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ON THE HYPERBOLICITY DOMAIN OF THE POLYNOMIAL $x^n + a_1x^{n-1} + \cdots + a_n$

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ABSTRACT. We consider the polynomial $P_n = x^n + a_1x^{n-1} + \cdots + a_n$, $a_i \in \mathbb{R}$. We represent by figures the projections on $Oa_1 \ldots a_k$, $k \leq 6$, of its hyperbolicity domain $\Pi = \{a \in \mathbb{R}^n \mid \text{all roots of } P_n \text{ are real}\}$. The set Π and its projections Π^k in the spaces $Oa_1 \ldots a_k$, $k \leq n$, have the structure of stratified manifolds, the strata being defined by the multiplicity vectors. It is known that for $k > 2$ every non-empty fibre of the projection $\Pi^k \to \Pi^{k-1}$ is a segment or a point. We prove that this is also true for the strata of Π of dimension $\geq k$. This implies that for any two adjacent strata there always exist a space $Oa_1 \ldots a_k$, $k \leq n$, such that from the projections of the strata in it one is "above" the other w.r.t. the axis Oa_k. We show

1) how to find this k and which stratum is "above" just by looking at the multiplicity vectors of the strata;

2) how to obtain the relative position of a stratum of dimension l and of all strata of dimension $l + 1$ and $l + 2$ to which it is adjacent.

In the present paper we consider the polynomial $P_n(x) = x^n + a_1x^{n-1} + \ldots + a_n$, $x, a_j \in \mathbb{R}$. Call the set $\Pi \subset \mathbb{R}^n$ of values of $a = (a_1, \ldots, a_n)$ for which all roots x_i of P_n are real its *hyperbolicity domain.*

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This set has the structure of a stratified manifold, the *strata* being defined by the corresponding *multiplicity vectors*. For example, for $n = 7$ the multiplicity vector $(1, 2, 1, 3)$ defines the stratum S of Π such that if $a \in S$, then one has $x_1 > x_2 = x_3 > x_4 > x_5 = x_6 = x_7$. The stratum S_0 of Π defined by the multiplicity vector $(1, \ldots, 1)$ (i.e. all roots are real and simple) is the interior of Π and Π is the closure of S_0. The dimension of a stratum as a variety in $\mathbb{R}^n$ equals the number of components of its multiplicity vector.

Remark 1. A stratum S_1 of lower dimension lies in the closure of a stratum S_2 of higher dimension if the multiplicity vector of S_1 is obtained from the one of S_2 by replacing groups of consecutive components of the vector by their sums. E.g., the strata $(5, 8)$ and $(6, 4, 3)$ lie in the closure of the stratum $(2, 3, 1, 4, 2, 1)$ whereas the strata $(1, 4, 7, 1)$ and $(4, 2, 7)$ do not.

The geometry of Π, of its boundary and of its projections Π^k in the spaces $Oa_1 \ldots a_k$ was studied in [1], [3], [4] [5] and [6] (the projections also have the structure of stratified manifolds and their boundaries are unions of projections of strata of Π). The following theorem contains the most important of these results:

Theorem 2 (Arnold, Meguerditchian, Kostov)**.**

1) *Let $k > 2$. Then the boundary of the set $\Pi^k \subset Oa_1 \ldots a_k$ is the union of the graphs H_k^+ and H_k^- of two piecewise-smooth and locally Lipschitz functions defined on Π^{k-1} ($H_k^+ \geq H_k^-$). (One has $\Pi^1 = \mathbb{R}$, $\Pi^2 = \{a_2 \leq ((n-1)/2n))a_1^2\} \subset \mathbb{R}^2$.)*

2) *Let $k \geq 3$. The fibre of the projection $\Pi^k \to \Pi^{k-1}$ is a point exactly if the fibre lies over the boundary of Π^{k-1} (i.e. on $H_{k-1}^+ \cup H_{k-1}^-$) and is a segment exactly if it lies over the interior of Π^{k-1}. Thus for $k \geq 3$ the projection of $H_k^+ \cap H_k^-$ in $Oa_1 \ldots a_{k-1}$ is $(H_{k-1}^+ \cup H_{k-1}^-)$.*

