in the next paragraph), which directly gives the required claim. In fact, conditions (2.83) and (2.87)) imply that either $p < b < a < c < r$ ($p < b$), or $a = b = c = p = r$ and $q = \infty$.

We do not have a suitable counterexample showing the necessity of the upper bound $r^*(c)$ in Theorem 2.3.7 for the validity of the generalized homogeneous estimates (in the case when $b \ne c$, $n > 1$).

2.9.5 Global inhomogeneous estimates

Let us consider now the inhomogeneous Strichartz estimate

$$\|W(t)F\|_{L_t^q L_x^r L_v^p} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}}. \tag{2.14}$$

By scaling, see Lemma 2.4.5, we obtain that the restrictions

$$\frac{1}{q} + \frac{1}{\tilde{q}} = n\left(1 - \frac{1}{r} - \frac{1}{\tilde{r}}\right), \quad \mathrm{HM}(p,r) = \mathrm{HM}(\tilde{p}', \tilde{r}') \stackrel{def}{=} a,$$

are necessary.

Consider $F(t,x,v) = \chi\left(0 \le t \le 1, |x| \le 1, |v| \le 1\right)$. When $t \gg 1$ we have that

$$\{TT^*F\}(t) = W(t)F \approx \chi\left(\left|v - \frac{x}{t}\right| \le \frac{1}{t}, |v| \le 1\right) \approx \chi\left\{v \sim \frac{1}{t}, x \sim t\right\}.$$

Hence,

$$\|W(t)F\|_{L_x^r L_v^p} \sim t^{\frac{n}{r} - \frac{n}{p}}, \quad t \gg 1.$$

It follows that $\|W(t)F\|_{L_t^q L_x^r L_v^p} < \infty$ only if

$$\left(\frac{n}{r} - \frac{n}{p}\right) q < -1, \quad \text{or if } q = \infty, r = p. \tag{2.88}$$

By the duality Lemma 2.5.3, the dual exponents $(\tilde{q}, \tilde{r}, \tilde{p})$ must also satisfy (2.88). Thus we have that the conditions $p \le r$ and $\tilde{p} \le \tilde{r}$ are necessary for the validity of estimate (2.14). The same conclusion applies for the TT^*-operator.

We now show that conditions

$$\frac{1}{q}+\frac{1}{\tilde{q}}\leq 1,\quad \frac{1}{r}+\frac{1}{\tilde{r}}\leq 1,$$

are necessary for the validity of estimate (2.14). Indeed, the claim follows from the translation invariance of $W(t)$ in t and x and Hörmander's lemma 2.9.7. Let us check this fact for t. Consider $F_\tau(t)=F(t-\tau)$ and $W(t)F_\tau$. We have

$$\int_{-\infty}^{t} U(t-s)F(s-\tau)ds=\int_{-\infty}^{t-\tau} U(t-\tau-\sigma)F(\sigma)d\sigma,$$

or in other words $W(t)F_\tau = W(t-\tau)F$. Thus we have verified the necessity of the condition that (q,r,p) and $(\tilde{q},\tilde{r},\tilde{p})$ must be two jointly KT-acceptable exponent triplets, apart from the additional restrictions in Definition 2.2.4.

We do not have a suitable counterexample showing the necessity of condition

$$\frac{n-1}{p'}\leq\frac{n}{\tilde{r}},\quad \frac{n-1}{\tilde{p}'}\leq\frac{n}{r},\quad n>1. \tag{2.21}$$

However, we can show that the similar condition

$$\frac{n}{p'}<\frac{1}{\tilde{q}}+\frac{n}{\tilde{r}},\quad \frac{n}{\tilde{p}'}<\frac{1}{q}+\frac{n}{r}, \tag{2.21a}$$

is sharp. Indeed, (2.21a) is a direct consequence of the two conditions (2.18), (2.19). Condition (2.21a) implies (2.21) whenever $p'\leq\tilde{q}$ and $\tilde{p}'\leq q$. Thus, if there are some other global inhomogeneous estimates for $W(t)$ not included in Theorem 2.3.2, they must belong to the range $\tilde{q}<p'$ or $q<\tilde{p}'$.

2.9.6 Local inhomogeneous estimates

In this paragraph we show the fact that in the context of the KT equation the local inhomogeneous estimates do not exist in a "full neighborhood" around a given local inhomogeneous Strichartz estimate. This presents an obstruction for the application of the perturbative techniques of Keel and Tao [30], and their improvement by Foschi [20]. Since no other method have been found to treat the endpoint estimates (of the type that lie on the hypothenuse AB in fig. 2.1), these estimates remain unresolved.

For example, consider the estimate

$$\|W(t)F\|_{L_t^q([2,3];L_x^rL_v^p)}\lesssim\|F\|_{L_t^{\tilde{q}'}([1,2];L_x^{\tilde{r}'}L_v^{\tilde{p}'})}. \tag{2.89}$$

Take $F(t,x,v)=\chi\,(t\in[0,1],\ (x,v)\in Q_R)$, where by Q_R we denote the cube of side length $2R$ centered at the origin of $\mathbb{R}^{2n}$. If we denote $\|x\|_\infty=\sup_{1\leq i\leq n}|x_i|$, for $x=(x_1,..,x_n)$, we can write the latter as

$$Q_R=\{(x,v):\|x\|_\infty\leq R,\ \|v\|_\infty\leq R\}.$$

Hence,

$$\|F\|_{L_t^{\tilde{q}'}([1,2];L_x^{\tilde{r}'}L_v^{\tilde{p}'})}\sim R^{\frac{n}{\tilde{p}'}+\frac{n}{\tilde{r}'}}.$$

We now set $\tau=t-s$, and consider the set $Q_R(\tau)$ given by

$$\|x-(t-s)v\|_\infty\leq R,\ \|v\|_\infty\leq R.$$

Then, for $t\in[2,3]$, $s\in[0,1]$, equivalently for $\tau\in[1,3]$, we have the inclusions

$$Q_{R/4}\subset Q_R(\tau)\subset Q_{4R}.$$

Hence,

$$\|W(t)F\|_{L^q_t([2,3];L^r_x L^p_v)} \sim R^{\frac{n}{p}+\frac{n}{r}}.$$

We conclude that condition

$$\frac{1}{r} + \frac{1}{p} = \frac{1}{\tilde{r}'} + \frac{1}{\tilde{p}'}$$

is necessary for the validity of the local estimates (2.89).

Chapter 3

Application to Kinetic Chemotaxis

3.1 Introduction and main results

Chemotaxis is a process in which bacteria, or, more generally, cells, change their state of movement, reacting to the presence of chemical substance called chemoattractant, approaching chemically favorably environments and avoiding unfavorable ones. Generally, the movement of bacteria is composed of two different phases, a "run" phase and a "tumble" phase. The run phase consists of a directed movement in a straight line, while the "tumble" phase is the reorientation to a new direction.

We denote the chemoattractant $S(t,x)$ at time $t \in [0,\infty)$ and position $x \in \mathbb{R}^n$. The cell density in phase space is denoted by $u(t,x,v)$ and its integral over all possible velocities, which is assumed to be the bounded set $V \subset \mathbb{R}^n$, is the cell density

$$\rho(t,x) = \int_{v \in V} u(t,x,v)dv$$

in physical space.

The kinetic model of chemotaxis proposed by Othmer-Dunbar-Alt, see e.g. [14], reads

$$\partial_t u(t,x,v) + v \cdot \nabla_x u(t,x,v) = \int_{v' \in V} T[S](t,x,v,v')u(t,x,v')dv', \tag{3.1}$$

$$- \int_{v' \in V} T[S](t,x,v',v)u(t,x,v)dv', \quad t > 0,\ x \in \mathbb{R}^n, v \in V$$

$$- \Delta_x S(t,x) + S(t,x) = \rho(t,x) \stackrel{def}{=} \int_{v \in V} u(t,x,v)dv, \tag{3.2}$$

$$u(0,x,v) = f(x,v) \geq 0. \tag{3.3}$$

Here, the free transport operator $\partial_t u(t,x,v) + v \cdot \nabla_x u(t,x,v)$ describes the free runs of the bacteria which have velocity $v \in V$. The right hand side of (3.1) denotes a scattering operator whose first term describes turning into direction v, and the second term turning away from v. More specifically, in this model the tumble (the reorientation) is a Poisson process with rate

$$\lambda[S] = \int_V T[S](t,x,v',v)dv',$$

and $T[S](t,x,v',v)/\lambda[S]$ is the probability density for a change in velocity from v to v', given that a reorientation occurs for a cell at position x, velocity v, and time t.

The Cauchy problem (3.1)-(3.3) was first studied in [14] (2004) where global existence was proved in dimension $n = 3$ for nonnegative initial data $f \in L^1 \cap L^\infty$ under the assumption that the turning kernel satisfies the structural condition

$$0 \le T[S](t,x,v,v') \le C(1 + S(t,x+v) + S(t,x-v')).$$

The meaning of the term $S(t,x-v')$ is that the cells measure the concentration of the chemical S at position $x-v'$ before changing their direction at position x, because of an internal memory effect. The meaning of the term $S(t,x+v)$ is that the cells are able to measure the concentration at location $x+v$ thanks to censorial protrusions.

However, based on experimental data, it is believed that the reorientation of the bacteria depends on the changes in concentration of the chemoattractant. Thus, in a more realistic model the turning kernel should depend not only on S, but also on its gradient ∇S (the x variables). Let us consider the most general condition on T

$$\begin{aligned} 0 \le T[S](t,x,v,v') \le C_1 + C_2 S(t,x+v) + C_3 S(t,x-v') + \\ C_4 |\nabla S(t,x+v)| + C_5 |\nabla S(t,x-v')|. \end{aligned} \tag{3.4}$$

The method of [14] was adapted in [28] to include turning kernels satisfying

$$0 \le T[S](t,x,v,v') \le C(1 + S(t,x+v) + |\nabla S(t,x+v)|),$$

or

$$0 \le T[S](t,x,v,v') \le C(1 + S(t,x-v') + |\nabla S(t,x-v')|),$$

under the same assumptions for the initial data.

The first successful attempt to consider the most general kernel in 3d, i.e. (3.4), was made in [8]. The authors replace condition (3.4) (with $C_1 = 0$) with the more general

$$\|T[S](t,\cdot,\cdot,\cdot)\|_{L^r_x L^{p_1}_v L^{p_2}_{v'}} \lesssim_{|V|,p_1,p_2} \|S(t,\cdot)\|_{L^r_x} + \|\nabla S(t,\cdot)\|_{L^r_x}, \tag{3.5}$$

whenever $r \ge p_1$, p_2, see [8, Theorem 3]. They establish the existence of a global weak solution for small enough initial data $f \in L^1 \cap L^a$, where $a \in [3/2, 2]$. However, the authors do not prove uniqueness of the solution and work in data classes that are not preserved by the evolution of the system. The new feature of their approach is the use of Strichartz estimates for the kinetic transport equation derived in [11]. We shall adapt their method but based on the larger set of inhomogeneous Strichartz estimates that we derive in the present work. Our proof shall use more delicate spacetime estimates on the chemoattractant S, unlike the proof in [8] that uses estimates on S only for fixed time, and use a double bootstrap argument. Because our aim is to show global well-posedness of the solution we need to consider differences $T[S_1] - T[S_2]$, together with structural condition (3.5) we impose the natural condition

$$\begin{aligned} \|T[S_1](t,\cdot,\cdot,\cdot) - T[S_2](t,\cdot,\cdot,\cdot)\|_{L^r_x L^{p_1}_v L^{p_2}_{v'}} \lesssim_{|V|,p_1,p_2} \\ \|S_1(t,\cdot) - S_2(t,\cdot)\|_{L^r_x} + \|\nabla S_1(t,\cdot) - \nabla S_2(t,\cdot)\|_{L^r_x}, \end{aligned} \tag{3.6}$$

whenever $r \ge p_1$, p_2.

Our result is presented in

Theorem 3.1.1. *The Cauchy problem* (3.1)-(3.3), (3.5), (3.6), *is globally well-posed for small data $f \ge 0$ in the class $f \in L^1(\mathbb{R}^n \times V) \cap L^a(\mathbb{R}^n \times V)$ for $3n/4 < a < 3$ and $n = 2,3$. More specifically, for every $3n/4 < a < 3$ there exist a fixed positive constant M, depending only on the space dimension n, the constants in the structural conditions* (3.5), (3.6), *and on the Lebesgue exponent a, such that whenever $\|f\|_{L^a_{x,v}} < M$ the considered problem admits a unique nonnegative solution*

$$u(t) \in C([0,\infty); L^1(\mathbb{R}^n \times V) \cap L^a(\mathbb{R}^n \times V))$$

for which

$$\|\rho\|_{L^3_t L^{3na/(3n-a)}_x} < \infty, \quad \|S\|_{L^3_t L^\infty_x} + \|\nabla S\|_{L^3_t L^\infty_x} < \infty.$$

3.2 Maximum principle for the kinetic transport equation

In this paragraph we present the maximum principle for the Cauchy problem for the inhomogeneous kinetic transport equation

$$\begin{aligned}\partial_t u(t,x,v) + v\cdot\nabla_x u(t,x,v) + g(t,x,v)u(t,x,v) = \\ \int_V h(t,x,v,v')u(t,x,v')dv' + F(t,x,v),\end{aligned} \tag{3.7}$$

$$u(0,x,v) = f(x,v). \tag{3.8}$$

The results are standard, see for example the references cited in [40, pp. 226, 227]. However, we have adapted their presentation to the current context, and in order to make the exposition self-contained, we shall give proofs.

Let us assume that $(t,x,v)\in\mathbb{R}\times\mathbb{R}^n\times V$, $V\subseteq\mathbb{R}^n$, and that the kernel $h(t,x,v,v')$ in (3.7) is nonnegative.

Lemma 3.2.1 (Maximum principle for the kinetic transport equation)**.** *Suppose that $u(t)$ satisfies* (3.7), (3.8) *with $F(t,x,v)\geq 0$ and $f(x,v)\geq 0$, which are smooth enough to guarantee the construction of solution to* (3.7), (3.8) *by means of an iteration scheme. Then $u(t,x,v)$ remains nonnegative for all time.*

Proof. We shall use the notation

$$Q(t,x,v) = \int_V h(t,x,v,v')u(t,x,v')dv' + F(t,x,v).$$

The claim follows directly from the representation of (3.7), (3.8) as an integral equation

$$\begin{aligned}u(t,x,v) = f(x-tv,v)\exp\left(-\int_0^t g(s,x-(t-s)v,v)ds\right) + \\ \int_0^t Q(s,x-(t-s)v,v)\exp\left(-\int_0^{t-s} g(\tau+s,x-(t-s-\tau)v,v)d\tau\right)ds.\end{aligned} \tag{3.9}$$

The solution to (3.9) can be constructed by an iteration scheme whenever f, g, h and F are regular enough. It is easy to see that on each step we obtain nonnegative solutions. □

Corollary 3.2.2 (Comparison principle for the kinetic transport equation)**.** *Suppose that $u_1(t)$ solves*

$$\partial_t u_1(t,x,v) + v\cdot\nabla_x u_1(t,x,v) = F_1(t,x,v), \quad u(0) = f_1(x,v),$$

and that $u_2(t)$ solves

$$\partial_t u_2(t,x,v) + v\cdot\nabla_x u_2(t,x,v) = F_2(t,x,v), \quad u(0) = f_2(x,v),$$

where $F_1(t,x,v)\leq F_2(t,x,v)$ and $f_1(x,v)\leq f_2(x,v)$. Then $u_1(t)\leq u_2(t)$ for all time.

Proof. Consider the difference of the above two equations and apply the maximum principle of Lemma 3.2.1. □

We shall call the following kinetic transport equation

$$\begin{aligned}\partial_t u(t,x,v) + v\cdot\nabla_x u(t,x,v) = \\ \int_{\mathbb{R}^n} K(v,v')u(t,x,v') - K(v',v)u(t,x,v)dv'\end{aligned} \tag{3.10}$$

the scattering kinetic equation, and the function $K(v,v')\geq 0$ — a scattering kernel. For simplicity we have suppressed any other arguments in K, but it may also depend on u, t, and x. In particular, for the kernel K we can choose the operator $T[S]$ in equation (3.1).

Corollary 3.2.3. *The solution $u(t)$ to the Cauchy problem* (3.1) - (3.3) *satisfies the following bound*

$$0 \le u(t,x,v) \le u_1(t,x,v) + u_2(t,x,v), \tag{3.11}$$

where

$$\partial_t u_1(t,x,v) + v \cdot \nabla_x u_1(t,x,v) = 0, \quad u_1(0) = f,$$

and

$$\partial_t u_2(t,x,v) + v \cdot \nabla_x u_2(t,x,v) = \int_{v' \in V} T[S](t,x,v,v')u(t,x,v')dv', \quad u_2(0) = 0.$$

Proof. The first inequality in (3.11) follows from the maximum principle of Lemma 3.2.1. Indeed, set $g(t,x,v) = \int_V K(v,v')dv'$, and $h(t,x,v,v') = K(v',v)$. The second one follows from the comparison principle of Corollary 3.2.2. □

3.3 Proof of Theorem 3.1.1

In view of Corollary 3.2.3 the estimate

$$\|u\|_{L_t^q L_x^r L_v^p} \lesssim \|f\|_{L_{x,v}^a} + \left\| \int_V T[S]u'dv' \right\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'} L_v^{\tilde{p}'}}. \tag{3.12}$$

holds whenever the exponent triplets (q,r,p) and $(\tilde{q},\tilde{r},\tilde{p})$ are KT-admissible. Here we have used the abbreviation $u' = u(t,x,v')$. Let us now apply Hölder's inequality to the integral term above to get

$$\int_V T[S](t,x,v,v')u(t,x,v')dv' \le \|T[S](t,x,v,\cdot)\|_{L_{v'}^{p'}} \|u(t,x,\cdot)\|_{L_{v'}^p}.$$

We next take the $L^{\tilde{p}'}$-norm in v, and then — the $L^{\tilde{r}'}$-norm in x, and use Hölder's inequality with $\frac{1}{\tilde{r}'} = \frac{1}{r} + \frac{1}{w}$, to get

$$\left\| \int_V T[S]u(t,x,v')dv' \right\|_{L_x^{\tilde{r}'} L_v^{\tilde{p}'}} \le \|T[S](t,\cdot,\cdot,\cdot)\|_{L_x^w L_v^{\tilde{p}'} L_{v'}^{p'}} \|u(t,\cdot,\cdot)\|_{L_x^r L_v^p}.$$

We impose $w \ge \tilde{p}'$, p' so that we can use the structural condition (3.5) on the turning kernel T. We have

$$\|T[S](t,\cdot,\cdot,\cdot)\|_{L_x^w L_v^{\tilde{p}'} L_{v'}^{p'}} \lesssim \|G * \rho\|_{L_x^w} + \|\nabla G * \rho\|_{L_x^w}.$$

Here, $S = G * \rho$, and by G we have denoted the Bessel potential

$$G(x) = \int_0^\infty e^{-\pi \frac{|x|^2}{4s}} s^{\frac{-n+2}{2}} \frac{ds}{s}.$$

We now recall that $G \in L^b(\mathbb{R}^n)$, for $1 \le b < \frac{n}{n-2}$, and that $\nabla G(x) \in L^c(\mathbb{R}^n)$, for $1 \le c < \frac{n}{n-1}$, see [28]. Hence, applying Young's inequality, we obtain the following two estimates on the chemoattractant

$$\|G * \rho(t)\|_{L_x^\infty} \lesssim \|G\|_{L^c} \|\rho(t)\|_{L^r},$$
$$\|\nabla G * \rho(t)\|_{L_x^\infty} \lesssim \|\nabla G\|_{L^c} \|\rho(t)\|_{L^r},$$

where

$$1 - \frac{1}{n} < \frac{1}{c} = 1 - \frac{1}{w} - \frac{1}{r} = 1 + \frac{1}{\tilde{r}'} - \frac{2}{r}.$$

We next use the fact that $\|\rho\|_{L^r} \lesssim_{|V|} \|u(t,\cdot,\cdot)\|_{L^r_x L^p_v}$, and choose $\tilde{q}' = q/2$, to obtain

$$\left\| \int_V T[S]u'dv' \right\|_{L^q_t L^r_x L^p_v} \lesssim \left\| \|u(t,\cdot,\cdot)\|^2_{L^r_x L^p_v} \right\|_{L^{q/2}_t} \lesssim \|u\|^2_{L^q_t L^r_x L^p_v}.$$

Hence, we obtain the following a-priori estimate

$$\|u\|_{L^q_t L^r_x L^p_v} \lesssim \|f\|_{L^a_{x,v}} + \|u\|^2_{L^q_t L^r_x L^p_v}, \tag{3.13}$$

whenever the following system

$$\begin{gathered} \mathrm{HM}(r,p) = \mathrm{HM}(\tilde{r}',\tilde{p}') = a, \\ \frac{1}{q} = \frac{n}{2}\left(\frac{1}{p} - \frac{1}{r}\right), \quad \frac{1}{\tilde{q}} = \frac{n}{2}\left(\frac{1}{\tilde{p}} - \frac{1}{\tilde{r}}\right), \\ \tilde{q}' = q/2, \quad q > a, \quad \tilde{q} \geq a', \quad n \geq 2, \quad r > n, \\ \left(\frac{1}{w} :=\right) 1 - \frac{1}{r} - \frac{1}{\tilde{r}} \leq \frac{1}{\tilde{p}'}, \frac{1}{p'}, \quad 1 + \frac{1}{\tilde{r}'} - \frac{2}{r} > 1 - \frac{1}{n}, \end{gathered}$$

admits a solution. By direct inspection we see that

$$\begin{gathered} \left(\frac{1}{q}, \frac{1}{r}, \frac{1}{p}\right) = \left(\frac{1}{3}, \frac{1}{a} - \frac{1}{3n}, \frac{1}{a} + \frac{1}{3n}\right) \\ \left(\frac{1}{\tilde{q}'}, \frac{1}{\tilde{r}'}, \frac{1}{\tilde{p}'}\right) = \left(\frac{2}{3}, \frac{1}{p}, \frac{1}{r}\right), \end{gathered}$$

for $3n/4 < a < 3$ and $n = 2, 3$ is a solution. By a standard bootstrap argument on estimate (3.13), there exist some fixed positive constant M, depending on the space dimension n and the constant in estimate (3.5), such that whenever $\|f\|_{L^a_{x,v}} < M$

$$\|u\|_{L^3_t L^{3na/(3n-a)}_x L^{3na/(3n+a)}_v} < \infty.$$

This estimate trivially implies that

$$\|\rho\|_{L^3_t L^{3na/(3n-a)}_x} < \infty, \quad \|S\|_{L^3_t L^\infty_x} + \|\nabla S\|_{L^3_t L^\infty_x} < \infty. \tag{3.14}$$

Hence, considering again equation (3.1), and using the estimate on S above, we obtain the following Gronwall's inequality

$$\sup_{t\in[0,T]} \|u(t)\|^{3/2}_{L^a_{x,v}} \lesssim_T \|f\|^{3/2}_{L^a_{x,v}} + \int_0^T \|u(s)\|^{3/2}_{L^a_{x,v}} \, ds,$$

for all $0 < T < \infty$. Hence,

$$\sup_{t\in[0,T]} \|u(t)\|_{L^a_{x,v}} < \infty, \quad \forall T \in (0,\infty). \tag{3.15}$$

We next sketch the proof of the local well-posedness of the system. Coupled with the global estimate (3.15), the full claim will then follow easily. Let us write equations (3.1) - (3.3) in the equivalent integral form

$$u(t) = U(t)f + \int_0^t U(t-s)F(u(s))ds, \tag{3.16}$$

where $F(u)$ is the right hand side of (3.1). Define the right hand side of (3.16) as the operator

$$K : L^\infty([0,T], L^a(\mathbb{R}^{2n})) \to L^\infty([0,T], L^a(\mathbb{R}^{2n})),$$
$$K(u) = U(t)f + \int_0^t U(t-s)F(u(s))ds.$$

This nonlinear operator is bounded on the cited spaces as it can be easily seen from the estimate

$$\sup_{t\le T} \|K(u(t))\|_{L^a_{x,v}} \lesssim_T \|f\|_{L^a_{x,v}} + \int_0^t \|S(\tau)\|_{L^\infty_x} \|u(\tau)\|_{L^a_{x,v}} ds.$$

Hence, for $3n/4 < a < 3$, $a < M$, and any $0 < T < \infty$ the operator K is bounded.