3) *The strata of Π of maximal dimension whose projections in $Oa_1 \ldots a_k$ constitute H_k^+ have multiplicity vectors of the form $(r, 1, s, 1, \ldots)$ and the ones constituting H_k^- have multiplicity vectors of the form $(1, r, 1, s, \ldots)$, $r, s \in \mathbb{N}^+$. The other strata of H_k^+ and H_k^- (i.e. of non-maximal dimension) belong to the closures of these ones.*

4) *The closure of every k-dimensional stratum S of Π (of Π^l, $l > k$) is locally the graph of an $(n-k)$-dimensional (of an $(l-k)$-dimensional) vector-function (defined on the projection of the stratum in $Oa_1 \ldots a_k$) which is C^∞ over the interior of the projection. The field of tangent spaces to S is continuously extended on the strata of non-maximal dimension from the closure of S, the extension being everywhere transversal to the space $Oa_{k+1} \ldots a_n$ (to the space $Oa_{k+1} \ldots a_l$).*

Parts 1) and 4) of the theorem are respectively Theorems 1.11 and 1.8 from [4].

Part 2) follows from Theorem 1.13 from [4].

Part 3) follows from [1], Theorem 6 and corollary, and from [5], Proposition 9.

Remark 3. In the definition of a multiplicity vector in [4] the sense of the inequalities is opposite to the one used in this paper and this is not correct. However, this is relevant only to 3) of the theorem and does not affect 1), 2) and 4).

As one sees from the theorem, there exists the sequence of "privileged" projections

$$Proj(k) : Oa_1 \dots a_k \to Oa_1 \dots a_{k-1} \tag{1}$$

with the interesting property every non-empty fibre of Π^k to be contractible (i.e. a point or a segment). This remains true if one sets $a_1 = 0$. For any projections from a coordinate space of higher to one of lower dimension this will not be true (already for $n = 4$ and $a_1 = 0$ the well-known picture of the swallowtail shows that the projection $Oa_2a_3a_4 \to Oa_2a_4$ does not have this property).

The following fact (well-known at least to the specialists) can be deduced from the theorem as well:

Corollary 4. *The limit of the tangent spaces to all strata of* Π *of dimension* k *when the point of the stratum tends to 0 is the space* $Oa_1 \dots a_k$.

The corollary follows from 4) of the theorem and from the invariance of Π and its boundary under the one-parameter family of mappings

$$a_j \mapsto a_j e^{jt} \ , \ t \in \mathbb{R} \tag{2}$$

resulting from $x_i \mapsto e^t x_i$; recall that a_j is a symmetric function of the roots x_i. If one lets t tend to $-\infty$, one obtains the space $Oa_1 \dots a_k$ as limiting position of the tangent spaces to the strata of dimension k.

Remark 5 (M. Merle)**.** As we see, both the corollary and the sequence of projections mentioned above lead to the same flag of spaces $Oa_1 \dots a_k$, $k = 1, \dots, n$. One can explain this as follows: consider the curve $\Gamma \subset \mathbb{C}^n$, $\Gamma : x_1 = t, \dots, x_n = t^n$, $t \in \mathbb{C}$. Consider the set of hyperplanes $x_n + a_1 x_{n-1} + \cdots + a_n = 0$ tangent to it. By duality, such a tangent hyperplane is mapped onto a point from the dual curve $\check{\Gamma}$ (which is the discriminant set of the polynomial $P_n(x)$) and a hyperplane tangent to $\check{\Gamma}$ is mapped onto a point from Γ. These points, however, form only a one-parameter family which explains why $\check{\Gamma}$ consists of affine spaces of dimension $n-2$ and why a hyperplane tangent to $\check{\Gamma}$ is tangent to it not only at a point but at all points of such an $(n-2)$-dimensional space (a situation which is not at all typical). The limits of the tangent spaces to the strata at 0 can be explained by the increasing valuations at 0 of the coordinates of Γ. There remains to pass from Γ to its part corresponding to $t \in \mathbb{R}$.