Let $u_1(t)$, $u_2(t) \in L^\infty([0,T], L^a(\mathbb{R}^{2n}))$. Similarly, we have the estimate

$$\sup_{t\in[0,T]} \|K(u_1(t) - u_2(t))\|_{L^a_{x,v}} \le q \sup_{t\in[0,T]} \|u_1(t) - u_2(t)\|_{L^a_{x,v}},$$

for some $0 < q < 1$, whenever $T > 0$ is small enough. The theorem follows by standard arguments henceforth.

Remark 3.3.1. In the proof above we have constructed the solution and proved uniqueness to the reduced chemotaxis system where equation (3.1) is replaced by

$$\partial_t u(t,x,v) + v \cdot \nabla_x u(t,x,v) = \int_{v'\in V} T[S](t,x,v,v')u(t,x,v')dv'$$

and all other equations and conditions remain the same. We showed that the associated integral operator to that system is a contraction map and thus the solution can be constructed via an iteration scheme which is convergent. Recall formula (3.9) giving the solution to the scattering kinetic equation (3.1). In terms of the notation there we have

$$g(t,x,v) = \int T[S](t,x,v',v)dv'$$

and that the exponential terms in (3.9) are positive, away from zero, and less than one. Indeed, this follows from the fact that

$$\sup_{0\le t\le T} \left| \int_0^t g(s, x-(t-s)v, v)ds \right| \lesssim_T \|T[S]\|_{L^3_t L^\infty_{x,v',v}}$$

which is bounded in view of the key a-priori estimate (3.14) on the chemoattractant S and the structural assumption (3.5) on T. Thus the solutions to the two problems are comparable on every finite time interval and therefore the theorem follows from the proof above.

Chapter 4

Strichartz Estimates with Spherical Symmetry

Strichartz estimates with spherical symmetry have attracted a lot of interest recently. The gain of regularity of these estimates over the standard Strichartz estimates varies with the equation but, for example, in the context of the wave equation this gain is significant. Most attention has been dedicated to the homogeneous setting, see Sterbenz [43], Fang and Wang [19], Hidano and Kurokawa [26], Machihara et al [33], Tao [48], and Vilela [49], with only a few special inhomogeneous estimates being proved. Below, we produce a range of inhomogeneous Strichartz estimates with spherical symmetry analogous to the inhomogeneous Strichartz estimates in the standard setting. To this end we restrict the standard spaces of functions to their subspaces of spherically symmetric functions and proceed with the TT^*-argument.

4.1 Inhomogeneous Strichartz estimates with spherical symmetry

In order to present the proof of the Strichartz estimates with spherical symmetry in the abstract setting we need to introduce an appropriate framework.

Consider a Hilbert space H and an unitary operator R on H, that is for all $f, g \in H$, we have the identity

$$\langle Rf, g\rangle_H = \langle f, R^{-1}g\rangle_H. \tag{4.1}$$

Consider now the subspace $\mathcal{H}$ of H consisting of all elements in H that are invariant to R, that is $f = Rf$. We shall further assume that $\mathcal{H}$ is closed and thus a Hilbert space itself, with the same scalar product as H. The nature of the operator R is immaterial but we shall have the following model as an example. Take $H = L^2(\mathbb{R}^n)$, let $f \in L^2(\mathbb{R}^n)$, and let R be a spatial rotation on $\mathbb{R}^n$. We define the unitary operator $R : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by $Rf = f(Rx)$. The invariant functions are spherically symmetric. From now on, even in the abstract context, we shall call the operator R rotation, and all elements that are invariant with respect to R, spherically symmetric.

Suppose that we are given a family of Strichartz estimates

$$\|U(t)f\|_{L^q_t(I;\mathcal{B}_\theta)} \lesssim \|f\|_H \tag{4.2}$$

that hold for all spherically symmetric $f \in H$, where B_θ are a family of Banach spaces. An exponent pair (q, θ) for which estimate (4.2) holds under the assumptions we have made shall be called radially-admissible, and the full range of radially-admissible pairs shall be denoted by the set $\mathcal{A} \subseteq [1, \infty] \times [0, 1]$. We assume that in the family of estimates (4.2) there is the energy inequality

$$\|U(t)f\|_{H} \lesssim \|f\|_H, \quad \forall t \in I \subseteq \mathbb{R},$$

which shall be used to define the adjoint operator $U^*(t)$ to the evolution group $U(t)$ through the identity

$$\langle U(t)f, g\rangle_H = \langle f, U^*(t)g\rangle_H, \quad \forall f, g \in H, \; \forall t \in I. \tag{4.3}$$

We assume also the group property

$$U(t)U^*(s) = U(t-s)$$

and that $U(t)$ commutes with spatial rotations,

$$U(t)[Rf] = R[U(t)f], \quad \forall t \in I. \tag{4.4}$$

As an easy consequence of (4.4), (4.3), and (4.1), we obtain that the dual $U^*(t)$ also commutes with R.

And finally, let us assume that the intersection $B_\theta \cap H$ is non-empty and is a dense subspace to B_θ, for all $\theta \in [0,1]$. Thus, R extends by density to a bounded linear operator $R : B_\theta \to B_\theta$, with $\|Rf\|_{B_\theta} = \|f\|_{B_\theta}$, for all $\theta \in [0,1]$. Similarly, we shall also assume that the intersection between ${B_\theta}^*$ and H is nonempty and is dense in ${B_\theta}^*$ for all $\theta \in [0,1]$. As a consequence, the operator R is also defined on the dual spaces ${B_\theta}^*$ and satisfies the identity

$$\langle Rf, g\rangle = \langle f, R^{-1}g\rangle, \quad \forall f \in B_\theta, \; \forall g \in {B_\theta}^*. \tag{4.5}$$

Denote by $\mathcal{B}_\theta$ the Banach subspace of B_θ consisting of all spherically symmetric elements of B_θ, for all $\theta \in [0,1]$.

It is immaterial to us what extra properties of $U(t)$ are assumed, apart from the energy and the decay (dispersive) estimates, to derive the family of estimates (4.2) under the assumption of spherical symmetry on the data f. In the situation we have in mind (4.2) are not direct consequences of the energy and the decay estimates for $U(t)$ and thus there is a need to address the question of what family of inhomogeneous Strichartz estimates are implied by (4.2).

The main difficulty is that $T : H \to L^q(I; B_\theta)$, where $Tf = U(t)f$, is in general unbounded on the given spaces when (q,θ) is radially admissible. The first natural step shall be to restrict the domain of T to the subspace $\mathcal{H}$, and the space B_θ to $\mathcal{B}_\theta$. Thus, we have that the operator

$$T : \mathcal{H} \to L^q(I; \mathcal{B}_\theta), \quad Tf = U(t)f,$$

is bounded. The formal adjoint $T^* : L^{q'}(\mathbb{R}; {\mathcal{B}_\theta}^*) \to \mathcal{H}$ is defined through the identity

$$\int_{-\infty}^{\infty} \langle U(t)f, F(t)\rangle dt = \int_{-\infty}^{\infty} \langle f, U^*(s)F(s)\rangle_{\mathcal{H}} ds = \Big\langle f, \int_{-\infty}^{\infty} U^*(s)F(s)ds \Big\rangle_{\mathcal{H}}, \tag{4.6}$$

or explicitly

$$T^*F = \int_{-\infty}^{\infty} U^*(s)F(s)ds.$$

This computation proves also the following

Lemma 4.1.1. *The boundedness of*

$$T : \mathcal{H} \to L^q(\mathbb{R}; \mathcal{B}_\theta), \quad Tf = U(t)f$$

is equivalent to the boundedness of

$$T^* : L^{q'}(\mathbb{R}; {\mathcal{B}_\theta}^*) \to \mathcal{H}, \quad T^*F = \int_{-\infty}^{\infty} U^*(s)F(s)ds.$$

Our final concern is whether ${\mathcal{B}_\theta}^*$ can be replaced (or identified) with the Banach subspace of ${B_\theta}^*$ consisting of the spherically symmetric functions in ${B_\theta}^*$, for all $\theta \in [0,1]$, in the domain of T^*. A useful criterion for that is the following one. Suppose that B_θ and its dual ${B_\theta}^*$ satisfy

$$\|f\|_{B_\theta} = \sup\Big\{ |\langle f, \psi\rangle| : \psi \text{ is spherically symmetric with } \|\psi\|_{B_\theta{}^*} \le 1\Big\}, \tag{4.7}$$

$$\|\phi\|_{B_\theta{}^*} = \sup\Big\{ |\langle \phi, g\rangle| : g \text{ is spherically symmetric with } \|g\|_{B_\theta} \le 1\Big\}, \tag{4.8}$$

whenever f and ϕ are spherically symmetric functions. Then, obviously, the desired property holds. As an example of such spaces we have

Lemma 4.1.2. *$L^p(\mathbb{R}^n)$ and $L^{p'}(\mathbb{R}^n)$ are associate spaces satisfying* (4.7), (4.8), *for $1 \le p \le \infty$ and $n \ge 2$.*

Proof. Whenever $1 \le p < \infty$, the supremum in (4.7) is reached on

$$\psi = sgn(f)\,|f|^{p-1} \,/\, \|f\|^{p-1}_{L^p_x}\,.$$

Apparently, $\psi \in L^{p'}(\mathbb{R}^n)$, $\|\psi\|_{L^{p'}_x}$ =1, and ψ is spherically symmetric if f is.

In the case when $p = \infty$ the supremum generally is not reached on a concrete function in $L^1(\mathbb{R}^n)$ but on a sequence of functions that approximate the identity. They can be taken to be spherically symmetric. ☐

We conclude this section by

Theorem 4.1.3 (Inhomogeneous Strichartz estimates with spherical symmetry)**.** *Suppose that the homogeneous Strichartz estimate* (4.2) *holds for all spherically symmetric $f \in H$ whenever $(q, r) \in \mathcal{A}$ and the spaces B_θ satisfy conditions* (4.7), (4.8), *for all $\theta \in [0, 1]$. Then we have that the following inhomogeneous Strichartz estimate*

$$\Big\| \int_0^t U(t-s)F(s)ds \Big\|_{L^q_t(I;B_\theta)} \lesssim \|F\|_{L^{\tilde q'}(I;B_{\tilde\theta}{}^*)} \tag{4.9}$$

holds for all spherically symmetric $F(t) \in L^{\tilde q'}(I; B_{\tilde\theta}{}^*)$, whenever (q, θ), $(\tilde q, \tilde\theta) \in \mathcal{A}$, and $q > \tilde q'$.

Proof. As usual, we consider the TT^*-operator

$$TT^* : L^{\tilde q'}(\mathbb{R}; \mathcal{B}_{\tilde\theta}{}^*) \to L^q(\mathbb{R}; \mathcal{B}_\theta), \quad TT^*F = U(t)\int_{-\infty}^{\infty} U^*(s)F(s)ds,$$

where (q, θ), $(\tilde q, \tilde\theta) \in \mathcal{A}$. Obviously, TT^* commutes with spatial rotations and is bounded on the cited spaces due to the preceding lemmas. Whenever $q > \tilde q'$ we can apply the Christ-Kiselev Lemma 7.1.5 and conclude the proof. ☐

4.2 Strichartz estimates for the wave equation

Define the operators

$$\widehat{U_\pm(t)f} = e^{\pm i(t|\xi|)}\hat f(\xi),$$
$$U(t)f = (U_+(t) - U_-(t))/2iD,$$
$$W(t)F = \int_0^t U(t-s)F(s)ds,$$

where the fractional derivative D has Fourier symbol $|\xi|$. Note that D commutes with rotations and thus preserves spherical symmetry. Then the solution to the IVP for the wave equation

$$\Box u(t,x) = F(t,x), \quad t \in [0,\infty) \times \mathbb{R}^n, \tag{4.10}$$
$$u(0) = f, \quad \partial_t u(0) = g. \tag{4.11}$$

is given by the formula

$$u(t) = \partial_t U(t)f + U(t)g + W(t)F.$$

For simplicity we shall make the following short-hand notation for the wave propagator

$$U(t)[f,g] = \partial_t U(t)f + U(t)g.$$

Definition 4.2.1. We say that the exponent pair (q,r) is radially wave-admissible if

$$\frac{1}{q} + \frac{n-1}{r} < \frac{n-1}{2}, \qquad n > 1, \tag{4.12}$$

where $2 \le q, r \le \infty$, $(q,r) \neq (\infty,\infty)$, or if (q,r) coincides with $(\infty, 2)$.

Theorem 4.2.2 ([44], [19])**.** *The following estimate*

$$\|U(t)[f,g]\|_{L^q_t L^r_x} \lesssim \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-1}}, \tag{4.13}$$

holds for all spherically symmetric $f \in \dot{H}^s(\mathbb{R}^n)$, $g \in \dot{H}^{s-1}(\mathbb{R}^n)$, whenever the exponent pair (q,r) is radially wave-admissible and the Sobolev exponent s satisfies the scaling condition

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - s.$$

Often, we consider the wave equation in the inhomogeneous Sobolev norms. In such case we can use the following corollary to Theorem 4.2.2

Corollary 4.2.3. *The following local estimate*

$$\|U(t)[f,g]\|_{L^q_t([0,T];L^r_x)} \lesssim_T \|f\|_{H^s} + \|g\|_{H^{s-1}}, \tag{4.14}$$

holds for all spherically symmetric $f \in H^s(\mathbb{R}^n)$, $g \in H^{s-1}(\mathbb{R}^n)$, whenever the exponent pair (q,r) is radially wave-admissible and the Sobolev exponent s satisfies the scaling condition

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - s.$$

Proof. It is enough to consider only the case when $f = 0$. We separate the low and high frequencies in g by writing

$$\hat{g}(\xi) = \chi\{|\xi| \le 1\}\hat{g}(\xi) + \chi\{|\xi| > 1\}\hat{g}(\xi) = \hat{g_1}(\xi) + \hat{g_2}(\xi).$$

Then, $U_\pm(t)g_1$ is infinitely smooth in x and continuous in t, and therefore is in the class $L^q_t([0,T];L^r_x)$, for any $0 < T < \infty$. Furthermore, from the inequality

$$\frac{\sin t\,|\xi|}{|\xi|} \le t$$

we obtain the estimate

$$\|U_\pm(t)g_1\|_{L^q_t([0,T];L^r_x)} \lesssim_T \|g_1\|_{L^2_x}.$$

For the high frequencies the homogeneous Sobolev and the inhomogeneous Sobolev norm coincide. Thus, estimate (4.14) follows. □

Theorem 4.2.4. *Let $u(t)$ be the solution to the IVP for the wave equation* (4.10), (4.11), *where f, g, and $F(t)$ are spherically symmetric. Then the following estimate*

$$\begin{aligned} \|D^{\sigma_1} u(t)\|_{L^q_t L^r_x} + \left\|D^{\sigma_1 - 1}\partial_t u(t)\right\|_{L^q_t L^r_x} \lesssim \\ \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-1}} + \|D^{\sigma_2} F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x} \end{aligned} \tag{4.15}$$

holds for all $f \in \dot{H}^s(\mathbb{R}^n)$, $g \in \dot{H}^{s-1}(\mathbb{R}^n)$, and $D^{\sigma_2}F(t) \in L^{\tilde{q}'}_t L^{\tilde{r}'}_x$ whenever (q,r), $(\tilde{q},\tilde{r})$ are two *radially wave-admissible pairs and satisfy the following scaling condition*

$$\frac{1}{q} + \frac{n}{r} - \sigma_1 = \frac{n}{2} - s = \frac{1}{\tilde{q}'} + \frac{n}{\tilde{r}'} - 2 - \sigma_2, \tag{4.16}$$

except in the case when ($n \geq 3$) $q = \tilde{q} = 2$, $r \neq \tilde{r}$, and either r or $\tilde{r}$ is equal to ∞, which is open.

Proof. The homogeneous Strichartz estimates of Theorem 4.2.2 hold for each of the operators $U_{\pm}$ separately. For simplicity let us consider $U_{-}(t)$ first. In view of Theorem 4.2.2, the operators $T_1 : H^s(\mathbb{R}^n) \to L_t^q L_x^r$, $T_1 f = D^{\sigma_1} U_{-}(t) f$, and $T_2 : H^s(\mathbb{R}^n) \to L_t^{\tilde{q}} L_x^{\tilde{r}}$, $T_2 f = D^{s-\beta} U_{-}(t) f$ are bounded on spherically symmetric data $f \in H^s(\mathbb{R}^n)$, where

$$s = \frac{n}{2} - \frac{n}{r} - \frac{1}{q} + \sigma_1, \quad \beta = \frac{n}{2} - \frac{n}{\tilde{r}} - \frac{1}{\tilde{q}},$$

and (q, r), $(\tilde{q}, \tilde{r})$ are two radially wave-admissible pairs and $q > \tilde{q}'$. Note that the dual operator ${T_2}^* : L_t^{\tilde{q}'}(\mathbb{R}; L_x^{\tilde{r}'}) \to H^s$ is given by

$${T_2}^* F = \int_{-\infty}^{\infty} D^{s-\beta-2s} U_{+}(\tau) F(\tau) d\tau.$$

Hence, in view of Theorem 4.1.3, we obtain the estimate

$$\left\| \int_0^t U_{-}(t-\tau) D^{\sigma_1 + s - \beta - 2s} F(\tau) d\tau \right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}.$$

Repeating the same argument for $U_{+}(t)$, we obtain the estimate

$$\left\| D^{\sigma_1} W_0(t) F + D^{\sigma_1 - 1} \partial_t W_0(t) F \right\|_{L_t^q L_x^r} \lesssim \left\| D^{s+\beta-1} F \right\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}.$$

Setting $\sigma_2 = s + \beta - 1$ gives condition (4.16).

The case when (q, r), $(\tilde{q}, \tilde{r})$ are two radially wave-admissible pairs with $q = \tilde{q} = 2$ and $r = \tilde{r}$ follows directly by symmetry, see Theorem 1.3.2.

And finally, the case when (q, r), $(\tilde{q}, \tilde{r})$ are two radially wave-admissible pairs with $q = \tilde{q} = 2$ and $r \neq \tilde{r}$ is reduced to the previous one by Sobolev embedding. □

Corollary 4.2.5. *If the initial data in Theorem 4.2.4 is given in the inhomogeneous Sobolev norms, then we have the following local analogue to estimate* (4.15)

$$\begin{aligned} \|D^{\sigma_1} u(t)\|_{L_t^q([0,T]; L_x^r)} + \left\| D^{\sigma_1 - 1} \partial_t u(t) \right\|_{L_t^q([0,T]; L_x^r)} &\lesssim_T \\ \|f\|_{H^s} + \|g\|_{H^{s-1}} + \|D^{\sigma_2} F\|_{L_t^{\tilde{q}'}([0,T]; L_x^{\tilde{r}'})}&, \end{aligned} \tag{4.17}$$

under the same other assumptions in the theorem.