In the present paper we give two applications of these results to the further study of the geometry of Π and its projections Π^k:

1) The local study of the boundaries of Π *and* Π^k*, see Section 1.*

Theorem 7 is the analog of 1) and 2) from Theorem 2 for a stratum of the boundary of Π instead of Π itself. Theorem 8 gives the mutual position of strata adjacent to one another — if k is the greater of the two dimensions, then there exists $k_1 \leq k$ such that for the projections of the strata in $Oa_1 \ldots a_{k_1}$ one can say which stratum is "above" the other along the axis Oa_{k_1} just by looking at the multiplicity vectors of the strata. As application of Theorem 8 one can consider the intersection of an l-dimensional stratum S of Π with a plane parallel to $Oa_{l+1}a_{l+2}$. By 4) of Theorem 2, the intersection is transversal, hence, it is locally a point A. The intersections of the strata of dimension $l+1$ adjacent to S with the plane will be curves etc. We show in Section 1 how to determine the mutual positions of these curves (i.e. which is "above" and which is "below" w.r.t. the projections $Proj(l+1)$ and $Proj(l+2)$, along resp. Oa_{l+1} and Oa_{l+2}).

2) To show on figures the disposition of the projections in $Oa_1 \ldots a_k$, $k = 3, 4, 5$ *and* 6*, of the strata forming the boundaries of* Π^3, Π^4, Π^5 *and* Π^6*, see Section 2.*

1. Relative positions of strata.

1.1. Theorem 8 and its corollaries. The basic result of this section are Theorems 7 and 8.

Notation 6. An upper index l to a stratum means its projection in $Oa_1 \ldots a_l$. The notation $\Pi(m)$ means "the domain Π defined for n replaced by m in its definition". One has $\Pi(m) \subset \mathbb{R}^m$, $\Pi(n) = \Pi$. Denote by S_2 a stratum from Π, of dimension l, and by S_1 a stratum from its closure, $\dim S_1 = m < l$.

Explain how to define the relative position of S_1 and S_2 and how to deduce it from their multiplicity vectors $\vec{v}_1$ and $\vec{v}_2$. For the projections S_2^ν, $\nu \leq l$ of S_2 in $Oa_1 \ldots a_\nu$ there holds the analog of 1) and 2) of Theorem 2:

Theorem 7. 1) *Let* $\nu > 2$*. Then the boundary of the set* $S_2^\nu \subset Oa_1 \ldots a_\nu$ *is the union of the graphs* R_ν^+ *and* R_ν^- *of two piecewise-smooth and locally Lipschitz functions defined on* $S_2^{\nu-1}$ *(*$R_\nu^+ \geq R_\nu^-$*). (One has* $S_2^1 = \mathbb{R}$, $S_2^2 = \{a_2 \leq ((n-1)/2n)a_1^2\} \subset \mathbb{R}$*.)*

2) *The fibre of the projection $S_2^\nu \to S_2^{\nu-1}$ is a point exactly if the fibre lies over the boundary of $S_2^{\nu-1}$ (i.e. on $R_{\nu-1}^+ \cup R_{\nu-1}^-$) and is a segment exactly if it lies over the interior of $S_2^{\nu-1}$.*

The theorem is proved in the next but one subsection. For the projections S_2^ν with $\nu > l$ see 4) of Theorem 2.

Define two new multiplicity vectors — $\vec{w}_1$, $\vec{w}_2$ — as follows: let $\vec{v}_2 = (d_1, \ldots, d_l)$, $\vec{v}_1 = (b_1, \ldots, b_m)$. Set $\vec{w}_2 = (1, \ldots, 1)$ (l times). Recall that $b_1 = d_1 + \ldots + d_{i_1}$, $b_2 = d_{i_1+1} + \ldots + d_{i_2}$, ..., $b_m = d_{i_{m-1}+1} + \ldots + d_l$, $1 \leq i_1 < i_2 < \ldots < i_{m-1} \leq l-1$ (because S_1 is from the closure of S_2, see Remark 1). Set $\vec{w}_1 = (i_1, i_2 - i_1, \ldots, i_{m-1} - i_{m-2}, l - i_{m-1})$.