Proof. We repeat the same argument as in the proof of Theorem 4.2.4 but instead of using the homogeneous estimate (4.13), we use the local analogue (4.14) for inhomogeneous Sobolev norms. □

Chapter 5

Applications to the Dirac-Klein-Gordon System

5.1 Introduction

The central topic of this chapter is the question of the global well-posedness of the DKG system in two spatial dimensions. This is a relativistic field model for the interactions of subatomic particles and plays an important role in the relativistic quantum electrodynamics, see [5]. The system generates a significant mathematical interest too. This is a system for two unknown quantities, a spinor field ψ and a real scalar field ϕ, but there is no positive definite energy and only one a priori bound that states the conservation of L^2-norm of the spinor. At the same time there is a special null-form structure in both nonlinearities allowing the system to be studied at very low regularities, see [23], [16] [42], [39], [32], [17]. This delicate balance makes the system very interesting mathematically.

The new and more powerful estimates that we have developed in the preceding chapter require spherical symmetry. However, it is well-known that the Dirac operator does not preserve spherical symmetry in the standard sense. Therefore, the main challenge here shall be to define spherical symmetry for spinors in a natural way that shall allow us to exploit our Strichartz estimates in the spherically symmetric setting.

The basic local existence result of the DKG system is the following

Theorem 5.1.1 (D'Ancona, Foschi, Selberg [16])**.** *Consider the IVP for the DKG system* (5.1), (5.2) *for initial data in the class* $\psi_{|t=0} = \psi_0 \in L^2$, $\phi_{|t=0} = \phi_0 \in H^r$ *and* $\partial_t \phi_{|t=0} = \phi_1 \in H^{r-1}$, *where* $1/4 < r < 3/4$. *Then there exist a time* $T > 0$, *depending continuously on the* $L^2 \times H^r \times H^{r-1}$*-norm of the data, and a solution*

$$\psi \in C([0,T]; L^2), \quad \phi \in C([0,T]; H^r) \cap C^1([0,T]; H^{r-1}),$$

of the DKG system (5.1), (5.2) *on* $(0,T) \times \mathbb{R}^2$, *satisfying the initial condition above. Moreover, the solution is unique in this class, and depends continuously on the data.*

5.2 Global well-posedness of spherically symmetric solutions in 2-D

The two-dimensional DKG system reads

$$(\partial_t + \sigma_1 \partial_x + \sigma_2 \partial_y + iM\sigma_3)\psi(t,x,y) = i\phi\sigma_3\psi, \quad (t,x,y) \in [0,\infty) \times \mathbb{R} \times \mathbb{R}, \tag{5.1}$$

$$(\partial_t^2 - \Delta + m^2)\phi(t,x,y) = \langle \sigma_3\psi, \psi \rangle, \tag{5.2}$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5.3}$$

are the Pauli spin matrices and M and m are nonnegative constants. The unknown quantities are a two-spinor $\psi(t,x,y) : [0,\infty) \times \mathbb{R}^2 \to \mathbb{C}^2$, and a real scalar field $\phi(t,x,y) : [0,\infty) \times \mathbb{R}^2 \to \mathbb{R}$.

Let us recall that the system (5.1), (5.2) is form covariant with respect to Lorentzian transformations and in particular to space rotations. Suppose that the coordinate system Oxy is changed into $Ox'y'$ by a space rotation $R(\varphi)$ of an angle φ

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then we want to find a rule $\psi \to \psi'$ as $Oxy \to Ox'y'$ of the form $\psi'(t,z') = S(\varphi)\psi(t,z)$, where $S(\varphi)$ is a 2×2 matrix and z denotes (x,y), that leaves (5.1), (5.2) form invariant. Of course, for the scalar field ϕ we have $\phi'(t,z') = \phi(t,z)$. Substituting in (5.1), (5.2)

$$\begin{aligned} \psi(t,z) &= S^{-1}(\varphi)\psi'(t, R(\varphi)z), \\ \phi(t,z) &= \phi'(t, R(\varphi)z), \end{aligned}$$

we obtain

$$(\partial_t + \sigma_1'\partial_{x'} + \sigma_2'\partial_{y'} + iM\sigma_3')\psi'(t,z') = i\phi\sigma_3'\psi', \tag{5.4}$$

$$(\partial_t^2 - \Delta + m^2)\phi'(t,z') = \langle \sigma_3'\psi', \psi' \rangle, \tag{5.5}$$

where

$$\sigma_1' = S(\varphi)\,(\sigma_1\cos\varphi - \sigma_2\sin\varphi)\,S^{-1}(\varphi) \tag{5.6}$$

$$\sigma_2' = S(\varphi)\,(\sigma_1\sin\varphi + \sigma_2\cos\varphi)\,S^{-1}(\varphi) \tag{5.7}$$

$$\sigma_3' = \ S(\varphi)\sigma_3 S^{-1}(\varphi). \tag{5.8}$$

Thus the matrix $S(\varphi)$ must be such that $\sigma_j' = \sigma_j$, for $j = 1,2,3$. One can check that if we set

$$S(\varphi) = \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & 1 \end{pmatrix}$$

all of the above conditions are satisfied. Note that the Klein-Gordon part of the system is form invariant as $\langle \sigma_3'\psi', \psi' \rangle = \langle \sigma_3\psi, \psi \rangle$ due to the fact that $S(\varphi)$ is unitary and the well-known invariance of the Laplacian Δ with respect to rotations. Thus we come with the following

Definition 5.2.1. We say that the two-spinor $\psi_0(z) : \mathbb{R}^2 \to \mathbb{C}^2$ is spherically symmetric if it satisfies

$$\psi_0(R(\varphi)z) = S(\varphi)\psi_0(z). \tag{5.9}$$

Lemma 5.2.2. *A function $\psi_0(z) : \mathbb{R}^2 \to \mathbb{C}^2$ satisfies* (5.9) *if and only if it has the form*

$$\psi_0(z) = S(\varphi)\chi(|z|),$$

where φ is the argument of the complex number $x + iy$ and $\chi(\rho) : [0,\infty) \to \mathbb{C}^2$.

Proof. Trivial. □

Remark 5.2.3. From the explicit representation above and the fact that $e^{-i\varphi} = -(x+iy)/|z| \in C^\infty(\mathbb{R}^2 \setminus O)$, we see that the smoothness of ψ_0 depends on the smoothness of χ and the behavior of χ around the origin.

Lemma 5.2.4. *Suppose that IVP for* (5.1), (5.2) *has a unique classical solution for some class $\mathcal{C}$ of initial data. Then for a spherically symmetric data from $\mathcal{C}$ the solution to* (5.1), (5.2) *remains spherically symmetric for all time of existence.*

Proof. Consider the change of coordinates $z = (x, y) \to z' = (x', y')$ by a spatial rotation $R(\varphi)$ around the origin of an angle φ and the induced transformation on spin space

$$\psi'(t, z') = S(\varphi)\psi(t, z). \tag{5.10}$$

In view of the fact that DKG is invariant under rotations (of the form (5.10) for the spinor) then $\psi'(t, z')$ is also a solution. The uniqueness of the solution for the considered class $\mathcal{C}$ of initial data and the identity (5.10) have the following implication. Take any initial data $\psi_0(z) \in \mathcal{C}$ and consider its image

$$\psi_0'(z') = S(\varphi)\psi_0(z)$$

under $R(\varphi)$. Let us suppose that $\psi_0'(z')$ also belongs to the class $\mathcal{C}$ or otherwise that the system has a unique solution corresponding to $\psi_0'(z')$. Then the same relation remains valid for the whole lifespan of the two solutions, that is we have identity (5.10). Note that generally ψ_0 and ψ_0' are different functions. They coincide only if ψ_0 is spherically symmetric, i.e. if

$$\psi_0(z') = S(\varphi)\psi_0(z)$$

for all φ and z. In such case $\psi_0'(z') \in \mathcal{C}$ and $\psi'(t, \cdot) = \psi(t, \cdot)$ for the whole of their lifespan. Replacing the latter in (5.10), we obtain the desired identity

$$\psi(t, z') = S(\varphi)\psi(t, z)$$

for all time of existence.

By the same token we also show that

$$\phi(t, z') = \phi(t, z)$$

for all time of existence. □

We would next like to extend the result of Lemma 5.2.4 to distributional solutions. To that end, we shall consider the pair $(\psi(t), \phi(t))$ to be a solution the DKG system (5.1), (5.2) if it solves in a sense of distributions the associated system of integral equations

$$\psi(t) = U(t)\psi_0 + \int_0^t U(t-s) i\phi(s)\sigma_3\psi(s)ds, \tag{5.11}$$

$$\phi(t) = \partial_t V(t)\phi_0 + V(t)\phi_1 + \int_0^t V(t-s)\bar{\psi}\psi(s)ds. \tag{5.12}$$

Here, we have suppressed the dependence on z and have used the notation

$$U(t)\psi_0 = e^{-(\sigma_1\partial_x + \sigma_2\partial_y + iM\sigma_3)t}\psi_0,$$

$$\partial_t V(t)\phi_0 = \cos(t\sqrt{m^2 - \Delta})\phi_0, \quad V(t)\phi_1 = \frac{\sin(t\sqrt{m^2 - \Delta})}{\sqrt{m^2 - \Delta}}\phi_1,$$

$$\bar{\psi}\psi = \langle \sigma_3\psi, \psi\rangle.$$

The operators $U(t)$, $V(t)$, and $\partial_t V(t)$ are of course rigorously defined on Fourier space. For example, for $U(t)$ we have

$$U(t)\psi_0 = \mathcal{F}^{-1}\left(e^{-i(\sigma_1\xi + \sigma_2\eta + M\sigma_3)t}\mathcal{F}\psi_0\right).$$

Although we have no immediate use of the following fact, for the sake of completeness we give the explicit form of the symbol of $U(t)$. We have

$$e^{-i(\sigma_1\xi+\sigma_2\eta+M\sigma_3)t} = \begin{pmatrix} \cos(\xi t) & -i\sin(\xi t) \\ -i\sin(\xi t) & \cos(\xi t) \end{pmatrix} \begin{pmatrix} \cos(\eta t) & -\sin(\eta t) \\ \sin(\eta t) & \cos(\eta t) \end{pmatrix} \begin{pmatrix} e^{-iMt} & 0 \\ 0 & e^{iMt} \end{pmatrix}.$$

Let us show that the propagators $U(t)$, $V(t)$, and $\partial_t V(t)$ commute with spherical rotations in the sense introduced above. We shall first show the following identity

$$U(t)\psi_{0|z} = S^{-1}(\varphi)U(t)\psi'_{0|R(\varphi)z}. \tag{5.13}$$

Notice that for $t=0$ the above expresses the law of change of coordinates on spin space induced by spherical rotations $R(\varphi)$, and by an argument similar to that of Lemma 5.2.4 if the solution is smooth enough to satisfy equations (5.1), (5.2) in the classical sense, then the same relation remains true for all time $t>0$. This time, however, we want to show (5.13) directly and without any assumptions of smoothness. We have

$$\begin{aligned} U(t)\psi_0 &= U(t)S^{-1}(\varphi)\psi'_0(R(\varphi)z) = \\ &\mathcal{F}^{-1}\left(e^{-i\left(\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}\cdot\zeta + M\sigma_3\right)t} S^{-1}(\varphi)\mathcal{F}\psi_0(R(\varphi)\zeta)\right) = \\ &\mathcal{F}^{-1}S^{-1}(\varphi)\left(S(\varphi)e^{-i\left(R(\varphi)\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}\cdot R(\varphi)\zeta + M\sigma_3\right)t} S^{-1}(\varphi)\mathcal{F}\psi_0(R(\varphi)\zeta)\right) = \\ &S^{-1}(\varphi)\mathcal{F}^{-1}\left(e^{-i\left(\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}\cdot R(\varphi)\zeta + M\sigma_3\right)t} \mathcal{F}\psi_0(R(\varphi)\zeta)\right) = \\ &S^{-1}(\varphi)U(t)\psi'_{0|R(\varphi)z}. \end{aligned}$$

Here we have used the abbreviation $\zeta = (\xi,\eta)$ and the identities (5.6)–(5.8) for the Pauli matrices σ_1, σ_2, and σ_3. In more detail, we have used the following identity

$$\begin{aligned} &S(\varphi)e^{-i\left(R(\varphi)\begin{pmatrix}\sigma_1\\ \sigma_2\end{pmatrix}\cdot R(\varphi)\zeta + M\sigma_3\right)t} S^{-1}(\varphi) = \\ &S(\varphi)e^{-i\left((\sigma_1\cos\varphi - \sigma_2\sin\varphi)\,\xi' + (\sigma_1\sin\varphi + \sigma_2\cos\varphi)\,\eta' + M\sigma_3\right)t} S^{-1}(\varphi) = \\ &S(\varphi)e^{-i\,(\sigma_1\cos\varphi - \sigma_2\sin\varphi)\,\xi' t} S^{-1}(\varphi)S(\varphi)e^{-i\,(\sigma_1\sin\varphi + \sigma_2\cos\varphi)\,\eta' t} \\ &S^{-1}(\varphi)S(\varphi)e^{-i\,(M\sigma_3)\,t} S^{-1}(\varphi) = e^{-i\,(\sigma_1\xi' + \sigma_2\eta' + M\sigma_3)\,t}. \end{aligned}$$

In the last computation we have used the notation

$$\begin{pmatrix}\xi'\\ \eta'\end{pmatrix} = R(\varphi)\begin{pmatrix}\xi\\ \eta\end{pmatrix}.$$

Similarly, for $V(t)$ we have

$$V(t)\phi_{1|z} = V(t)\phi'_{1|R(\varphi)z}, \tag{5.14}$$

and for $\partial_t V(t)$ we have

$$\partial_t V(t)\phi_{0|z} = \partial_t V(t)\phi'_{0|R(\varphi)z}. \tag{5.15}$$

Let us check for example (5.14). We have

$$V(t)\phi_{1|z} =$$
$$V(t)[\phi_1'(R(\varphi)z)] = \mathcal{F}^{-1}\left(\frac{\sin(t\langle\zeta\rangle_m)}{\langle\zeta\rangle_m}\mathcal{F}(\phi_1')(R(\varphi)\zeta)\right) =$$
$$\mathcal{F}^{-1}\left(\frac{\sin(t\langle R(\varphi)\zeta\rangle_m)}{\langle R(\varphi)\zeta\rangle_m}\mathcal{F}(\phi_1')(R(\varphi)\zeta)\right) = V(t)\phi'_{1|R(\varphi)z}.$$

Here we have used the abbreviation

$$\langle\zeta\rangle_m = \sqrt{m^2 + |\zeta|^2}.$$

Analogously for (5.15).

Lemma 5.2.5. *Suppose that IVP for* (5.1), (5.2) *has a unique distributional solution for some class $\mathcal{C}$ of initial data. Then for a spherically symmetric data from $\mathcal{C}$ the solution to* (5.1), (5.2) *remains spherically symmetric for all time of existence.*

Proof. Suppose now that the pair $(\psi(t,z), \phi(t,z))$ solves the system of integral equations (5.11), (5.12) with initial data $\psi_0(z) = \psi(0,z)$, $\phi_0(z) = \phi(0,z)$, and $\phi_1(z) = \partial_t\phi(0,z)$. Suppose that $R(\varphi)$ is a rotation of the coordinate system and denote $z' = R(\varphi)z$. Then the pair $(\psi'(t,z'), \phi'(t,z'))$ solves the system of integral equations (5.11), (5.12) with initial data $\psi_0'(z') = \psi'(0,z')$, $\phi_0'(z') = \phi'(0,z')$, and $\phi_1'(z') = \partial_t\phi'(0,z')$. This follows immediately from (5.13), (5.14), and (5.15). The details are left to the interested reader. We only remark that in this computation one may use the fact that the matrices σ_3 and $S(\varphi)$ commute as they both are diagonal. If the initial data is spherically symmetric then $\psi_0'(z) = \psi_0(z)$, $\phi_0'(z) = \phi_0(z)$, and $\phi_1'(z) = \phi_1(z)$. By uniqueness we have that the same relation holds for the whole lifespan of the two solutions. We conclude the argument in the same way as in Lemma 5.2.4. ☐

Lemma 5.2.6. *Suppose that $u(t)$ is the solution to the IVP for the wave equation (4.10), (4.11) in space dimension $n = 2$. Suppose that the data f and g and the forcing term $F(t)$ are spherically symmetric with $f \in H^s(\mathbb{R}^2)$, $g \in H^{s-1}(\mathbb{R}^2)$, and $F(t) \in L_t^\infty L_x^1(\mathbb{R}^2)$. Then we have the estimate*

$$\|D^s u(t)\|_{L_t^\infty([0,T];L_x^2)} + \left\|D^{s-1}\partial_t u(t)\right\|_{L_t^\infty([0,T];L_x^2)} \lesssim_T \|f\|_{H^s(\mathbb{R}^2)} + \|g\|_{H^{s-1}(\mathbb{R}^2)} + \|F\|_{L_t^{\tilde{q}'}([0,T];L_x^1)}, \tag{5.16}$$

for $s \in [0, 1/2)$ and $1/\tilde{q} = s$.

Proof. We apply Corollary 4.2.5 with $(q,r) = (\infty, 2)$, $(\tilde{q}, \tilde{r}) = (\tilde{q}, \infty)$, $\tilde{q} > 2$, $\sigma_1 = s$, and $\sigma_2 = 0$. ☐

Theorem 5.2.7. *Consider the IVP for the DKG system* (5.1), (5.2), *with $m = 0$, for initial data in the class $\psi_{|t=0} = \psi_0 \in L^2$, $\phi_{|t=0} = \phi_0 \in H^s$ and $\partial_t\phi_{|t=0} = \phi_1 \in H^{s-1}$, where $1/4 < s < 1/2$ and ψ_0, ϕ_0, and ϕ_1 are spherically symmetric. Then there exist a spherically symmetric solution*

$$\psi \in C((0,\infty); L^2), \quad \phi \in C((0,\infty); H^s) \cap C^1((0,\infty); H^{s-1}),$$

of the DKG system (5.1), (5.2) *on $(0,\infty) \times \mathbb{R}^2$, satisfying the initial condition above. Moreover, the solution is unique in this class, and depends continuously on the data.*

Proof. The fundamental conserved property of the system is the charge estimate

$$\|\psi(t)\|_{L_x^2} = \|\psi_0\|_{L_x^2}.$$

In view of Lemma 5.2.6, this gives the bound

$$\sup_{t\in[0,T]} \|\phi(t)\|_{H^s} + \|\partial_t\phi(t)\|_{H^{s-1}} \le C(T)$$

for each $T > 0$, $s \in [0, 1/2)$. Using this, the proof follows by standard arguments from Theorem 5.1.1. $\square$

Chapter 6

Strichartz Estimates for Some Particular Dispersive Equations

In this chapter we present the Strichartz estimates for concrete equations. The intent is to have this chapter as an exhaustive reference for Strichartz estimates in a form ready for applications. Since there is a significant overlap between the proof of the theorems for each equation we shall give a detailed proof only in the most abstract context in the next chapter. Then, the explicit form of the estimate for each concrete equation can be derived from the general case. We deliberately call this process a "derivation" as once one becomes familiar with the main ideas and key results of the subject, which are formulated in the abstract setting, the process of translating them to each context is entirely mechanical. However, to aid the reader along that path we have sketched some points and principles in Section 7.4.

6.1 The Schrödinger equation

In this section we present the Strichartz estimates for the Schrödinger equation

$$i\partial_t u + \Delta u = F(t,x), \qquad (t,x) \in [0,\infty) \times \mathbb{R}^n, \tag{6.1}$$

$$u(0,x) = f(x). \tag{6.2}$$

Let us recall that the Schrödinger evolution group has the form $\widehat{U(t)}f = e^{it|\xi|^2}\hat{f}$ in Fourier space, and

$$U(t)f = \frac{1}{(4\pi t)^{-n/2}} \int_{-\infty}^{\infty} f(y) e^{i|x-y|/(4t)} dy, \tag{6.3}$$

in physical space. These two representations immediately yield the next two fundamental estimates

(i) the energy estimate:

$$\|U(t)f\|_{L^2_x} = \|f\|_{L^2_x}, \qquad \forall f \in \mathcal{S}, \tag{6.4}$$

(ii) the dispersive estimate:

$$\|U(t)f\|_{L^\infty_x} \lesssim \frac{1}{|t|^{n/2}} \|f\|_{L^1_x}, \qquad \forall f \in \mathcal{S}, \tag{6.5}$$

where by $\mathcal{S}$ we denote the Schwartz class on $\mathbb{R}^n$. Another fundamental property of $U(t)$ that shall play a role in our arguments is

(iii) the group property:

$$U^*(t) = U(-t), \quad U(t)U^*(s) = U(t-s). \tag{6.6}$$

We next proceed with the various definitions that shall describe the range of validity of the known Strichartz estimates for (6.1), (6.2).

Definition 6.1.1. Set

$$r^* = \begin{cases} \infty & \text{if } \sigma \le 1, \\ \frac{\sigma}{\sigma-1} & \text{if } \sigma > 1, \end{cases} \tag{6.7}$$

Definition 6.1.2 (Keel and Tao [30])**.** We say that the exponent pair (q, r) is *σ-admissible*, whenever

$$\frac{1}{q} + \frac{\sigma}{r} = \frac{\sigma}{2}, \quad 2 \le q \le \infty, \quad 2 \le r \le r^*, \tag{6.8}$$

apart from the case $\sigma = 1$, $(q, r) = (2, \infty)$.

The next two definitions pertain to the inhomogeneous estimates.

Definition 6.1.3 (Foschi [20])**.** We say that the exponent pair (q, r) is *σ-acceptable*, whenever

$$\frac{1}{q} + \frac{2\sigma}{r} < \sigma, \quad 1 \le q \le \infty, \quad 2 \le r \le \infty, \tag{6.9}$$

or if $(q, r) = (\infty, 2)$.

We introduce the following definition.

Definition 6.1.4. We say that the two σ-acceptable exponent pairs (q, r) and $(\tilde{q}, \tilde{r})$ are *jointly σ-acceptable*, whenever

$$\frac{1}{q} + \frac{1}{\tilde{q}} = \sigma\left(1 - \frac{1}{r} - \frac{1}{\tilde{r}}\right), \quad \frac{1}{q} + \frac{1}{\tilde{q}} \le 1, \tag{6.10}$$

and if further satisfy the following restrictions

(i) if $\sigma \ge 1$: then $r, \tilde{r} < \infty$,

(ii) whenever $q > \tilde{q}'$, $1 < q, \tilde{q} < \infty$: then

$$(\sigma - 1)r \le \sigma\tilde{r}, \quad (\sigma - 1)\tilde{r} \le \sigma r,$$

otherwise

$$(\sigma - 1)r < \sigma\tilde{r}, \quad (\sigma - 1)\tilde{r} < \sigma r.$$

Note that for $\sigma \le 1$ condition (ii) is void. We also have the two consequences that (i) if $q = \infty$, then $r < \tilde{r}$, and (ii) if $\tilde{q} = \infty$, then $\tilde{r} < r$. They follow directly from (6.9) and (6.10).

Definition 6.1.5. In the case when $\sigma = n/2$ we shall call an exponent pair that is σ-admissible, σ-acceptable, ... etc, Schrödinger-admissible, Schrödinger-acceptable, respectively, ... etc.

We are now ready to formulate the Strichartz estimates for the Schrödinger equation.

Theorem 6.1.6 (Strichartz estimates for admissible exponents [30])**.** *Let u be the solution to the IVP for* (6.1), (6.2)*. Then the estimate*

$$\|u\|_{L_t^q L_x^r} \lesssim \|f\|_{L_x^2} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}, \tag{6.11}$$

holds for all $f \in L^2(\mathbb{R}^n)$, $F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'}$, if and only if (q, r) and $(\tilde{q}, \tilde{r})$ are two Schrödinger-admissible exponent pairs.

Proposition 6.1.7 (Generalized homogeneous estimates)**.** *Suppose that (q, r) is an exponent pair satisfying*

$$\frac{2}{q} + \frac{n}{r} = \frac{n}{p}, \tag{6.12}$$

for some $p \in (1, 2]$. Then the estimate

$$\|U(t)f\|_{L_t^{q,p} L_x^r} \lesssim \|f\|_{L_x^p}, \tag{6.13}$$

holds for every $f \in L^p(\mathbb{R}^n)$ whenever the exponents r and p are in the range

- *if* $n = 1$, $\quad 1 < p \leq 2, \quad p' < r \leq \infty$,
- *if* $n = 2$, $\quad 1 < p \leq 2, \quad p' < r < \infty$,
- *if* $n \geq 3$, $\quad 1 < p \leq 2, \quad p' < r < \frac{n}{n-2}p'$,

or if $(q, r, p) = (\infty, 2, 2)$.

Remark 6.1.8. Note that for $q \geq p$ estimate (6.13) implies the estimate

$$\|U(t)f\|_{L_t^q L_x^r} \lesssim \|f\|_{L_x^p}. \tag{6.14}$$

This condition always holds for $n \leq 2$.

Theorem 6.1.9 (Global inhomogeneous estimates)**.** *The estimate*

$$\|W(t)F\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \tag{6.15}$$

holds for all $F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'}$*, if and only if* (q, r) *and* $(\tilde{q}, \tilde{r})$ *are two jointly Schrödinger-admissible exponent pairs for* $n = 1, 2$*. Suppose that* $n \geq 3$ *and that* (q, r) *and* $(\tilde{q}, \tilde{r})$ *are two jointly Schrödinger-admissible pairs with exponents in the range*

(i) $1 < q, \tilde{q} < \infty$, $q > \tilde{q}'$*: then estimate* (6.15) *holds for all* $F \in L_t^{\tilde{q}'}(\mathbb{R}; L_x^{\tilde{r}'})$,

(ii) $1 < q, \tilde{q} < \infty$, $q = \tilde{q}'$*: then estimate*

$$\|W(t)F\|_{L_t^q L_x^{r,q}} \lesssim \|F\|_{L_t^q L_x^{\tilde{r}',q}} \tag{6.16}$$

holds for all $F \in L_t^q(\mathbb{R}; L_x^{\tilde{r}'})$

(iii) $\tilde{q} = \infty$, $1 < q < \infty$*: then estimate*

$$\|W(t)F\|_{L_t^{q,\tilde{r}'} L_x^r} \lesssim \|F\|_{L_t^1 L_x^{\tilde{r}'}} \tag{6.17}$$

holds for every $F \in L_t^1(\mathbb{R}; L_x^{\tilde{r}'})$,

(iv) $q = \infty$, $1 < \tilde{q} < \infty$*: then estimate*

$$\|W(t)F\|_{L_t^\infty L_x^r} \lesssim \|F\|_{L_t^{\tilde{q}',r} L_x^{\tilde{r}'}} \tag{6.18}$$

holds for every $F \in L_t^{\tilde{q}',r}(\mathbb{R}; L_x^{\tilde{r}'})$.

Note that whenever $\tilde{r}' \leq q \leq r$, then estimate (6.16) implies (6.15), whenever $q \geq \tilde{r}'$ estimate (6.17) implies estimate (6.15) and similarly, whenever $\tilde{q}' \leq r$ estimate (6.18) implies estimate (6.15).

6.2 Generalized Schrödinger-type equations

In this section we shall generalize the results of the preceding section to linear operators $U(t)$ with the following properties

(i) $U(t)$ obeys the dispersive estimate:

$$\|U(t)f\|_{L_x^\infty} \lesssim \frac{1}{|t|^\sigma} \|f\|_{L_x^1}, \quad \forall f \in \mathcal{S},\ t \in \mathbb{R}, \tag{6.19}$$

for any $\sigma > 0$.

(ii) $U(t)$ obeys the energy estimate:

$$\|U(t)f\|_{L_x^2} \lesssim \|f\|_{L_x^2}, \quad \forall f \in \mathcal{S},\ t \in \mathbb{R}. \tag{6.20}$$

(iii) $U(t)$ enjoys the group property:

$$U^*(t) = U(-t), \quad U(t)U^*(s) = U(t-s). \tag{6.21}$$

The next statements are a direct consequence of this definition.

Theorem 6.2.1 (Strichartz estimates for admissible exponents, [30])**.** *The estimate*

$$\|U(t)f\|_{L_t^q L_x^r} + \|W(t)F\|_{L_t^q L_x^r} \lesssim \|f\|_{L_x^2} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}, \tag{6.22}$$

holds for all $f \in L^2(\mathbb{R}^n)$, $F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'}$, whenever (q,r) and $(\tilde{q},\tilde{r})$ are two σ-admissible exponent pairs.

Proposition 6.2.2 (Generalized homogeneous estimates)**.** *Suppose that (q,r) is an exponent pair satisfying*

$$\frac{1}{q} + \frac{\sigma}{r} = \frac{\sigma}{p}, \tag{6.23}$$

for some $p \in (1,2]$. Then the estimate

$$\|U(t)f\|_{L_t^{q,p} L_x^r} \lesssim \|f\|_{L_x^p}, \tag{6.24}$$

holds for every $f \in L^p(\mathbb{R}^n)$ whenever the exponents r and p are in the range

- *if $\sigma < 1$, $\quad 1 < p \le 2, \quad p' < r \le \infty$,*
- *if $\sigma = 1$, $\quad 1 < p \le 2, \quad p' < r < \infty$,*
- *if $\sigma > 1$, $\quad 1 < p \le 2, \quad p' < r < \frac{\sigma}{\sigma-1}p'$,*

or if $(q,r,p) = (\infty, 2, 2)$.

Remark 6.2.3. Note that for $q \ge p$ estimate (6.24) implies the estimate

$$\|U(t)f\|_{L_t^q L_x^r} \lesssim \|f\|_{L_x^p}. \tag{6.25}$$

This condition always holds for $\sigma \le 1$.

Theorem 6.2.4 (Global inhomogeneous estimates)**.** *Suppose that (q,r) and $(\tilde{q},\tilde{r})$ are two jointly σ-admissible exponent pairs and that $\sigma \le 1$. Then the estimate*

$$\|W(t)F\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \tag{6.26}$$

holds for all $F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'}$.For $\sigma > 1$, we consider the following cases

(i) $1 < q, \tilde{q} < \infty$, $q > \tilde{q}'$*: then estimate* (6.26) *holds for all* $F \in L_t^{\tilde{q}'}(\mathbb{R}; L_x^{\tilde{r}'})$,

(ii) $1 < q, \tilde{q} < \infty$, $q = \tilde{q}'$*: then the estimate*

$$\|W(t)F\|_{L_t^q L_x^{r,q}} \lesssim \|F\|_{L_t^q L_x^{\tilde{r}',q}} \tag{6.27}$$

holds for all $F \in L_t^q(\mathbb{R}; L_x^{\tilde{r}'})$

(iii) $\tilde{q} = \infty$, $1 < q < \infty$*: then the estimate*

$$\|W(t)F\|_{L_t^{q,\tilde{r}'} L_x^r} \lesssim \|F\|_{L_t^1 L_x^{\tilde{r}'}} \tag{6.28}$$

holds for every $F \in L_t^1(\mathbb{R}; L_x^{\tilde{r}'})$,

(iv) $q = \infty$, $1 < \tilde{q} < \infty$*: then the estimate*

$$\|W(t)F\|_{L_t^\infty L_x^r} \lesssim \|F\|_{L_t^{\tilde{q}',r} L_x^{\tilde{r}'}} \tag{6.29}$$

holds for every $F \in L_t^{\tilde{q}',r}(\mathbb{R}; L_x^{\tilde{r}'})$.

Note that whenever $\tilde{r}' \le q \le r$, estimate (6.27) implies (6.26), whenever $q \ge \tilde{r}'$, estimate (6.28) implies estimate (6.26) and similarly, whenever $\tilde{q}' \le r$, estimate (6.29) implies estimate (6.26).

Remark 6.2.5. Parts (i) and (ii) are originally proven by Foschi [20] and independently in the context of the Schrödinger equation by Vilela [50]. There is an earlier result by Kato [29] in the context of the Schrödinger equation that contains estimates similar to parts (i) - (iv) but, however, in more restricted range and based on a less sophisticated method.

6.3 The wave equation

The IVP for the wave equation reads

$$\Box u(t,x) = F(t,x), \qquad (t,x) \in [0,\infty) \times \mathbb{R}^n, \tag{6.30}$$

$$u(0) = f, \quad \partial_t u(0) = g. \tag{6.31}$$

Then the solution to the homogeneous wave equation is given by the formula

$$\frac{U(t)+U(-t)}{2} f + \frac{U(t)-U(-t)}{2iD} g,$$

where D is the operator of fractional differentiation with symbol $|\xi|$.

The inhomogeneous operator $W(t)$ is defined in the usual way, see (1.3), and thus the solution $w(t)$ to the inhomogeneous wave equation with zero initial conditions is given by

$$w(t) = \frac{W(t)-W(-t)}{2iD} F,$$

provided that supp $F \subseteq [0,\infty) \times \mathbb{R}^n$.

Typically, the dispersive estimate for the wave equation is given by

$$\|U(t)f\|_{L_x^\infty} \lesssim \frac{1}{|t|^\sigma} \|f\|_{L_x^1}, \quad \sigma = (n-1)/2, \tag{6.32}$$

where f is a frequency localized initial data away from the origin, e.g. supp $\hat{f} \subseteq \{1 \le |\xi| \le 2\}$. Let $\{\phi_k\}_{-\infty}^{\infty}$ be a homogeneous Littlewood-Paley dyadic decomposition on $\mathbb{R}^n$. By standard scaling arguments, see [41], estimate (6.32) can be sharpened to

$$\|U(t)\phi_k * f\|_{L_x^\infty} \lesssim \frac{2^{(n-\sigma)k}}{|t|^\sigma} \|\phi_k * f\|_{L_x^1}, \tag{6.33}$$

for all $k \in \mathbb{Z}$ and any $f \in L^1(\mathbb{R}^n)$. We further rework the dispersive estimate by multiplying (6.33) by $2^{-(n-\sigma)k/2}$ and take the l^2-norm to obtain the Besov norm formulation of dispersive estimate

$$\|U(t)f\|_{\dot{B}^{-\beta}_{\infty,2}} \lesssim \frac{1}{|t|^\sigma} \|f\|_{\dot{B}^{\beta}_{1,2}}, \tag{6.34}$$

where $\beta = (n+1)/4$, $\sigma = (n-1)/2$, and $f \in \dot{B}^{\beta}_{1,2}$.

It is not hard to see that $U(t)$ obeys the energy estimate $\|U(t)f\|_{L_x^2} = \|f\|_{L_x^2}$, enjoys the group property $U^*(t) = U(-t)$, $U(t)U^*(s) = U(t-s)$, and that $U(t)$ commutes with fractional differentiation, i.e. $U(t)D^\alpha f = D^\alpha U(t) f$.

Definition 6.3.1. We say that the exponent pair (q,r) is *nonsharply σ-admissible* whenever

$$\frac{1}{q} + \frac{\sigma}{r} \le \frac{\sigma}{2}, \quad 2 \le q \le \infty, \quad 2 \le r \le \infty, \tag{6.35}$$

apart from

- $\sigma \le 1$, $(q,r) = (2/\sigma, \infty)$,

- $\sigma > 1$, $(q,r) = (2,\infty)$,
- $(q,r) = (\infty,\infty)$.

Remark 6.3.2. Note that definition 6.3.1 generally allows $r = \infty$ apart from the three endpoint cases given above. The estimates with $r = \infty$ are proven by making use of an Gargliano-type interpolation inequality that first appeared in [18], see Proposition 7.4.2.

Definition 6.3.3. We say that the two jointly σ-acceptable exponent pairs (q,r) and $(\tilde{q},\tilde{r})$ are *nonsharply jointly σ-acceptable*, whenever

$$\frac{1}{q} + \frac{1}{\tilde{q}} \le \sigma\left(1 - \frac{1}{r} - \frac{1}{\tilde{r}}\right). \tag{6.36}$$

Definition 6.3.4. We shall call an exponent pair that is σ-admissible, σ-acceptable, ...etc, wave-admissible, wave-acceptable, respectively, ...etc, in the case when $\sigma = (n-1)/2$.

Theorem 6.3.5 (Strichartz estimates for wave-admissible exponents [30])**.** *The estimate*

$$\|u(t)\|_{L_t^q L_x^r} \lesssim \|f\|_{\dot{H}^s} + \|g\|_{\dot{H}^{s-1}} + \|D^\rho F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}},$$

holds for all $f \in \dot{H}^s(\mathbb{R}^n)$, $g \in \dot{H}^{s-1}(\mathbb{R}^n)$ and $D^\rho F \in L_t^{\tilde{q}'} L_x^{\tilde{r}'}$, if and only if (q,r) and $(\tilde{q},\tilde{r})$ are two nonsharply wave-admissible exponent pairs and the Sobolev exponents s and ρ fulfill condition

$$\frac{1}{q} + \frac{n}{r} = \frac{n}{2} - s = \frac{1}{\tilde{q}'} + \frac{n}{\tilde{r}'} - 2 - \rho. \tag{6.37}$$

Proposition 6.3.6 (Generalized homogeneous estimates)**.** *Let $u(t)$ be the solution to the IVP for the wave equation* (6.30)*,* (6.31)*. The estimate*

$$\|u(t)\|_{L_t^{q,p} L_x^r} \lesssim \|D^s f\|_{L_x^p} + \left\|D^{s-1} g\right\|_{L_x^p}, \tag{6.38}$$

holds for all f and g such that $D^s f \in L^p(\mathbb{R}^n)$, $D^{s-1} g \in L^p(\mathbb{R}^n)$ and $n > 1$, whenever the Lebesgue exponent q, r and p are such that

$$\frac{1}{q} + \frac{n-1}{2r} = \frac{n-1}{2p},$$

and according to the dimension n, lie in the range

- $n = 2, \quad 1 < p \le 2, \quad p' < r \le \infty,$
- $n = 3, \quad 1 < p \le 2, \quad p' < r < \infty,$
- $n > 3, \quad 1 < p \le 2, \quad p' < r < \frac{n-1}{n-3}p',$

and the Sobolev exponent s satisfies

$$s = \frac{n}{p} - \frac{1}{q} - \frac{n}{r}.$$

Theorem 6.3.7 (Global inhomogeneous estimates)**.** *Suppose that (q,r) and $(\tilde{q},\tilde{r})$ are two nonsharply jointly wave-acceptable exponent pairs with $r, \tilde{r} < \infty$, and $n = 2,3$ (i.e. $\sigma \le 1$). Then the solution $w(t)$ to the IVP for the wave equation* (6.30)*,* (6.31)*, with $f = g = 0$, enjoys the estimate*

$$\|w(t)\|_{L_t^q L_x^r} \lesssim \|D^\rho F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}},$$

for all F such that $D^\rho F \in L_t^{\tilde{q}'}(\mathbb{R}; L_x^{\tilde{r}'})$, whenever the Sobolev exponent ρ fulfills the dimensional condition

$$\frac{1}{q} + \frac{n}{r} = \frac{1}{\tilde{q}'} + \frac{n}{\tilde{r}'} - 2 - \rho. \tag{6.39}$$

The interpolation argument that allowed us to include $r, \tilde{r} = \infty$ in the context of Theorem 6.3.5 cannot be reproduced for pairs that are not nonsharply admissible, hence the restriction $r, \tilde{r} < \infty$.

The inhomogeneous Strichartz estimates for the wave equation of Theorem 6.3.7 are formulated only in the small dimensions $n = 2, 3$, where they can be stated especially simply. For the corresponding estimates in higher dimensions see section 6.6.

6.4 The Klein-Gordon equation

The IVP for the Klein-Gordon (KG) equation reads

$$\Box u(t,x) + u(t,x) = F(t,x), \qquad (t,x) \in [0,\infty) \times \mathbb{R}^n, \tag{6.40}$$

$$u(0) = f, \quad \partial_t u(0) = g. \tag{6.41}$$

Equation (6.40) deserves a special attention since it has a stronger decay than the wave equation. However, this does not necessarily mean that the Strichartz estimates for the Klein-Gordon equation are better than those for the wave equation in a sense of a bigger gain of regularity. The reason for that lies in the fact that the gain in decay rate is payed for a greater regularity assumptions on the initial data. This might not be always desirable, especially if one is interested in solutions of low regularity. However, the dispersive estimate for the KG equation offers a flexibility to trade between the rate of decay at large times and the initial regularity of the data. We shall base the Strichartz estimates for the KG equation on that ground and obtain a whole family of estimates for a given space dimension. Originally, this fact was exploited by Machihara et al [34] to circumvent the lack of an L_t^2-type estimate in $\mathbb{R}^3$ (when $\sigma = 1$ for the wave equation) that had obstructed the study of a nonlinear Dirac equation at almost critical regularity. The estimates of this section extend the estimates of [34] to non-admissible exponents.

We define the Klein-Gordon evolution group U by $\widehat{U}(t)f = e^{it\langle\xi\rangle}\hat{f}$ and by Λ^α we denote the inhomogeneous operator of fractional differentiation with symbol $\langle\xi\rangle^\alpha$, where $\langle x\rangle = (1+|x|^2)^{1/2}$. It is not hard to see that $U(t)$ has the group property $U^*(t) = U(-t)$, $U(t)U^*(s) = U(t-s)$, and that $U(t)$ commutes with fractional differentiation, i.e. $U(t)\Lambda^\alpha f = \Lambda^\alpha U(t) f$.

We now recall the following dispersive estimate

$$\|U(t)\phi_k * f\|_{L_x^\infty} \lesssim \frac{2^{2\beta(\theta)k}}{|t|^{2\beta(\theta)-1}} \|\phi_k * f\|_{L_x^1}, \tag{6.42}$$

for the Klein-Gordon equation from [34] and the references therein, where

$$\beta(\theta) = \frac{n+1+\theta}{4}, \quad 0 \le \theta \le 1,$$

and $\{\phi_k\}_0^\infty$ is an inhomogeneous Littlewood-Paley dyadic decomposition on $\mathbb{R}^n$. As in the case with the wave equation we multiply both sides by $2^{-\beta(\theta)}$ and take the l^2-norm to get the dispersive estimate in terms of Besov norms

$$\|U(t)f\|_{B_{\infty,2}^{-\beta(\theta)}} \lesssim \frac{1}{|t|^{2\beta(\theta)-1}} \|f\|_{B_{1,2}^{\beta(\theta)}}. \tag{6.43}$$

Thus we can vary the rate of dispersion

$$\sigma(\theta) = \frac{n-1+\theta}{2} =: \sigma_\theta \tag{6.44}$$

in (6.43) with $\theta \in [0,1]$. Note that in the dispersive estimate (6.43) the difference between the decay rate $\sigma_\theta = 2\beta(\theta) - 1$ and the regularity of the initial data equal to $2\beta(\theta)$, if measured in terms of generalized derivatives in the Besov space $\dot{B}_{1,2}^0$, remains constant with θ. However, observe also that as the forbidden L_t^2-type estimate occurs for $\sigma_\theta = 1$ it is not anymore fixed to the spatial dimension $n = 3$.