Thus $\vec{w}_1$, $\vec{w}_2$ define two strata Σ_1, Σ_2 of $\Pi(l)$; Σ_2 is the analog of S_0 from the beginning of the paper. Denote by G_ν^+, G_ν^- the graphs H_ν^+, H_ν^- defined for $\Pi(l)$ instead of $\Pi(n) = \Pi$.

Theorem 8. *One has $S_1^\nu \subset R_\nu^+$ (or $S_1^\nu \subset R_\nu^-$) if and only if $\Sigma_1^\nu \subset G_\nu^+$ (resp. $\Sigma_1^\nu \subset G_\nu^-$).*

The theorem is proved in the next but one subsection. The answer to the question whether $\Sigma_1^\nu \subset G_\nu^+$ or whether $\Sigma_1^\nu \subset G_\nu^-$ is given by Theorem 11 formulated below. It is part of the basic result of [5] (but formulated in a different way).

Remark 9. One can answer the question which strata of maximal dimension constitute the graphs $R_\nu^\pm$ from Theorem 7. For such strata S_1 construct the corresponding vectors $\vec{w}_1$. Then $\vec{w}_1$ must be of the form $(r, 1, s, 1, \ldots)$ for R_ν^+ and of the form $(1, r, 1, s, \ldots)$ for R_ν^-. This is the analog of 3) of Theorem 2. Having understood the proofs of Theorems 7 and 8, the reader will be able to prove this analog oneself.

Definition 10. For a stratum T defined by a multiplicity vector $\vec{m} = (m_1, \ldots, m_s)$ set $s_T = q + \sum_{m_i \geq 2} (m_i - 2)$ where q is the number of odd sequences of consecutive units in $\vec{m}$, only sequences between two multiplicities ≥ 2 are counted. Call T *even* if $\vec{m}$ begins with an even sequence of units and *odd* otherwise. Denote by T^k the projection of T in $Oa_1 \ldots a_k$.

Theorem 11 (Meguerditchian)**.**
If $s_T > n-k$, then $T^k \subset H_k^+$ and $T^k \subset H_k^-$.
If $s_T = n-k$ and T is even, then $T^k \subset H_k^+$ and $T^k \not\subset H_k^-$.
If $s_T = n-k$ and T is odd, then $T^k \subset H_k^-$ and $T^k \not\subset H_k^+$.
If $s_T < n-k$, then $T^k \not\subset H_k^+$ and $T^k \not\subset H_k^-$.

Remark 12. If for a stratum $T \subset \Pi$ one has $T^k \subset H_k^+$ or $T^k \subset H_k^-$, then one trivially has $T^l \subset H_l^+$ and $T^l \subset H_l^-$ for $l > k$, see 2) of Theorem 2.

Example 13. For $\vec{m} = (1,1,1,7,1,1,4,1,3,1)$ the stratum T is odd, $n = 21$, $q = 1$ and $s_T = 1 + (7-2) + (4-2) + (3-2) = 9$. Hence, $T^{12} \subset H_{12}^-$ and $T^{12} \not\subset H_{12}^+$.

Example 14. For $\vec{m} = (b,c,d)$, $b \geq 2$, $c \geq 2$, $d \geq 2$ one has $n = b+c+d$, $q = 0$, T is even and $s_T = (b-2) + (c-2) + (d-2) = n - 6$. Hence, $T^6 \subset H_6^+$ and $T^6 \not\subset H_6^-$.