Theorem 6.4.1 (Strichartz estimates for admissible exponents)**.** *Let $u(t)$ be the solution to the IVP for the Klein-Gordon equation* (6.40), (6.41)*. The estimate*

$$\|u\|_{L^q_t L^r_x} \lesssim \|f\|_{H^s} + \|g\|_{H^{s-1}} + \|\Lambda^\rho F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x},$$

holds for all $f \in H^s(\mathbb{R}^n)$, $g \in H^{s-1}(\mathbb{R}^n)$ and $\Lambda^\rho F \in L^{\tilde{q}'}_t L^{\tilde{r}'}_x$, whenever (q, r) and $(\tilde{q}, \tilde{r})$ are two nonsharply σ_θ-admissible exponent pairs ($\sigma_\theta > 0$) and the Sobolev exponents s and ρ fulfill the dimensional condition

$$\frac{1}{q} + \frac{n}{r} - \frac{\theta}{\sigma_\theta q} = \frac{n}{2} - s = \frac{1}{\tilde{q}'} + \frac{n}{\tilde{r}'} - \frac{\theta}{\sigma_\theta \tilde{q}'} + \frac{\theta}{\sigma_\theta} - 2 - \rho. \tag{6.45}$$

Proposition 6.4.2 (Generalized homogeneous estimates)**.** *Let $u(t)$ be the solution to the IVP for the KG equation* (6.40), (6.41)*. The estimate*

$$\|U(t)f\|_{L^{q,p}_t L^r_x} \lesssim \|\Lambda^s f\|_{L^p_x} + \left\|\Lambda^{s-1} g\right\|_{L^p_x}, \tag{6.46}$$

holds for all f and g such that $\Lambda^s f \in L^p(\mathbb{R}^n)$, $\Lambda^{s-1} g \in L^p(\mathbb{R}^n)$, whenever the Lebesgue exponent q, r and p are such that

$$\frac{1}{q} + \frac{\sigma_\theta}{r} = \frac{\sigma_\theta}{p},$$

and according to $\sigma_\theta > 0$, lie in the range

- $\sigma_\theta < 1, \quad 1 < p \leq 2, \quad p' < r \leq \infty,$
- $\sigma_\theta = 1, \quad 1 < p \leq 2, \quad p' < r < \infty,$
- $\sigma_\theta > 1, \quad 1 < p \leq 2, \quad p' < r < \frac{\sigma_\theta}{\sigma_\theta - 1} p',$

and the Sobolev exponent s satisfies

$$s = \frac{n}{p} - \frac{1}{q} - \frac{n}{r} + \frac{\theta}{\sigma_\theta q}.$$

Theorem 6.4.3 (Global inhomogeneous estimates)**.** *Suppose that (q, r) and $(\tilde{q}, \tilde{r})$ are two nonsharply jointly σ_θ-acceptable exponent pairs with $r, \tilde{r} < \infty$, and $0 < \sigma_\theta \leq 1$. Then the solution $w(t)$ to the IVP for the Klein-Gordon equation* (6.40), (6.41)*, with $f = g = 0$, enjoys the estimate*

$$\|w(t)\|_{L^q_t L^r_x} \lesssim \|\Lambda^\rho F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x},$$

for all F such that $\Lambda^\rho F \in L^{\tilde{q}'}_t(\mathbb{R}; L^{\tilde{r}'}_x)$, whenever the Sobolev exponent ρ fulfills the dimensional condition

$$\frac{1}{q} + \frac{n}{r} - \frac{\theta}{\sigma_\theta q} = \frac{1}{\tilde{q}'} + \frac{n}{\tilde{r}'} - \frac{\theta}{\sigma_\theta \tilde{q}'} + \frac{\theta}{\sigma_\theta} - 2 - \rho. \tag{6.47}$$

Remark 6.4.4. The examiners have asked for a better and a more detailed comparison between the results in this paragraph and the result of Machihara [34]. In that regard, the basic idea that we take from [34] is the use of the generalized dispersive estimate in Besov norms (6.43), (6.44). From this point on, the two works diverge both in scope and in philosophy. While the main goal of [34] is the well-posedness of a nonlinear (Dirac) system, and therefore they only need to prove some new inhomogeneous Strichartz estimates that are sufficient to that end, our thesis is structured to contain a systematic and exhaustive derivation of all Strichartz estimates for a given equation. Also, there are differences in presentation, style, and scope. There is a big overlap (in essence) between the estimates of Theorem 6.4.1 and in the results of [34]. The other two statements of this paragraph are new.

6.5 The Dirac equation

The Dirac equation is a first-order wave equation with matrix-valued coefficients. We shall distinguish the following two cases

- massless Dirac

$$i\partial_t\psi + i\alpha \cdot \nabla\psi = 0, \tag{6.48}$$

- massive Dirac

$$i\partial_t\psi + i\alpha \cdot \nabla\psi + \beta\psi = 0. \tag{6.49}$$

In both equations the spinor field $\psi(t,x) : \mathbb{R}^{1+n} \to \mathbb{C}^N$ maps $\mathbb{R}^{1+n}$ to a column vector $\psi(t,x) = (\psi_1(t,x), \dots, \psi_N(t,x))^t$ in $\mathbb{C}^N$, where $N = 2^{\lfloor \frac{n+1}{2} \rfloor}$. We use the abbreviations

$$\alpha \cdot \nabla = \alpha_1\partial_1 + ... + \alpha_n\partial_n, \quad \partial_j = \partial/\partial x_j.$$

The matrices $\alpha_j \in M_N(\mathbb{C})$, $j = 1 \dots n$ and $\beta \in M_N(\mathbb{C})$ are the well-known Dirac matrices. If we multiply the Dirac equations (6.48), (6.49) by β we obtain a new set of coefficients γ^μ, $\mu = 0 \dots n$, where $\gamma^0 = \beta$ and $\gamma^j = \beta\alpha_j$, $j = 1 \dots n$. The commutation properties of the gamma matrices give the identities

$$(\gamma^\mu\partial_\mu)^2 = \Box I_N, \quad (\gamma^\mu\partial_\mu + I_N)^2 = (\Box + 1)I_N. \tag{6.50}$$

where I_N is the identity in $M_N(\mathbb{C})$ and Einstein's summation convention is used. One can use the identities (6.50) to define the gamma matrices and through them α and β but there are more than one (equivalent) sets of matrix representations that satisfy (6.50).

The unknown $\psi(t,x)$ is a spinor, which loosely speaking means that ψ changes under change of coordinates by a specific rule. For more details about the nature of that rule see Bjorken and Drell [5] or chapter 5. In a fixed coordinate system, however, and with a fixed representation of the Dirac matrices ψ can be regarded as an ordinary vector-valued function.

Due to the identities (6.50) a free wave to (6.48) satisfies the homogeneous wave equation componentwise and a free wave to (6.49) satisfies the homogeneous Klein-Gordon equation componentwise. Therefore, the Strichartz estimates to the wave and the Klein-Gordon equation from the previous subsections also apply to the Dirac equation after some minor adjustment.

Let us for the sake of concreteness consider the IVP for the massive Dirac equation

$$i\partial_t\psi + i\alpha \cdot \nabla\psi + \beta\psi = F(t,x), \qquad (t,x) \in [0,\infty) \times \mathbb{R}^n, \tag{6.51}$$

$$\psi(0) = \psi_0. \tag{6.52}$$

Note that the rate of dispersion σ_θ of the massive Dirac equation is given by (6.44).

Theorem 6.5.1 (Strichartz estimates for admissible exponents)**.** *Let $\psi(t)$ be the solution to the IVP for the massive Dirac equation* (6.51)*,* (6.52)*. The estimate*

$$\|\psi\|_{L^q_t L^r_x} \lesssim \|\psi_0\|_{H^s} + \|\Lambda^\rho F\|_{L^{\tilde{q}'}_t L^{\tilde{r}'}_x},$$

holds for all $\psi_0 \in H^s(\mathbb{R}^n)$ and $\Lambda^\rho F \in L^{\tilde{q}'}_t L^{\tilde{r}'}_x$, whenever (q,r) and $(\tilde{q},\tilde{r})$ are two nonsharply σ_θ-admissible exponent pairs ($\sigma_\theta > 0$) and the Sobolev exponents s and ρ fulfill condition

$$\frac{1}{q} + \frac{n}{r} - \frac{\theta}{\sigma_\theta q} = \frac{n}{2} - s = \frac{1}{\tilde{q}'} + \frac{n}{\tilde{r}'} - \frac{\theta}{\sigma_\theta \tilde{q}'} + \frac{\theta}{\sigma_\theta} - 1 - \rho. \tag{6.53}$$

Proposition 6.5.2 (Generalized homogeneous estimates)**.** *Let $\psi(t)$ be the solution to the IVP for the massive Dirac equation* (6.51)*,* (6.52)*. The estimate*

$$\|\psi(t)\|_{L^{q,p}_t L^r_x} \lesssim \|\Lambda^s\psi_0\|_{L^p_x}, \tag{6.54}$$

holds for all ψ_0 such that $\Lambda^s\psi_0 \in L^p(\mathbb{R}^n)$, whenever the Lebesgue exponent q, r and p are such *that*

$$\frac{1}{q} + \frac{\sigma_\theta}{r} = \frac{\sigma_\theta}{p},$$

and according to $\sigma_\theta > 0$, *lie in the range*

- $\sigma_\theta < 1, \quad 1 < p \le 2, \quad p' < r \le \infty,$
- $\sigma_\theta = 1, \quad 1 < p \le 2, \quad p' < r < \infty,$
- $\sigma_\theta > 1, \quad 1 < p \le 2, \quad p' < r < \frac{\sigma_\theta}{\sigma_\theta - 1} p',$

and the Sobolev exponent s *satisfies*

$$s = \frac{n}{p} - \frac{1}{q} - \frac{n}{r} + \frac{\theta}{\sigma_\theta q}.$$

Theorem 6.5.3 (Global inhomogeneous estimates)**.** *Suppose that* (q, r) *and* $(\tilde{q}, \tilde{r})$ *are two nonsharply jointly* σ_θ*-acceptable exponent pairs with* $r, \tilde{r} < \infty$, *and* $0 < \sigma_\theta \le 1$. *Then the solution* $\chi(t)$ *to the IVP for the massive Dirac equation* (6.51), (6.52), *with* $\psi_0 = 0$, *enjoys the estimate*

$$\|\chi(t)\|_{L_t^q L_x^r} \lesssim \|\Lambda^\rho F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}},$$

for all F *such that* $\Lambda^\rho F \in L^{\tilde{q}'} L^{\tilde{r}'}$, *whenever the Sobolev exponent* ρ *fulfills condition*

$$\frac{1}{q} + \frac{n}{r} - \frac{\theta}{\sigma_\theta q} = \frac{1}{\tilde{q}'} + \frac{n}{\tilde{r}'} - \frac{\theta}{\sigma_\theta \tilde{q}'} + \frac{\theta}{\sigma_\theta} - 1 - \rho. \tag{6.55}$$

6.6 Generalized wave-type equations

In this section we shall generalize the results of the preceding sections to abstract linear operators $U(t)$ with the following properties

(i) $U(t)$ obeys the dispersive estimate

$$\|U(t)f\|_{B^{-\beta}_{\infty,2}} \lesssim \frac{1}{|t|^\sigma} \|f\|_{B^{\beta}_{1,2}}, \tag{6.56}$$

where $\sigma > 0,\ 0 \le \beta < \frac{n}{2}$, and $f \in \mathcal{S}$, the Schwartz class of rapidly decaying functions on $\mathbb{R}^n$.

(ii) $U(t)$ obeys the energy estimate

$$\|U(t)f\|_{L^2_x} \lesssim \|f\|_{L^2_x}, \quad \forall f \in \mathcal{S},\ t \in \mathbb{R}. \tag{6.57}$$

(iii) $U(t)$ enjoys the group property

$$U^*(t) = U(-t), \quad U(t)U^*(s) = U(t-s). \tag{6.58}$$

(iv) $U(t)$ commutes with fractional differentiation

$$U(t)\Lambda^\alpha = \Lambda^\alpha U(t). \tag{6.59}$$

Again, for the sake of concreteness we formulate the estimates in terms of inhomogeneous Besov norms. The case of homogeneous norms is completely analogous and can be treated by replacing everywhere the inhomogeneous Besov norms with homogeneous ones.

Theorem 6.6.1 (Strichartz estimates for admissible exponents)**.** *The estimate*

$$\|U(t)f\|_{L^q(\mathbb{R};B^{-\rho}_{r,2})} + \|W(t)F\|_{L^q(\mathbb{R};B^{-\rho}_{r,2})} \lesssim \|f\|_{L^2_x} + \|F\|_{L^{\tilde{q}'}(\mathbb{R};B^{\tilde{\rho}}_{\tilde{r}',2})}, \tag{6.60}$$

holds for all $f \in L^2$, $F \in L^{\tilde{q}'} B^{\tilde{\rho}}_{\tilde{r}',2}$, whenever (q,r) and $(\tilde{q},\tilde{r})$ are two σ-admissible exponent pairs and the smoothness exponents $\rho(r)$ and $\tilde{\rho}(\tilde{r})$ fulfill condition

$$\rho(r) = 2\beta\left(\frac{1}{2} - \frac{1}{r}\right). \tag{6.61}$$

Corollary 6.6.2. *The estimate*

$$\|U(t)f\|_{L^q_t L^r_x} + \|W(t)F\|_{L^q_t L^r_x} \lesssim \|f\|_{H^s} + \|\Lambda^\alpha F\|_{L^{\tilde{q}'} L^{\tilde{r}'}},$$

holds for all $f \in H^s$, $\Lambda^\alpha F \in L^{\tilde{q}'} L^{\tilde{r}'}$, whenever (q,r) and $(\tilde{q},\tilde{r})$ are two nonsharply σ-admissible exponent pairs and the Sobolev exponents α and s fulfill condition

$$\frac{n-2\beta}{\sigma q} + \frac{n}{r} = \frac{n}{2} - s = \frac{n-2\beta}{\sigma\tilde{q}'} + \frac{n}{\tilde{r}'} - \frac{n-2\beta}{\sigma} - \alpha. \tag{6.62}$$

Proposition 6.6.3 (Generalized homogeneous estimates)**.** *The estimate*

$$\|U(t)f\|_{L^{q,p}_t L^r_x} \lesssim \|\Lambda^s f\|_{L^p_x},$$

holds for all f such that $\Lambda^s f \in L^p(\mathbb{R}^n)$, whenever the Lebesgue exponent q, r and p are such that

$$\frac{1}{q} + \frac{\sigma}{r} = \frac{\sigma}{p},$$

and according to $\sigma > 0$, lie in the range

- $\sigma < 1, \quad 1 < p < 2, \quad p' < r \leq \infty,$
- $\sigma = 1, \quad 1 < p < 2, \quad p' < r < \infty,$
- $\sigma > 1, \; r^{*\prime} < p < 2, \quad p' < r < \frac{\sigma}{\sigma-1}p',$

and the Sobolev exponent s satisfies

$$s = \frac{n}{p} - \frac{n}{r} + \frac{n-2\beta}{\sigma q}.$$

Remark 6.6.4. In particular, for $\sigma \leq 1$ we have that

$$\|U(t)f\|_{L^q_t L^r_x} \lesssim \|\Lambda^s f\|_{L^p_x}$$

since then the inequality $p \leq q$ always holds.

Theorem 6.6.5 (Global inhomogeneous estimates)**.** *Suppose that (q,r) and $(\tilde{q},\tilde{r})$ are two jointly σ-acceptable exponent pairs and the smoothness exponents $\rho = \rho(r)$ and $\tilde{\rho} = \rho(\tilde{r})$ fulfill condition (6.61). Then the operator $W(t)$ obeys the estimate*

$$\|W(t)F\|_{L^q(\mathbb{R};B^{-\rho}_{r,2})} \lesssim \|F\|_{L^{\tilde{q}'}(\mathbb{R};B^{\tilde{\rho}}_{\tilde{r}',2})}, \tag{6.63}$$

for all $F \in L^{\tilde{q}'}(\mathbb{R}; B^{\tilde{\rho}}_{\tilde{r}',2})$, whenever $\sigma \leq 1$. If $\sigma > 1$ we consider different cases

(i) The point $(1/q, 1/\tilde{q})$ lies inside $\triangle OAB$ in fig. 2.1, that is $1 < q, \tilde{q} < \infty$ and $q > \tilde{q}'$: then $W(t)$ satisfies (6.63) for all $F \in L^{\tilde{q}'}(\mathbb{R}; B^{\tilde{\rho}}_{\tilde{r}',2})$.

(ii) The point $(1/q, 1/\tilde{q})$ lies on the hypotenuse AB, that is $1 < q, \tilde{q} < \infty$ and $q = \tilde{q}'$: then $W(t)$ satisfies (6.63) *if $q \leq 2$, or otherwise*

$$\|W(t)F\|_{L^q(\mathbb{R};B^{-\rho}_{r,q})} \lesssim \|F\|_{L^q(\mathbb{R};B^{\tilde{\rho}}_{\tilde{r}',q})}$$

for all $F \in L^q(\mathbb{R}; B^{\tilde{\rho}}_{\tilde{r}',2})$.

(iii) The point $(1/q, 1/\tilde{q})$ *lies on the side OA, that is* $1 < q < \infty$, $\tilde{q} = \infty$*: then* $W(t)$ *satisfies*

$$\|W(t)F\|_{L^{q,\tilde{r}'}(\mathbb{R};B^{-\rho}_{r,2})} \lesssim \|F\|_{L^1(\mathbb{R};B^{\tilde{\rho}}_{\tilde{r}',2})}$$

for all $F \in L^{\tilde{q}'}(\mathbb{R}; B^{\tilde{\rho}}_{\tilde{r}',2})$.

(iv) The point $(1/q, 1/\tilde{q})$ *lies on the side OB, that is* $1 < \tilde{q} < \infty$, $q = \infty$*, then* $W(t)$ *satisfies*

$$\|W(t)F\|_{L^\infty(\mathbb{R};B^{-\rho}_{r,2})} \lesssim \|F\|_{L^{\tilde{q}',r}(\mathbb{R};B^{\tilde{\rho}}_{\tilde{r}',2})}$$

for all $F \in L^{\tilde{q}',r}(\mathbb{R}; B^{\tilde{\rho}}_{\tilde{r}',2})$.

Corollary 6.6.6. *By the usual embeddings between the Besov and Sobolev spaces, estimate* (6.63) *implies estimate*

$$\|W(t)F\|_{L^q H^{-\rho}_r} \lesssim \|F\|_{L^{\tilde{q}'} H^{\tilde{\rho}}_{\tilde{r}'}}, \tag{6.64}$$

whenever $2 \leq r, \tilde{r} < \infty$.

6.7 Abstract vector-valued equations

In this section we shall present the Strichartz estimates in the abstract setting. Let us first fix the abstract framework in which we shall be working.

Let us consider two Banach spaces $\mathcal{B}_0$ and $\mathcal{B}_1$ that are compatible for interpolation such that $\mathcal{B}_0 \cap \mathcal{B}_1$ is dense in $\mathcal{B}_0$ and $\mathcal{B}_1$. For more details on interpolation spaces the reader may consult the standard references [3, 2]. Informally, we may think that the two spaces have a common underlying topology, i.e. that they embed into a larger topological space. We shall also assume that the space $\mathcal{B}_1$ is a Hilbert space. We remark here that the reason we are not using the typical notation $\mathcal{H}$ for Hilbert space is because we shall consider in the sequel the interpolation spaces $\mathcal{B}_\theta = (\mathcal{B}_0, \mathcal{B}_1)_\theta$, $\theta \in (0,1)$, obtained by complex interpolation. Occasionally, we shall also use the real method of interpolation and the interpolation spaces obtained by it will be denoted by $\mathcal{B}_{\theta,q}$ which stands for $(\mathcal{B}_0, \mathcal{B}_1)_{\theta,q}$, for $\theta \in (0,1)$, $1 \leq q \leq \infty$.

We shall also consider the space ${\mathcal{B}_0}^*$ which is the topological dual space to $\mathcal{B}_0$ or more generally the space ${\mathcal{B}_0}' \subset {\mathcal{B}_0}^*$, where the embedding is continuous and isometric. A typical example is the space $\mathcal{B} = L^\infty(\mathbb{R}^n)$. The space ${\mathcal{B}_0}' = L^1(\mathbb{R}^n)$ is a proper subspace of $(L^\infty(\mathbb{R}^n))^*$ that embeds isometrically into the latter. The claim follows from the fact that the space $L^\infty(\mathbb{R}^n)^*$ consists of all elements ϕ for which the norm

$$\|\phi\|_{L^\infty(\mathbb{R}^n)^*} = \sup\Big\{ |\langle \phi, g\rangle| : \|g\|_{L^\infty_x} = 1\Big\}.$$

is finite, while the space $L^1(\mathbb{R}^n)$ consists only of those ϕ for which there exist a proper function $f : \mathbb{R}^n \to \mathbb{C}$ such that

$$\langle \phi, g\rangle = \int_{\mathbb{R}^n} fg dx, \quad \forall g \in L^\infty(\mathbb{R}^n).$$

Let us recall the following result.

Lemma 6.7.1 (Duality lemma for the complex method, see p. 98 of [3])**.** *Assume that* $(\mathcal{B}_0, \mathcal{B}_1)$ *is a compatible Banach couple and that* $\mathcal{B}_0 \cap \mathcal{B}_1$ *is dense in* $\mathcal{B}_0$ *and* $\mathcal{B}_1$*. Suppose also that* $\mathcal{B}_1$ *is reflexive or a fortiori a Hilbert space. Then*

$$(\mathcal{B}_0, \mathcal{B}_1)_\theta{}^* = ({\mathcal{B}_0}^*, {\mathcal{B}_1}^*)_\theta, \quad 0 < \theta < 1, \tag{6.65}$$

with equal norms.

Thus under our assumptions the dual space ${\mathcal{B}_\theta}^*$ to $\mathcal{B}_\theta$ will always be the space defined in the right hand side of (6.65). The analogous result for the real method is the following one.

Lemma 6.7.2 (Duality lemma for the real method, see p. 54 of [3])**.** *Assume that* $(\mathcal{B}_0, \mathcal{B}_1)$ *is a compatible Banach couple and that* $\mathcal{B}_0 \cap \mathcal{B}_1$ *is dense in* $\mathcal{B}_0$ *and* $\mathcal{B}_1$*. Suppose also that* $1 \le q < \infty$*,* $0 < \theta < 1$*, and that* $1/q + 1/q' = 1$*. Then*

$$(\mathcal{B}_0, \mathcal{B}_1)_{\theta,q}{}^* = ({\mathcal{B}_0}^*, {\mathcal{B}_1}^*)_{\theta,q'}, \tag{6.66}$$

with equal norms.

Clearly, we have the continuous isometric embeddings

$$({\mathcal{B}_0}', \mathcal{B}_1)_\theta \subset {\mathcal{B}_\theta}^*, \quad 0 < \theta < 1, \tag{6.67}$$

$$({\mathcal{B}_0}', \mathcal{B}_1)_{\theta,q} \subset {\mathcal{B}_{\theta,q}}^*, \quad 0 < \theta < 1, \quad 1 \le q < \infty. \tag{6.68}$$

Under the above conventions the dispersive estimate has the form

$$\|U(t)f\|_{\mathcal{B}_0} \lesssim \frac{1}{|t|^\sigma} \|f\|_{{\mathcal{B}_0}'}, \quad \sigma > 0, \quad \forall t \in \mathbb{R}.$$

Furthermore, its validity may only be proven for all $f \in \mathcal{S}$, where $\mathcal{S} \subset {\mathcal{B}_0}'$ is dense. In order to be able to interpret the operator $U(t)$ in an unique way on the space ${\mathcal{B}_0}^*$ we require that $\mathcal{S}$ is also dense in ${\mathcal{B}_0}^*$ or alternatively, if that is difficult to show, at least to be dense in the spaces ${\mathcal{B}_\theta}^*$, $0 < \theta < 1$. The abstract energy estimate for $U(t)$ has the form

$$\|U(t)f\|_{\mathcal{B}_1} \lesssim \|f\|_{\mathcal{B}_1}, \quad \forall t \in \mathbb{R}.$$

By complex interpolation between the two estimates and (6.67), (6.68) we obtain the family of estimates

$$\|U(t)f\|_{\mathcal{B}_\theta} \lesssim \frac{1}{|t|^{\sigma(1-\theta)}} \|f\|_{{\mathcal{B}_\theta}^*}, \quad 1 < \theta < 1.$$

Analogously, be real interpolation we obtain the family of estimates

$$\|U(t)f\|_{\mathcal{B}_{\theta,2}} \lesssim \frac{1}{|t|^{\sigma(1-\theta)}} \|f\|_{{\mathcal{B}_{\theta,2}}^*}, \quad 1 < \theta < 1.$$

Below we summarize all the requirements on $U(t)$ that will be used in the derivation of the Strichartz estimates.

a) $U(t)$ obeys the dispersive estimate

$$\|U(t)f\|_{\mathcal{B}_0} \lesssim \frac{1}{|t|^\sigma} \|f\|_{{\mathcal{B}_0}'}, \quad \sigma > 0. \tag{6.69}$$

b) $U(t)$ obeys the energy estimate

$$\|U(t)f\|_{\mathcal{B}_1} \lesssim \|f\|_{\mathcal{B}_1}, \quad \forall t \in \mathbb{R}. \tag{6.70}$$

c) $U(t)$ enjoys the group property

$$U^*(t) = U(-t), \quad U(t)U^*(s) = U(t-s). \tag{6.71}$$

d) $U(t)$ enjoys the following regularity property

$$U(t)f \in C(\mathbb{R}; {\mathcal{B}_\theta}^*), \tag{6.72}$$

for all $0 < \theta < 1$, $f \in \mathcal{S}$.

The Strichartz estimates for $U(t)$ have the form

$$\|U(t)f\|_{L^q_t(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|f\|_{\mathcal{B}_1}, \tag{6.73}$$

and more generally

$$\|U(t)f\|_{L^q_t(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|f\|_{\mathcal{B}_{\tilde{\theta}}^*} . \tag{6.74}$$

For comparison estimate (6.73) correspond to estimate (6.14) with $p = 2$ and estimate (6.74) corresponds to estimate (6.14) with $1 < p \leq 2$ in the context of the Schrödinger equation. The inhomogeneous estimates have the form

$$\|W(t)F\|_{L^q_t(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|F\|_{L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}^*)} . \tag{6.75}$$

The global in time estimates (6.74), (6.75) have local counterparts

$$\|U(t)f\|_{L^q_t([0,T];\mathcal{B}_\theta)} \lesssim \|f\|_{\mathcal{B}_1} , \tag{6.76}$$

$$\|W(t)F\|_{L^q_t([0,T];\mathcal{B}_\theta)} \lesssim \|F\|_{L^{\tilde{q}'}([0,T];\mathcal{B}_{\tilde{\theta}}^*)} , \tag{6.77}$$

for any $0 < T \leq \infty$. This follows immediately if we consider the localized operator $U(t)\chi_{[0,T]}$, where $\chi_{[0,T]}$ is the characteristic function of the interval $[0,T]$, and see that it as well as $U(t)$ satisfies conditions (6.69) - (6.72).

Let us make another introductory remark. In Keel and Tao [30] the energy estimate (6.70) is given a slightly more general formulation

$$\|U(t)f\|_{\mathcal{B}_1} \lesssim \|f\|_{\mathcal{H}}, \quad \forall f \in \mathcal{S},\ t \in \mathbb{R}, \tag{6.78}$$

for some Hilbert space $\mathcal{H}$ and some Banach space $\mathcal{B}_1$ with $\mathcal{H} \neq \mathcal{B}_1$ in general. This formulation, i.e. the assumptions (6.69), (6.78), (6.71), and (6.72) lead to the same family of homogeneous Strichartz estimates as the assumptions (6.69), (6.70), (6.71), (6.72).

However, this is not the case with regard to the inhomogeneous estimate. For example, we have

$$\|W(t)F\|_{L^\infty_t(\mathbb{R};\mathcal{B}_1)} \lesssim \|F\|_{L^1(\mathbb{R};\mathcal{H})} , \tag{6.79}$$

together with the estimates

$$\|W(t)F\|_{L^q_t(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|F\|_{L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}^*)} . \tag{6.80}$$

Then, if we interpolate between the two estimates we shall obtain another family of estimates which is difficult to describe in simple terms. Therefore, to save ourselves from having to deal with such technical difficulties we have adopted the more restricted version of the energy inequality (6.70). In fact, from the viewpoint of applications formulation (6.70) reflects the physics principle of energy conservation more closely than (6.78).

We continue with several definitions that are used to describe the range of validity of the homogeneous and inhomogeneous Strichartz estimates (6.73), (6.74) and (6.75).

Remark 6.7.3 (Mnemonic rule). The abstract definitions below can be easily remembered from the more familiar definitions for the Schrödinger equation if one replaces $2/r$ by θ.

Definition 6.7.4. Set

$$\begin{cases} \theta^* = 0, & \text{if} \quad \sigma \leq 1, \\ \theta^* = (\sigma - 1)/\sigma, & \text{if} \quad \sigma > 1. \end{cases} \tag{6.81}$$

Definition 6.7.5 ([30])**.** We say that the exponent pair (q,θ) is σ-*admissible*, whenever

$$\frac{1}{q} = \frac{\sigma}{2}(1-\theta), \quad 2 \leq q \leq \infty, \quad \theta^* \leq \theta \leq 1, \tag{6.82}$$

apart from the case $\sigma = 1$, $(q,\theta) = (2,0)$.

We shall call a pair (q,θ) endpoint if $\sigma \geq 1$ and $(q,\theta) = (2,\theta^*)$. Note that definition 6.7.5 forbids the endpoint $\sigma = 1$, $(q,\theta) = (2,0)$ but allows all higher-dimensional endpoints for $\sigma > 1$. The following definitions pertain to the inhomogeneous estimates.

Definition 6.7.6 ([20])**.** We say that the exponent pair (q, θ) is *σ-acceptable*, whenever

$$\frac{1}{q} < \sigma(1-\theta), \quad 1 \leq q \leq \infty, \quad \theta \in [0,1), \tag{6.83}$$

or if $(q, \theta) = (\infty, 1)$.

We introduce the following definition.

Definition 6.7.7. We say that the two σ-acceptable exponent pairs (q, θ) and $(\tilde{q}, \tilde{\theta})$ are *jointly σ-acceptable*, whenever

$$\frac{1}{q} + \frac{1}{\tilde{q}} = \frac{\sigma}{2}\left(2 - \theta - \tilde{\theta}\right), \tag{6.84}$$

and if further satisfy the following restrictions

(i) if $\sigma \geq 1$: then $0 < \theta, \tilde{\theta}$,

(ii) whenever $q > \tilde{q}'$, $1 < q, \tilde{q} < \infty$: then

$$(\sigma - 1)\theta \leq \sigma\tilde{\theta}, \quad (\sigma - 1)\tilde{\theta} \leq \sigma\theta,$$

otherwise

$$(\sigma - 1)\theta < \sigma\tilde{\theta}, \quad (\sigma - 1)\tilde{\theta} < \sigma\theta.$$

Note that for $\sigma \leq 1$ condition (ii) is void. We also have the two consequences: (i) if $q = \infty$, then $\tilde{\theta} < \theta$, and (ii) if $\tilde{q} = \infty$, then $\theta < \tilde{\theta}$. They follow directly from (6.83) and (6.84). We shall call an inhomogeneous Strichartz estimate with exponent pairs (q, θ), $(\tilde{q}, \tilde{\theta})$ *endpoint* if (i) $q = \tilde{q}'$, which can only happen if $\sigma \geq 1$, (ii) if $q = \infty$, and (iii) if $\tilde{q} = \infty$. We next formulate the Strichartz estimates in the abstract setting.

Theorem 6.7.8 (Estimates for admissible exponents)**.** *The estimate*

$$\|U(t)f\|_{L^q_t(\mathbb{R};\mathcal{B}_\theta)} + \|W(t)F\|_{L^q_t(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|f\|_{\mathcal{B}_1} + \|F\|_{L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)}, \tag{6.85}$$

holds for all $f \in \mathcal{B}_1$, $F \in L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)$, whenever (q, θ) and $(\tilde{q}, \tilde{\theta})$ are two σ-admissible exponent pairs, and (q, θ) is not an endpoint pair.

Proposition 6.7.9 (Generalized homogeneous estimates)**.** *Suppose that (q, θ) is an exponent pair satisfying*

$$\frac{1}{q} = \frac{\sigma}{2}\left(2 - \theta - \tilde{\theta}\right),$$

for some $\tilde{\theta} \in (0,1]$. Then the estimate

$$\|U(t)f\|_{L^{q,\infty}_t(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|f\|_{\mathcal{B}_{\tilde{\theta}}{}^*}, \tag{6.86}$$

holds for every $f \in \mathcal{B}^*_{\tilde{\theta}}$ whenever the exponents θ and $\tilde{\theta}$ are in the range

- $\sigma < 1$, $\quad 0 < \tilde{\theta} \leq 1$, $\quad 0 \leq \theta < \tilde{\theta}$,
- $\sigma = 1$, $\quad 0 < \tilde{\theta} \leq 1$, $\quad 0 < \theta < \tilde{\theta}$,
- $\sigma > 1$, $\quad 0 < \tilde{\theta} \leq 1$, $\quad \frac{\sigma-1}{\sigma}\tilde{\theta} \leq \theta < \tilde{\theta}$,

or if $(q, \theta, \tilde{\theta}) = (\infty, 1, 1)$.

Theorem 6.7.10 (Global inhomogeneous estimates)**.** *Suppose that (q, θ) and $(\tilde{q}, \tilde{\theta})$ are two jointly σ-acceptable exponent pairs. Then the operator $W(t)$ obeys the estimate*

$$\|W(t)F\|_{L^q_t(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|F\|_{L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)} \tag{6.87}$$

for all $F \in L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)$ whenever $\sigma \leq 1$. If $\sigma > 1$ we consider different cases *(i) the point $(1/q, 1/\tilde{q})$ lies inside $\triangle OAB$ in fig. 2.1, that is $1 < q, \tilde{q} < \infty$ and $q > \tilde{q}'$: then $W(t)$ satisfies (6.87) for all $F \in L^{\tilde{q}'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}{}^*)$.*

(ii) The point $(1/q, 1/\tilde{q})$ lies on the hypotenuse AB, that is $1 < q, \tilde{q} < \infty$ and $q = \tilde{q}'$: then $W(t)$ satisfies

$$\|W(t)F\|_{L^q_t(\mathbb{R};\mathcal{B}_{\theta,q})} \lesssim \|F\|_{L^q(\mathbb{R};\mathcal{B}_{\tilde{\theta},\tilde{q}}{}^*)}$$

for all $F \in L^q(\mathbb{R}; \mathcal{B}_{\tilde{\theta},\tilde{q}}{}^*)$.

(iii) The point $(1/q, 1/\tilde{q})$ lies on the side OA, that is $1 < q < \infty$, $\tilde{q} = \infty$: then $W(t)$ satisfies

$$\|W(t)F\|_{L^{q,\infty}_t(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|F\|_{L^1(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)}\,, \tag{6.88}$$

for all $F \in L^1(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}{}^*)$.

(iv) The point $(1/q, 1/\tilde{q})$ lies on the side OB, that is $1 < \tilde{q} < \infty$, $q = \infty$: then $W(t)$ satisfies

$$\|W(t)F\|_{L^\infty_t(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|F\|_{L^{\tilde{q}',1}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)} \tag{6.89}$$

for all $F \in L^{\tilde{q}',1}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}{}^*)$.

Remark 6.7.11. The appearance of Lorentz norms in some of the estimates above is not a great obstacle to applications. Indeed, if we restrict to finite time intervals $[0, T]$, we have the continuous embeddings

$$\begin{aligned} L^{q,r}([0,T]) &\hookrightarrow L^p([0,T]), \quad q > p,\ 1 \le q, p, r \le \infty, \\ L^p([0,T]) &\hookrightarrow L^{q,r}([0,T]), \quad p > q,\ 1 \le q, p, r \le \infty, \end{aligned}$$

see [2, p. 217]. For example, let q, θ and $\tilde{\theta}$ be such that estimate (6.86) holds and let $1 \le Q < q$. Then we have the local homogeneous estimate

$$\|U(t)f\|_{L^Q_t([0,T];\mathcal{B}_\theta)} \lesssim_T \|f\|_{\mathcal{B}_{\tilde{\theta}}{}^*}\,.$$

Similarly, if for example $\tilde{q}$, θ and $\tilde{\theta}$ are such that estimate (6.89) holds and $1 \le \tilde{Q} < \tilde{q}$, then we have the local inhomogeneous estimate

$$\|W(t)F\|_{L^\infty_t([0,T];\mathcal{B}_\theta)} \lesssim_T \|F\|_{L^{\tilde{Q}'}([0,T];\mathcal{B}_{\tilde{\theta}}{}^*)}\,.$$

Chapter 7

Proofs

In this chapter we shall prove the estimates we presented in the sections of the preceding chapter. We shall give complete and rigorous proofs only in the abstract formulation of the estimates. The estimates for the concrete equations we considered so far shall be derived as consequences of the abstract estimates.

We begin with a revision of the TT^*-principle in the abstract setting.

7.1 The TT^*-principle

Lemma 7.1.1. *[22, p. 56], [41, p. 113] Let T be a linear operator, $\mathcal{B}$ be a Banach space, and $\mathcal{H}$ be a Hilbert space. The following statements are equivalent:*

(i) $T : \mathcal{H} \to \mathcal{B}$ is bounded,

(ii) $T^* : \mathcal{B}^* \to \mathcal{H}$ is bounded,

(iii) $TT^* : \mathcal{B}^* \to \mathcal{B}$ is bounded.

Furthermore, we have the following equality of operator norms $\|T\|^2 = \|T^*\|^2 = \|TT^*\|$.

A few remarks are due. The second source [41] contains the proof of this lemma in the important and technically uncomplicated setting of $\mathcal{B} = L^p$, for $1 \leq p \leq \infty$, $\mathcal{H} = L^2$. The general case is presented in the first source [22] and the references therein. The general proof is word for word the same as the L^p-case if one replaces the L^p-symbol with that of the Banach space $\mathcal{B}$. We notice, however, that in the context of the Lebesgue spaces the lemma holds with $\mathcal{B} = L^\infty$ but instead of the dual space to L^∞ we can use its associate L^1.

We next present an important consequence of the TT^*-principle that shall play a role in the proof of the inhomogeneous Strichartz estimates. Suppose that the bounded linear operator $T : \mathcal{B}_1 \to L^q(\mathbb{R}; \mathcal{B}_\theta)$ is of the form $Tf = U(t)f$. Then its formal adjoint $T^* : L^{q'}(\mathbb{R}; {\mathcal{B}_\theta}^*) \to \mathcal{B}_1$ is a bounded linear operator of the form $\int_{\mathbb{R}} U^*(t)F(t)dt$. We shall call the exponent pair (q, θ) admissible. Suppose that T is bounded for two admissible pairs (q, θ) and $(\tilde{q}, \tilde{\theta})$. Then the composition of $T : \mathcal{B}_1 \to L^q(\mathbb{R}; \mathcal{B}_\theta)$ with $T^* : L^{\tilde{q}'}(\mathbb{R}; {\mathcal{B}_{\tilde{\theta}}}^*) \to \mathcal{B}_1$ is the bounded operator

$$TT^* : L^{\tilde{q}'}(\mathbb{R}; {\mathcal{B}_{\tilde{\theta}}}^*) \to L^q(\mathbb{R}; \mathcal{B}_\theta), \quad TT^*F = \int_{\mathbb{R}} U(t-s)F(s)ds.$$

Notice the similarity between TT^*F and

$$W(t)F = \int_{-\infty}^{t} U(t-s)F(s)ds.$$

The boundedness of the operator $W(t) : L^{\tilde{q}'}(\mathbb{R}; {\mathcal{B}_{\tilde{\theta}}}^*) \to L^q(\mathbb{R}; \mathcal{B}_\theta)$ easily implies that of the $TT^* : L^{\tilde{q}'}(\mathbb{R}; {\mathcal{B}_{\tilde{\theta}}}^*) \to L^q(\mathbb{R}; \mathcal{B}_\theta)$ under some minor assumptions on $U(t)$. The proof of that statement follows if we consider that

$$\int_{\mathbb{R}} = \int_{-\infty}^{t} + \int_{t}^{\infty}$$

in the definition of the TT^*-operator and then make a change of variables in the second integral on the right to transform it to an integral like the first one on the right. The details are left as an exercise and we now address the more important question of when the boundedness of the TT^*-operator implies that of $W(t)$. In general this implication holds whenever $q > \tilde{q}'$ and there are known counterexamples to the limiting case $q = \tilde{q}'$. This is due to the celebrated Lemma 7.1.5 of Christ-Kiselev. This combination of the TT^*-principle with the Christ-Kiselev Lemma is the standard way of obtaining Strichartz estimates for $W(t)$ from the estimates for $U(t)$. In passing we remark that in the symmetric case $(q,\theta) = (\tilde{q},\tilde{\theta})$ the opposite is also true which is an easy consequence of Lemma 7.1.2.

The TT^*-principle for Strichartz estimates can be recast in a bilinear formulation which is more effective.

Lemma 7.1.2 (Keel and Tao [30])**.** *Consider the bilinear form*

$$B(F,G) = \iint_{s<t} \langle U^*(s)F, U^*(t)G\rangle dsdt. \tag{7.1}$$

(i) The boundedness of the operator $T : \mathcal{H} \to L^q(\mathbb{R};\mathcal{B}_\theta)$ of the form $Tf = U(t)f$ is equivalent to the boundedness of the bilinear mapping $B : L^{q'}(\mathbb{R};{\mathcal{B}_\theta}^*) \times L^{q'}(\mathbb{R};{\mathcal{B}_\theta}^*) \to \mathbb{C}$.

(ii) The boundedness of the operator $W(t) : L^{\tilde{q}'}(\mathbb{R};{\mathcal{B}_{\tilde{\theta}}}^*) \to L^q(\mathbb{R};\mathcal{B}_\theta)$ is equivalent to that of the bilinear mapping $B : L^{q'}(\mathbb{R};{\mathcal{B}_\theta}^*) \times L^{\tilde{q}'}(\mathbb{R};{\mathcal{B}_{\tilde{\theta}}}^*) \to \mathbb{C}$.

As with the KT equation we consider the decompositions

$$\Omega = \bigcup_{\lambda} \bigcup_{Q\in\mathcal{O}_\lambda}, \quad B(F,G) = \sum_{\lambda}\sum_{Q\in\mathcal{O}_\lambda} B_Q(F,G),$$

where

$$B_Q(F,G) = \iint_Q \langle U^*(s)F(s), U^*(t)G(t)\rangle dsdt.$$

The advantage of the above decomposition is that whenever $Q = J \times I$ and $Q \in \mathcal{O}_\lambda$ we have

$$\lambda = |I| = |J| \sim \operatorname{dist}(\Omega, \partial\Omega) \sim \operatorname{dist}(I,J). \tag{7.2}$$

The very special property (7.2) of this decomposition allows us to obtain the following scaling invariance

$$|B_Q(F,G)| \lesssim \lambda^{\beta(q,\theta,\tilde{q},\tilde{\theta})} \|F\|_{L^{\tilde{q}'}(J;{\mathcal{B}_{\tilde{\theta}}}^*)} \|G\|_{L^{q'}(I;{\mathcal{B}_\theta}^*)}, \tag{7.3}$$

of each dyadic piece B_Q in the bilinear form B. The latter shall be proved in section 7.3.1 and in particular Lemma 7.3.4 gives a certain range for the ordered 4-tuple of exponents (q,θ), $(\tilde{q},\tilde{\theta})$, where the local scaling (7.3) holds true. Another scaling invariant quantity is given by

Lemma 7.1.3. *If $\frac{1}{q} + \frac{1}{\tilde{q}} \le 1$, then*

$$\sum_{Q\in\mathcal{O}_\lambda} |B_Q(F,G)| \lesssim \lambda^{\beta(q,\theta,\tilde{q},\tilde{\theta})} \|F\|_{L^{\tilde{q}'}(\mathbb{R};{\mathcal{B}_{\tilde{\theta}}}^*)} \|G\|_{L^{q'}(\mathbb{R};{\mathcal{B}_\theta}^*)}.$$

Proof. In view of (7.3)

$$\sum_{Q\in\mathcal{O}_\lambda} |B_Q(F,G)| \lesssim \lambda^{\beta(q,\theta,\tilde{q},\tilde{\theta})} \sum_{Q\in\mathcal{O}_\lambda,\, Q=J\times I} \|F\|_{L^{\tilde{q}'}(J;{\mathcal{B}_{\tilde{\theta}}}^*)} \|G\|_{L^{q'}(I;{\mathcal{B}_\theta}^*)}.$$

An application of Lemma 7.1.4 below completes the proof. □

Lemma 7.1.4. *Suppose* $\frac{1}{p}+\frac{1}{\tilde{p}}\geq 1$. *Then*

$$\sum_{Q\in\mathcal{O}_\lambda,\ Q=J\times I}\|f\|_{L^{\tilde{p}}(J)}\|g\|_{L^p(I)}\leq\|f\|_{L^{\tilde{p}}(\mathbb{R})}\|g\|_{L^p(\mathbb{R})}.$$

Proof. The lemma follows directly from the inequality

$$\sum_j|a_jb_j|\leq\left(\sum_j|a_j|^{\tilde{p}}\right)^{\frac{1}{\tilde{p}}}\left(\sum_j|b_j|^p\right)^{\frac{1}{p}},$$

which holds in the range $\frac{1}{p}+\frac{1}{\tilde{p}}\geq 1$, and the fact that for each dyadic interval I there are at most two dyadic squares in $\mathcal{O}_\lambda$ with side I. ☐

Consider the bilinear operator $A: L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)\times L^{q'}(\mathbb{R};\mathcal{B}_\theta{}^*)\to l^\infty_s$, defined by the formula

$$A(F,G)=\{b_\lambda\}_{\lambda\in 2^{\mathbb{Z}}}=\left\{\sum_{Q\in\mathcal{O}_\lambda}|B_Q(F,G)|\right\}_{\lambda\in 2^{\mathbb{Z}}}.$$

Thus, in view of the bilinear formulation of the TT^* in Lemma 2.5.2, the estimate

$$\|\{b_\lambda\}\|_{l^1_0}\lesssim\|F\|_{L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)}\|G\|_{L^{q'}(\mathbb{R};\mathcal{B}_\theta{}^*)},\quad\forall F\in L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*),\ \forall G\in L^{q'}(\mathbb{R};\mathcal{B}_\theta{}^*),$$

implies the boundedness of $W(t): L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)\to L^q(\mathbb{R};\mathcal{B}_\theta)$.

The next result shall be useful in the sequel.

Lemma 7.1.5 (Christ-Kiselev, see Lemma 3.1 of [48], or [47])**.** *Suppose that the integral operator*

$$T[F](t)=\int_{-\infty}^{\infty}K(t,s)F(s)ds\tag{7.4}$$

is bounded from $L^p(\mathbb{R};\mathcal{B}_1)$ *to* $L^q(\mathbb{R};\mathcal{B}_2)$ *for some Banach spaces* $\mathcal{B}_1$, $\mathcal{B}_2$ *and* $1\leq p<q\leq\infty$. *The operator-valued kernel* $K(t,s):\mathbb{R}^2\to L(\mathcal{B}_1,\mathcal{B}_2)$ *maps* $\mathbb{R}^2$ *to the space of all bounded linear operators from* $\mathcal{B}_1$ *to* $\mathcal{B}_2$. *Assume also that the kernel* K *is regular enough to ensure that (7.4) is well-defined as a* $\mathcal{B}_2$*-valued Bochner integral for almost all* $t\in\mathbb{R}$. *Then the operator*

$$\tilde{T}[F](t)=\int_{-\infty}^{t}K(t,s)F(s)ds$$

is also bounded on the same spaces.

7.2 Estimates for admissible exponents

7.2.1 The basic case

Let us recall

Theorem 7.2.1. *The estimate*

$$\|U(t)f\|_{L^q_t(\mathbb{R};\mathcal{B}_\theta)}+\|W(t)F\|_{L^q_t(\mathbb{R};\mathcal{B}_\theta)}\lesssim\|f\|_{\mathcal{B}_1}+\|F\|_{L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)},\tag{7.5}$$

holds for all $f\in\mathcal{B}_1$, *and all* $F\in L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)$, *whenever* (q,θ) *and* $(\tilde{q},\tilde{\theta})$ *are two* σ*-admissible exponent pairs. However, in the double endpoint case* $(q,\theta)=(\tilde{q},\tilde{\theta})=(2,\theta^*)$, *the norm of the space* $L^2(\mathbb{R};\mathcal{B}_{\theta^*})$ *has to be replaced by that of the space* $L^2(\mathbb{R};\mathcal{B}_{\theta^*,2})$.

Proof. By complex interpolation between the dispersive estimate (6.69) and the energy estimate

(6.70), we obtain the following decay estimate

$$\|U(t)f\|_{\mathcal{B}_\theta} \lesssim \frac{1}{|t|^{\sigma(1-\theta)}} \|f\|_{\mathcal{B}_{\theta^*}}, \quad \theta \in [0,1].$$

Using this, we obtain

$$\|TT^*F\|_{\mathcal{B}_\theta} \lesssim \int_{-\infty}^{\infty} \|U(t-s)F(s)\|_{\mathcal{B}_\theta}\, ds \lesssim \int_{-\infty}^{\infty} \frac{\|F(s)\|_{\mathcal{B}_{\theta^*}}}{|t-s|^{\sigma(1-\theta)}} ds. \tag{7.6}$$

We now take the L^q-norm in t in both sides of the above inequality. To estimate the right hand side (RHS), we apply the Hardy-Littlewood-Sobolev theorem of fractional integration, see [2, pp. 228-229], [41]. Thus we obtain

$$\|TT^*F\|_{L^q_t(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|F\|_{L^{q'}(\mathbb{R};\mathcal{B}_{\theta^*})},$$

whenever $0 < \sigma(1-\theta) < 1$, $1 + 1/q = 1/q' + \sigma(1-\theta)$. The latter conditions are equivalent to $\theta^* < \theta < 1$, $2/q = \sigma(1-\theta)$. Remember that the exponent $\sigma(1-\theta) = \alpha$ must be in $(0,1)$ in order to apply the above argument. However, the left endpoint $\alpha = 0$, equivalent to $\theta = 1$, coincides with the energy estimate (6.70). In view of the TT^*-principle, this implies the estimate

$$\|U(t)f\|_{L^q_t(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|f\|_{\mathcal{B}_1},$$

whenever (q,θ) is σ-admissible non-endpoint exponent pair.

The right endpoint $\alpha = 1$, equivalent to $\theta = \theta^*$, (see (6.81) if necessary) is endpoint and it is too delicate to be resolved by the same argument. The corresponding estimate is the endpoint homogeneous Strichartz estimate

$$\|U(t)f\|_{L^2_t(\mathbb{R};\mathcal{B}_{\theta^*})} \lesssim \|f\|_{\mathcal{B}_1}, \quad \forall f \in \mathcal{B}_1. \tag{7.7}$$

Estimate (7.7) has been proved false for many concrete equations when $\sigma = 1$. In higher dimensions, when $\sigma > 1$, Keel and Tao showed that the modified estimate

$$\|U(t)f\|_{L^2_t(\mathbb{R};\mathcal{B}_{\theta^*,2})} \lesssim \|f\|_{\mathcal{B}_1}, \quad \forall f \in \mathcal{B}_1$$

always holds. Of course, in the special case when $\mathcal{B}_\theta$ are Lebesgue spaces L^r with $r \geq 2$, this estimate implies the original one.

In view of the TT^*-principle and the Christ-Kiselev Lemma 7.1.5, the above implies the inhomogeneous estimate

$$\|W(t)f\|_{L^q_t(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|F\|_{L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}^*})}, \tag{7.8}$$

whenever (q,θ), $(\tilde{q},\tilde{\theta})$ are two σ-admissible exponent pairs with $q > \tilde{q}'$. The double endpoint case $(q,\theta) = (\tilde{q},\tilde{\theta}) = (2,\theta^*)$ follows from Theorem 6.7.10, part (ii), whose proof can be found in Subsection 7.3.3 □

7.2.2 Generalized global homogeneous estimates

Proposition 7.2.2. *Suppose that (q,θ) is an exponent pair satisfying*

$$\frac{1}{q} = \frac{\sigma}{2}\left(2 - \theta - \tilde{\theta}\right),$$

for some $\tilde{\theta} \in (0,1]$. Then the estimate

$$\|U(t)f\|_{L^{q,c}_t(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|f\|_{\mathcal{B}_{\tilde{\theta},c'}{}^*}, \quad 1 \leq c \leq \infty, \tag{7.9}$$

holds for every $f \in \mathcal{B}_{\tilde{\theta},c'}{}'$ whenever the exponents θ and $\tilde{\theta}$ are in the range

- $\sigma < 1, \qquad 0 < \tilde{\theta} \le 1, \qquad 0 \le \theta < \tilde{\theta},$
- $\sigma = 1, \qquad 0 < \tilde{\theta} \le 1, \qquad 0 < \theta < \tilde{\theta},$
- $\sigma > 1, \qquad 0 < \tilde{\theta} \le 1, \qquad \theta^* \le \theta < \tilde{\theta},$

or if $(q, \theta, \tilde{\theta}) = (\infty, 1, 1)$.

Proof. Suppose at first that $\sigma \neq 1$. We interpolate with the real method with parameters η, c, for $0 < \eta < 1$, $1 \le c \le \infty$, in the two inequalities below

$$\|U(t)f\|_{L_t^{\xi(\theta),\infty}(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|f\|_{\mathcal{B}_\theta{}^*}, \quad \theta^* \le \theta \le 1, \tag{7.10}$$

$$\|U(t)f\|_{L_t^q(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|f\|_{\mathcal{B}_1{}^*}, \quad \theta^* \le \theta \le 1, \tag{7.11}$$

where $1/\xi(\theta) = \sigma(1-\theta)$. In view of the reiteration theorem, see [2], we obtain the estimate

$$\|U(t)f\|_{L_t^{Q,c}(\mathbb{R};\mathcal{B}_{[\theta]})} \lesssim \|f\|_{\mathcal{B}_{\tilde{\theta},c'}{}^*},$$

where

$$\frac{1}{Q} = \frac{1-\eta}{\xi} + \frac{\eta}{q}, \quad 0 < \eta < 1,$$
$$\tilde{\theta} = \theta(1-\eta) + \eta.$$

Expressing ξ and q in terms of θ and eliminating η from these equations, we obtain the equivalent conditions

$$\frac{1}{Q} = \frac{\sigma}{2}(2 - \theta - \tilde{\theta}), \quad \theta < \tilde{\theta} < 1, \quad \theta^* \le \theta \le 1.$$

Relabeling Q by q and reformulating the inequalities above as $\theta^* < \tilde{\theta} < 1$, $\theta^* \le \theta < \tilde{\theta}$, we finish the proof in the case $\sigma \neq 1$.

The case $\sigma = 1$ is treated in exactly the same way but this time estimates (7.10), (7.11) are valid only in the range $\theta^* < \theta \le 1$. □

Note that instead of real interpolation we can use the complex method, which yields the alternative estimate

$$\|U(t)f\|_{L_t^{q,\infty}(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|f\|_{\mathcal{B}_{\tilde{\theta}}{}^*}. \tag{7.12}$$

The argument of this section is a generalization of an argument of Kato [29], originally presented in the specific context of the Schrödinger equation. We shall further extend the range of these estimates for $\sigma > 1$ in section 7.3.4.

7.3 Estimates for acceptable exponents

This section is dedicated to the proof of the global inhomogeneous Strichartz estimates of Theorem 6.7.10 which is to be done considering several different case. We begin with the proof of some local estimates that shall be crucial in the sequel.

7.3.1 Local inhomogeneous estimates

Following Foschi [20], we want to find the range of local estimates for $W(t)$ that are invariant to the scaling

$$\|W(t)[\chi_{\lambda J}F]\|_{L_t^q(\lambda I;\mathcal{B}_\theta)} \lesssim \lambda^{\beta(q,\theta,\tilde{q},\tilde{\theta})} \|F\|_{L^{\tilde{q}'}(\lambda J;\mathcal{B}_{\tilde{\theta}}{}^*)}, \quad \forall \lambda > 0, \tag{7.13}$$

where I and J are two unit intervals separated by a unit distance and $\chi_{\lambda J}$ is the characteristic of the rescaled interval λJ and

$$\beta(q,\theta,\tilde{q},\tilde{\theta}) = \frac{1}{q} + \frac{1}{\tilde{q}} - \frac{\sigma}{2}(2-\theta-\tilde{\theta}). \tag{7.14}$$

The bilinear formulation of (7.13) is

$$|B_Q(F,G)| \lesssim \lambda^{\beta(q,\theta,\tilde{q},\tilde{\theta})} \|F\|_{L^{\tilde{q}'}(J;\mathcal{B}_{\tilde{\theta}}{}^*)} \|G\|_{L^{q'}(I;\mathcal{B}_\theta{}^*)}, \tag{7.15}$$

where Q is the square $I \times J$.

Lemma 7.3.1. *Estimate* (7.13) *holds for any two σ-admissible pairs* (q,θ) *and* $(\tilde{q},\tilde{\theta})$.

Proof. The proof follows trivially from Theorem 6.1.6 due to the fact that $\beta(q,\theta,\tilde{q},\tilde{\theta}) = 0$ under the hypothesis of the lemma. □

Lemma 7.3.2. *Estimate* (7.13) *holds with* $(q,\theta) = (\tilde{q},\tilde{\theta}) = (\infty,0)$.

Proof. By the dispersive estimate (6.69) we have that

$$\begin{aligned}\sup_{t\in\lambda I} \|W(t)[\chi_{\lambda J}F]\|_{\mathcal{B}_0} &\lesssim \sup_{t\in\lambda I} \int_{\lambda J} \frac{\|F(\tau)\|_{\mathcal{B}_0{}^*}}{|t-\tau|^\sigma} d\tau \\ &\lesssim \lambda^{\beta(\infty,0,\infty,0)} \|F\|_{L^1(\lambda I;\mathcal{B}_0{}^*)}.\end{aligned}$$

□

Lemma 7.3.3. *Whenever* (q,θ) *and* $(\tilde{q},\tilde{\theta})$ *are exponent pairs for which estimate* (7.13) *holds, we have that* (7.13) *also holds with* (Q,θ) *and* $(\tilde{Q},\tilde{\theta})$, *where* $1 \le Q \le q$, $1 \le \tilde{Q} \le \tilde{q}$.

Proof. A trivial application of Hölder's inequality

$$\begin{aligned}\|W(t)[\chi_{\lambda J}F]\|_{L_t^Q(\lambda I;\mathcal{B}_\theta)} &\lesssim \lambda^{\frac{1}{Q}-\frac{1}{q}} \|W(t)[\chi_{\lambda J}F]\|_{L_t^q(\lambda I;\mathcal{B}_\theta)} \\ &\lesssim \lambda^{\beta(Q,r,\tilde{q},\tilde{r})} \|F\|_{L^{\tilde{q}'}(\lambda J;\mathcal{B}_{\tilde{\theta}}{}^*)} \lesssim \lambda^{\beta(Q,r,\tilde{Q},\tilde{r})} \|F\|_{L^{\tilde{q}'}(\lambda J;\mathcal{B}_{\tilde{\theta}}{}^*)}.\end{aligned}$$

□

Let us define the range of validity of the local estimates (7.13) as the set $\mathcal{E}$ in $\mathbb{R}^4$. Each point in $\mathcal{E}$ corresponds to a 4-tuple of exponents $(1/q,\theta,1/\tilde{q},\tilde{\theta})$. Below we find the convex hull $\mathcal{E}^*$ ($\mathcal{E}^* \subseteq \mathcal{E}$) of the points in $\mathbb{R}^4$ that correspond to the estimates in the three lemmas above. We shall call any point or collection of points in $\mathcal{E}$ *acceptable*.

Lemma 7.3.4 (Local inhomogeneous estimates)**.** *Estimate* (7.13), *or equivalently* (7.15), *holds whenever the exponent pairs* (q,θ), $(\tilde{q},\tilde{\theta}) \in \mathcal{E}^*$ *given explicitly by the following conditions*

$$0 \le \frac{1}{q}, \frac{1}{\tilde{q}} \le 1, \quad 0 \le \theta, \tilde{\theta} \le 1, \tag{7.16}$$

$$\frac{\sigma}{2}(\tilde{\theta}-\theta) \le \frac{1}{q}, \quad \frac{\sigma}{2}(\theta-\tilde{\theta}) \le \frac{1}{\tilde{q}}, \tag{7.17}$$

$$(\sigma-1)\tilde{\theta} \le \sigma\theta, \quad (\sigma-1)\theta \le \sigma\tilde{\theta}. \tag{7.18}$$

If $\sigma \ge 1$ *then also* $\theta, \tilde{\theta} > 0$.

Remark 7.3.5. Condition (7.18) is void when $\sigma \le 1$.

Proof. We apply the Riesz-Thorin convexity theorem to interpolate between the already proven local estimates. In essence, we find the convex hull of the locally acceptable sets associated with Lemmas 7.3.1 and 7.3.2 and then expand that set by the rule given in Lemma 7.3.3.

When $\sigma \neq 1$ the set of acceptability S_1 of the local estimates in Lemma 7.3.1 is given by the system

$$S_1 = \begin{cases} \dfrac{1}{q} = \dfrac{\sigma}{2}(1-\theta), \quad \dfrac{1}{\tilde{q}} = \dfrac{\sigma}{2}(1-\tilde{\theta}), \\ \theta^* \leq \theta \leq 1, \quad \theta^* < \tilde{\theta} \leq 1. \end{cases} \tag{7.19}$$

Interpolating with $O = (0,0,0,0)$ we get

$$S_2 = \begin{cases} \dfrac{1}{Q} = \dfrac{\eta}{q}, \quad \dfrac{1}{\tilde{Q}} = \dfrac{\eta}{\tilde{q}}, \\ \Theta = \eta\theta \quad \tilde{\Theta} = \eta\tilde{\theta}, \quad 0 \leq \eta \leq 1. \end{cases}$$

And lastly, applying to S_2 the rule of Lemma 7.3.3 we get

$$S_3 = \begin{cases} 1 \geq \dfrac{1}{Q} \geq \dfrac{\eta}{q}, \quad 1 \geq \dfrac{1}{\tilde{Q}} \geq \dfrac{\eta}{\tilde{q}}, \\ \Theta = \eta\theta \quad \tilde{\Theta} = \eta\tilde{\theta}, \qquad 0 \leq \eta \leq 1. \end{cases} \tag{7.20}$$

We need to eliminate from the definition of S_3 the following variables $q, \tilde{q}, \theta, \tilde{\theta}$ and η. By expressing q and $\tilde{q}$ in terms of θ and $\tilde{\theta}$ respectively, see (7.19), we simplify the four inequalities in (7.20) to

$$0 \leq \frac{1}{Q}\frac{1}{\tilde{Q}} \leq 1,$$

$$\eta \leq \frac{2}{\sigma Q} + \Theta, \quad \eta \leq \frac{2}{\sigma \tilde{Q}} + \tilde{\Theta}.$$

The two equalities in (7.20) are simplified to

$$\eta\theta^* < \Theta \leq \eta, \quad \eta\theta^* < \tilde{\Theta} \leq \eta, \quad 0 \leq \eta \leq 1.$$

Let us now group all similar inequalities for η

$$0, \Theta, \tilde{\Theta} \leq \eta, \tag{7.21}$$

$$\eta < \frac{\Theta}{\theta^*}, \frac{\tilde{\Theta}}{\theta^*}, \tag{7.22}$$

$$\eta \leq \frac{2}{\sigma Q} + \Theta, \frac{2}{\sigma \tilde{Q}} + \tilde{\Theta}, 1. \tag{7.23}$$

There is $\eta \in [0,1]$ solving the inequalities in (7.21), (7.22), (7.23), if and only if any quantity on the left is bounded by any quantity on the right in these inequalities. This gives the lemma, i.e. that $S_3 = \mathcal{E}^*$. □

7.3.2 Non-endpoint global inhomogeneous estimates

Our goal in this subsection shall be to show the boundedness of

$$A : L^{\tilde{q}'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}{}^*) \times L^{q'}(\mathbb{R}; \mathcal{B}_{\theta}{}^*) \to l^1 \tag{7.24}$$

whenever the ordered 4-tuple (q,θ), $(\tilde{q},\tilde{\theta})$ is non-endpoint in a certain sense, recall the notation introduced at the end of section 7.1.

Suppose that $(1/q, 1/\tilde{q}) \in \Delta_0$, where $\Delta_0 = \{1/q > 0, 1/\tilde{q} > 0, 1/q + 1/\tilde{q} < 1\}$, and that the 4-tuple (q,θ), $(\tilde{q},\tilde{\theta}) \in \mathcal{E}^*$, together with a neighborhood of small perturbations in $(1/q, 1/\tilde{q})$. Then in virtue of Corollary 7.1.3 we have the estimate

$$|b_\lambda| \lesssim \lambda^{\beta(q,\theta,\tilde{q},\tilde{\theta})} \|F\|_{L^{\tilde{q}'}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)} \|G\|_{L^{q'}(\mathbb{R};\mathcal{B}_{\theta}{}^*)},$$

or in other words $\{b_\lambda\} \in l^\infty_{\beta(q,\theta,\tilde{q},\tilde{\theta})}$. Since Δ_0 is an open set (triangle) on the $(1/q, 1/\tilde{q})$-coordinate plane, we can always find a small enough open neighborhood of points in Δ_0 around $(1/q, 1/\tilde{q})$. Let us set

$$1/q_0 = 1/q + \epsilon, \quad 1/\tilde{q}_0 = 1/\tilde{q} + \epsilon, \quad 1/q_1 = 1/q - 3\epsilon, \quad 1/\tilde{q}_1 = 1/\tilde{q} - 3\epsilon.$$

Suppose that $\epsilon > 0$ is small enough so that $(1/q_0, 1/\tilde{q}_0), (1/q_1, 1/\tilde{q}_1) \in \Delta \cap \mathcal{E}^*$. Suppose also that, cf. (7.14), $\beta(q, \theta, \tilde{q}, \tilde{\theta}) = 0$. Then we have that $\beta(q_0, \theta, \tilde{q}_0, \tilde{\theta}) = 2\epsilon$, and $\beta(q_1, \theta, \tilde{q}_0, \tilde{\theta}) = \beta(q_0, \theta, \tilde{q}_1, \tilde{\theta}) = -2\epsilon$. Thus we obtain the maps

$$A : L^{\tilde{q}_0'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}{}^*) \times L^{q_0'}(\mathbb{R}; \mathcal{B}_\theta{}^*) \to l^\infty_{2\epsilon},$$
$$A : L^{\tilde{q}_0'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}{}^*) \times L^{q_1'}(\mathbb{R}; \mathcal{B}_\theta{}^*) \to l^\infty_{-2\epsilon},$$
$$A : L^{\tilde{q}_1'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}{}^*) \times L^{q_0'}(\mathbb{R}; \mathcal{B}_\theta{}^*) \to l^\infty_{-2\epsilon},$$

are bounded. In virtue of Lemma 2.5.17 we have that the map

$$A : (L^{\tilde{q}_0'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}{}^*), L^{\tilde{q}_1'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}{}^*))_{1/4,\tilde{q}'} \times (L^{q_0'}(\mathbb{R}; \mathcal{B}_\theta{}^*), L^{q_1'}(\mathbb{R}; \mathcal{B}_\theta{}^*))_{1/4,q'} \to (l^\infty_{2\epsilon}, l^\infty_{-2\epsilon})_{1/2,1}$$

is also bounded. Finally, in view of the well-known interpolation identities of the Lorentz spaces and that of Lemma 2.5.16, this implies (7.24)

Now let us recapitulate all conditions we have imposed so far on the exponents. We have the conditions of the local estimates (set $\mathcal{E}^*$) plus the scaling condition $\beta(q, \theta, \tilde{q}, \tilde{\theta}) = 0$. Remember that all inequalities in the definition of $\mathcal{E}^*$ that contain q or $\tilde{q}$ must be rewritten as strict inequalities to allow perturbation in these quantities. Also note that conditions (7.17) together with $\beta = 0$ are equivalent to (q, θ) and $(\tilde{q}, \tilde{\theta})$ being KT-acceptable.

7.3.3 Endpoint global inhomogeneous estimates, case of $q = \tilde{q}'$

We now proceed with the proof of the endpoint estimates with exponents that lie on the hypotenuse on ΔOAB, see figure 2.1. To that end we shall need the well-known interpolation identities

$$(L^p(\mathbb{R}; \mathcal{A}_0), L^p(\mathbb{R}; \mathcal{A}_1))_{\theta,p} = L^p(\mathbb{R}; (\mathcal{A}_0, \mathcal{A}_1)_{\theta,p}), \quad 1 < p < \infty, \tag{7.25}$$

see [3]. We fix an exponent 4-tuple $(1/q, \theta, 1/\tilde{q}, \tilde{\theta}) \in \mathcal{E}^*$ such that $\beta(q, \theta, \tilde{q}, \tilde{\theta}) = 0$. We perturb the exponents in estimate (7.3) by finding two 4-tuples $(1/q, \theta_0, 1/\tilde{q}, \tilde{\theta}_0), (1/q, \theta_1, 1/\tilde{q}, \tilde{\theta}_1) \in \mathcal{E}^*$ subject to

$$\theta_0 = \theta + \epsilon, \quad \tilde{\theta}_0 = \tilde{\theta} + \epsilon, \quad \theta_1 = 1/q - 3\epsilon, \quad \tilde{\theta}_1 = \tilde{\theta} - 3\epsilon.$$

Thus $\beta(q, \theta_0, \tilde{q}, \tilde{\theta}_0) = 2\sigma\epsilon$ and $\beta(q, \theta_1, \tilde{q}, \tilde{\theta}_0) = \beta(q, \theta_0, \tilde{q}, \tilde{\theta}_1) = -2\sigma\epsilon$. Hence the maps

$$A : L^{\tilde{q}'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}_0}{}^*) \times L^{q'}(\mathbb{R}; \mathcal{B}_{\theta_0}{}^*) \to l^\infty_{2\epsilon},$$
$$A : L^{\tilde{q}'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}_0}{}^*) \times L^{q'}(\mathbb{R}; \mathcal{B}_{\theta_1}{}^*) \to l^\infty_{-2\epsilon},$$
$$A : L^{\tilde{q}'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}_1}{}^*) \times L^{q'}(\mathbb{R}; \mathcal{B}_{\theta_0}{}^*) \to l^\infty_{-2\epsilon},$$

are bounded. In virtue of Lemma 2.5.17 we have that the map

$$A : (L^{\tilde{q}'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}_0}{}^*), L^{\tilde{q}'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}_1}{}^*))_{1/4,\tilde{q}'} \times (L^{q'}(\mathbb{R}; \mathcal{B}_{\theta_0}{}^*), L^{q'}(\mathbb{R}; \mathcal{B}_{\theta_1}{}^*))_{1/4,q'} \to (l^\infty_{2\epsilon}, l^\infty_{-2\epsilon})_{1/2,1}$$

is also bounded. Finally, in view of the interpolation identity (7.25), the above simplifies to

$$A : L^{\tilde{q}'}(\mathbb{R}; \mathcal{B}_{\tilde{\theta},\tilde{q}}{}^*) \times L^{q'}(\mathbb{R}; \mathcal{B}_{\theta,q}{}^*) \to l^1,$$

where by $\mathcal{B}_{\theta,q}$ we denote the real interpolation space $(\mathcal{B}_0, \mathcal{B}_1)_{\theta,q}$ for $\theta \in [0,1]$, $1 \le q \le \infty$.

The set of validity of these estimates is determined in the same way as in the previous section. Except that now all inequalities in the definition of $\mathcal{E}^*$ that contain θ or $\tilde{\theta}$ must be rewritten as strict inequalities as we perturb with respect to these quantities.

7.3.4 Endpoint global inhomogeneous estimates, case of $\tilde{q} = \infty$

Suppose now that $(1/q, 1/\tilde{q})$ lies on either one of the two catheti of ΔOAB in figure 2.1, for the sake of concreteness let us suppose that $1/\tilde{q} = 0$. We also exclude the two endpoints $O = (0,0)$ and $A = (1,0)$ (if $\sigma \geq 1$, otherwise $A = (\sigma, 0)$), so that we can perturb with respect to q. We also suppose that (q, θ) belongs to $\mathcal{E}^*$ together with a neighborhood of small perturbations in q as well as the 4-tuple (q, θ), $(\tilde{q}, \tilde{\theta})$. Assuming that the scaling condition $\beta(q, \theta, \tilde{q}, \tilde{\theta}) = 0$ holds we have that

$$A : L^1(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}{}^*) \times L^{q_0'}(\mathbb{R}; \mathcal{B}_\theta{}^*) \to l^\infty_\epsilon,$$
$$A : L^1(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}{}^*) \times L^{q_1'}(\mathbb{R}; \mathcal{B}_\theta{}^*) \to l^\infty_{-\epsilon},$$

where

$$\frac{1}{q_0} = \frac{1}{q} - \frac{1}{\epsilon}, \quad \frac{1}{q_1} = \frac{1}{q} + \frac{1}{\epsilon}.$$

The real method with parameters $(\theta, q) = (1/2, 1)$ gives that

$$A : L^1(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}{}^*) \times L^{q',1}(\mathbb{R}; \mathcal{B}_\theta{}^*) \to l^1_0.$$

By the TT^*-principle, this means that

$$\|W(t)F\|_{L^{q,\infty}_t(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|F\|_{L^1(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)} \tag{7.26}$$

for all $F \in L^1(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}{}^*)$ whenever (q, θ) and $(\infty, \tilde{\theta})$ satisfy the assumptions we have made so far.

In view of the Equivalence Theorem 1.3.2, we also have the following homogeneous estimate

$$\|U(t)f\|_{L^{q,\infty}_t(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|f\|_{\mathcal{B}_{\tilde{\theta}}{}^*}, \tag{7.27}$$

for all $f \in \mathcal{B}^*_{\tilde{\theta}}$, in the same range as the inhomogeneous estimate (7.26).

To summarize, we have

Proposition 7.3.6. *Suppose that (q, θ) and $(\tilde{q}, \tilde{\theta})$ are two jointly-acceptable exponent pairs. If $\tilde{q} = \infty$ then the estimate*

$$\|W(t)F\|_{L^{q,\infty}_t(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|F\|_{L^1(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)}, \tag{7.28}$$

holds for every $F \in L^1(\mathbb{R}; \mathcal{B}_{\tilde{\theta}})$. If analogously $q = \infty$, then by duality the estimate

$$\|W(t)F\|_{L^\infty_t(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|F\|_{L^{\tilde{q}',1}(\mathbb{R};\mathcal{B}_{\tilde{\theta}}{}^*)}, \tag{7.29}$$

holds for every $F \in L^{\tilde{q}',1}(\mathbb{R}; \mathcal{B}_{\tilde{\theta}}{}^*)$.

Proposition 7.3.7. *Suppose that (q, θ) is an exponent pair satisfying*

$$\frac{1}{q} = \frac{\sigma}{2}\left(2 - \theta - \tilde{\theta}\right),$$

for some $\tilde{\theta} \in (0, 1]$. Then the estimate

$$\|U(t)f\|_{L^{q,\infty}_t(\mathbb{R};\mathcal{B}_\theta)} \lesssim \|f\|_{\mathcal{B}_{\tilde{\theta}}{}^*}, \tag{7.30}$$

holds for every $f \in \mathcal{B}_{\tilde{\theta},c'}{}'$ whenever the exponents θ and $\tilde{\theta}$ are in the range

- $\sigma < 1, \qquad 0 < \tilde{\theta} \leq 1, \qquad 0 \leq \theta < \tilde{\theta},$

- $\sigma = 1, \qquad 0 < \tilde{\theta} \leq 1, \qquad 0 < \theta < \tilde{\theta},$
- $\sigma > 1, \qquad 0 < \tilde{\theta} \leq 1, \qquad \frac{\sigma-1}{\sigma}\tilde{\theta} \leq \theta < \tilde{\theta},$

or if $(q, \theta, \tilde{\theta}) = (\infty, 1, 1)$.

7.4 Derivations of Strichartz estimates for concrete equations

The derivation of the Strichartz estimates from the abstract setting to each of the concrete equations in Chapter 6 is completely straightforward. In a number of cases, though, we shall need to make some extra computations. That is sketched in the context of the generalized Schrödinger and wave equations and can be found in the two subsections that follow. The cases of the Klein-Gordon and Dirac equations are completely analogous to that of the wave equation and shall not be considered separately.

Note that in the careful exposition of Keel and Tao [30], one can find the derivation of the estimates for admissible exponents in the context of the Schrödinger and the wave equation, that is Theorem 6.1.6 and Theorem 6.3.5. The sharpness of these theorems is also given in [30]. Similarly, in Foschi [20] one can find a derivation of the inhomogeneous Strichartz estimates for the Schrödinger equation, that is Theorem 6.1.9. Foschi [20] and Vilela [50] present a number of counterexamples for the sharpness of Theorem 6.1.9. In Taggart [46] one can find a derivation of the inhomogeneous Strichartz estimates for the wave equation that are similar to those given in Theorem 6.3.7.

7.4.1 Generalized Schrödinger-type equations

The Strichartz estimates for the Schrödinger and the generalized Schrödinger-type equations follow from the abstract Strichartz estimates by the identification

$$\mathcal{B}_\theta = L^r, \quad \theta = \frac{2}{r}, \quad \mathcal{B}_{\tilde{\theta}} = L^{\tilde{r}}, \quad \tilde{\theta} = \frac{2}{\tilde{r}}, \quad p = \tilde{r}', \tag{7.31}$$

for $\theta, \tilde{\theta} \in [0, 1]$.

The only additional computation is that of the generalized homogeneous estimates of Proposition 6.2.2 where one needs to upgrade an $L_t^{q,\infty}$-norm to a $L_t^{q,p}$-norm, where L^p is the class of the initial data. For that see next subsection where this matter is discussed in the context of the wave equation.

7.4.2 Generalized wave-type equations

The Strichartz estimates for the generalized wave-type equations are contained as a special case in that of the abstract Strichartz estimates by the identification

$$\mathcal{B}_\theta = B^{-\rho}_{r,2}, \quad \theta = \frac{2}{r}, \quad \mathcal{B}_{\tilde{\theta}} = B^{-\tilde{\rho}}_{\tilde{r},2}, \quad \tilde{\theta} = \frac{2}{\tilde{r}}, \quad p = \tilde{r}', \tag{7.32}$$

for $\theta, \tilde{\theta} \in [0, 1]$.

In this section we suppose that $U(t)$ is a generalized wave evolution group satisfying conditions (6.56)-(6.59).

Let us for example prove the generalized homogeneous estimates of Proposition 6.6.3. In view of the abstract estimates of Proposition 6.7.9 and the identification above we have the estimate

$$\|U(t)f\|_{L^{q,\infty}(\mathbb{R};B^{-\rho}_{r,2})} \lesssim \|f\|_{B^{\tilde{\rho}}_{\tilde{r}',2}},$$

or equivalently

$$\left\|D^{-\rho-\tilde{\rho}}U(t)f\right\|_{L^{q,\infty}(\mathbb{R};B^{0}_{r,2})} \lesssim \|f\|_{B^{0}_{\tilde{r}',2}},$$

in the same range for the exponents as in Proposition 6.7.9. In fact, this estimate implies

$$\left\|D^{-\rho-\tilde{\rho}}U(t)f\right\|_{L_t^{q,\infty}L_x^r} \lesssim \|f\|_{L_x^p}\,, \tag{7.33}$$

by the usual embeddings and the fact that $r \geq 2$ and $p \leq 2$. Now we use a standard argument that shall be repeated in similar situations. We slightly perturb the exponents q and p and use real interpolation with (θ, p). Basically, in this way we sharpen (7.33) *inside* its range of validity to

$$\left\|D^{-\rho-\tilde{\rho}}U(t)f\right\|_{L_t^{q,p}L_x^r} \lesssim \|f\|_{L_x^p}\,. \tag{7.34}$$

The final result is summarized in Proposition 6.6.3.

In the next two propositions we assume (without loss of generality) that (6.56)-(6.59) are given in terms of homogeneous Besov norms.

Proposition 7.4.1. *Suppose that (q,r) is a nonsharply σ-admissible exponent pair. Then we have the estimate*

$$\|U(t)f\|_{L_t^qL_x^r} \lesssim \|f\|_{\dot{H}^\alpha}\,, \quad \alpha = \frac{n}{2} - \frac{n-2\beta}{\sigma q} - \frac{n}{r}$$

for all $f \in \dot{H}^\alpha$. Analogously, if (q,r) and $(\tilde{q},\tilde{r})$ are two nonsharply σ-admissible exponent pairs, then we have the estimate

$$\|W(t)F\|_{L_t^qL_x^r} \lesssim \|D^\alpha F\|_{L_t^{\tilde{q}'}L_x^{\tilde{r}'}} \tag{7.35}$$

for all $D^\alpha F \in L_t^{\tilde{q}'}(\mathbb{R}; L_x^{\tilde{r}'})$, where α is subject to (7.37).

Proof. If the pair (q,r) is nonsharply σ-admissible then we can always find an exponent R, $2 \leq R \leq r$, such that the pair (q,R) is σ-admissible. If $r < \infty$ we use the Sobolev embedding

$$\|U(t)f\|_{L_t^qL_x^r} \lesssim \left\|D^{\frac{n}{R}-\frac{n}{r}}U(t)f\right\|_{L_t^qL_x^R}$$

so that we can apply the Strichartz estimates of Theorem 6.6.1 to the right hand side. In view of 6.61, we obtain

$$\alpha = 2\beta\left(\frac{1}{2} - \left(\frac{1}{2} - \frac{1}{\sigma q}\right)\right) + n\left(\frac{1}{2} - \frac{1}{\sigma q}\right) - \frac{n}{r}.$$

If $r = \infty$ we use Proposition 7.4.2.

By the TT^*-principle, we have already proven that the inhomogeneous estimate (7.35) applies to the TT^*-operator. If $q > 2$ we use the Christ-Kiselev Lemma 7.1.5. If $q = \tilde{q} = 2$ we use Sobolev embedding and the Equivalence Theorem 1.3.2, part B. □

Proposition 7.4.2. *The estimate*

$$\|U(t)f\|_{L_t^qL_x^\infty} \lesssim \|f\|_{\dot{H}^\alpha}\,, \quad \alpha = \frac{n}{2} - \frac{n-2\beta}{\sigma q}$$

holds for every $f \in \dot{H}^\alpha$ whenever the exponent pair (q,∞) is nonsharply σ-admissible.

Proof. Consider the formula

$$\|U(t)f\|_{L_x^\infty} \lesssim \left\|D^\alpha U(t)f\right\|_{L^2}^{1-\theta}\left\|D^\gamma U(t)f\right\|_{L^R}^{\theta}, \tag{7.36}$$

which is a special case of the interpolation inequality of Proposition 7.4.4, for some $R < \infty$ fixed and big enough, and some θ, α, and γ, which we need to determine. To determine α we consult formula (6.62) with $r = \infty$ which suggests

$$\alpha = \frac{n}{2} - \frac{n-2\beta}{\sigma q} < \frac{n}{2}, \quad \gamma + \left(\frac{n}{2} - \frac{n-2\beta}{\sigma q\theta} - \frac{n}{R}\right) = \alpha.$$

Thus, $\gamma = -k + k/\theta + n/r$, $k = (n-2\beta)/\sigma q$, and $\gamma > n/r$ if we can choose $\theta < 1$. Substituting in (7.40) $\mu = \alpha$, $\lambda = \gamma$, we obtain the identity

$$(1-\theta)k + (1-\theta)(-k) = 0,$$

so that we are free to choose any $\theta < 1$. In particular, we choose θ so that the pair $(q\theta, R)$ is σ-admissible. Thus we first apply Proposition 7.4.4 to the term $\|U(t)f\|_{L_t^q L_x^\infty}$ to obtain (7.36). To each term on the right hand side of (7.36) we then apply Proposition 7.4.1 with the admissible pairs $(\infty, 2)$, $(q\theta, R)$, respectively, to conclude the proof. Note that this argument only works if q is away from the two endpoints $q = 2/\sigma$ ($q = 2$ when $\sigma > 1$), $q = \infty$.

As a final remark we note that since $\alpha > 0$ we can always replace the homogeneous norm $\dot{H}^\alpha$ with the inhomogeneous norm H^α in case that our estimates are based on inhomogeneous norms. □

We next give a corollary to Theorem 6.6.5 in the case when $\sigma \leq 1$.

Corollary 7.4.3. *Suppose that (q,r) and $(\tilde{q},\tilde{r})$ are two nonsharply jointly σ-acceptable exponent pairs with $\sigma \leq 1$. Then the operator $W(t)$ obeys the estimate*

$$\|W(t)F\|_{L_t^q L_x^r} \lesssim \|D^\alpha F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}, \quad r, \tilde{r} < \infty,$$

for all F, $D^\alpha F \in L_t^{\tilde{q}'}(\mathbb{R}; L_x^{\tilde{r}'})$, whenever the Sobolev exponent α fulfills condition

$$\frac{n-2\beta}{\sigma q} + \frac{n}{r} = \frac{n-2\beta}{\sigma \tilde{q}'} + \frac{n}{\tilde{r}'} - \frac{n-2\beta}{\sigma} - \alpha. \tag{7.37}$$

Proof. Consider two nonsharply jointly σ-acceptable pairs (q,r) and $(\tilde{q},\tilde{r})$. We can always find R and $\tilde{R}$ such that $2 \leq R \leq r$ and $2 \leq \tilde{R} \leq \tilde{r}$ and (q,R) and (q,R) are jointly σ-acceptable. Then as we did in the preceding propositions we use Sobolev embedding. The Sobolev exponent α in (7.37) is computed from

$$\alpha = \rho + \tilde{\rho} + \gamma + \tilde{\gamma},$$

where

$$\gamma = n/R - n/r, \quad \tilde{\gamma} = n/\tilde{R} - n/\tilde{r}.$$

□

Proposition 7.4.4 (Interpolation inequality, [35])**.** *Let $\lambda, \mu, p, q, r, \theta$ satisfy $\lambda, \mu \in \mathbb{R}$, $1 \leq p, q \leq r \leq \infty$, $0 < \theta < 1$,*

$$\lambda > \frac{n}{p} - \frac{n}{r}, \tag{7.38}$$

$$\mu < \frac{n}{q} - \frac{n}{r}, \tag{7.39}$$

$$\theta\left(\lambda - \frac{n}{p} + \frac{n}{r}\right) + (1-\theta)\left(\mu - \frac{n}{q} + \frac{n}{r}\right) = 0. \tag{7.40}$$

Then there exists a constant $C > 0$ such that

$$\|f\|_{L_x^r} \leq C\, \|f\|_{\dot{H}_p^\lambda}^\theta \, \|f\|_{\dot{H}_q^\mu}^{1-\theta} \tag{7.41}$$

for all $f \in \dot{H}_p^\lambda \cap \dot{H}_q^\mu$.

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