1.2. Application of Theorem 8. The aim of this subsection is to explain how to apply Theorem 8 to the local study of the boundary of Π and its projections Π^k. Denote by S_1 a stratum of Π of dimension l. Theorem 8 allows one to understand the disposition of S_1 and the sets Y_{l+1}, Y_{l+2} of strata of Π of dimension $l+1$ and $l+2$ to which S_1 is adjacent. We consider from now on and we call *strata* both the strata and their projections in $Oa_1 \ldots a_{l+2}$.

Namely, represent the intersection (at a point of the interior of S_1^{l+2}) of Π^{l+2} with a plane $L = \{a_i = \alpha_i, i \neq l+1, l+2, \alpha_i \in \mathbb{R}\}$ (i.e. with an affine plane parallel to $Oa_{l+1}a_{l+2}$). On this picture, $S_1^{l+2} \cap L$ will be a point (the tangent space to S_1 is everywhere transversal to $Oa_{l+1} \ldots a_n$ by 4) of Theorem 2); the strata from Y_{l+1} will be represented by curves whose tangent lines are nowhere vertical (including their limits at $S_1^{l+2} \cap L$, see again 4) of Theorem 2). The strata of Y_{l+2} will be the sectors delimited by these curves. In general, there will be always overlapping sectors, see the example at the end of the section.

So to represent correctly the disposition of the stratum S_1^{l+2} and the ones from Y_{l+1} and Y_{l+2}, it suffices to place correctly the intersections of the ones of Y_{l+1} with L on the picture. (We place the positive axes Oa_{l+1}, Oa_{l+2} respectively to the right and to above.) This means that one has to answer the following questions:

1. The intersections with L of which strata from Y_{l+1} are to the left and of which to the right of $S_1^{l+2} \cap L$?

2. What is the order from above to below of the set of intersections of strata from Y_{l+1} with L which are on one and same side of $S_1^{l+2} \cap L$?

3. What can be said about the limits at $S_1^{l+2} \cap L$ of the tangent spaces to the intersections of the strata from Y_{l+1} with L?

4. These tangent spaces are lines in $\mathbb{R}^2$. What is the order of their slopes?

We give the answers to these questions in the form of several lemmas. The last question was suggested by M. Coste.

Remark 15. If $\vec{v}_1 = (m_1, \ldots, m_l)$ is the multiplicity vector of S_1, then all multiplicity vectors $\vec{v}'$ of strata from Y_{l+1} are obtained from $\vec{v}$ by replacing a component m_i by m_i', m_i'', where $m_i',m_i'' \in \mathbb{N}^+$, $m_i' + m_i'' = m_i$. Applying to all such multiplicity vectors $\vec{v}'$ the above procedure again (i.e. replacing a component by two components, wherever this is possible) yields the multiplicity vectors of all strata from Y_{l+2}. The procedure can be applied only if $m_i > 1$.

The answer to the third question is given by

Lemma 16. *The limits of the tangent spaces to the strata $S_2, S_3 \in Y_{i+1}$ with multiplicity vectors*

$$\vec{v}_2 = (m_1, \ldots, m_{i-1}, m_i', m_i'', m_{i+1}, \ldots, m_l),$$

$$\vec{v}_3 = (m_1, \ldots, m_{j-1}, m_j', m_j'', m_{j+1}, \ldots, m_l)$$

at a point from S_1 are the same for $i = j$ and are different for $i \neq j$. In both cases these limits contain the tangent space to S_1 at its point.

P r o o f.

Proposition 17 (I. Meguerditchian)**.** *Denote by P, $P_1, \ldots, P_l$ monic polynomials of degrees d, $d_1, \ldots, d_l$, P_i and P_j having no common root for $i \neq j$. Then there exists an open neighbourhood U of P and open neighbourhoods U_i of P_i (considered as points in $\mathbb{R}^d$ and $\mathbb{R}^{d_i}$) such that the mapping*

$$\phi : U_1 \times \cdots \times U_l \to U, \quad (Q_1, \ldots, Q_l) \mapsto Q_1 \ldots Q_l$$

is a diffeomorphism. In particular, the hyperbolicity domain Π locally, at a point from a stratum $(m_1, \ldots, m_l)$, is diffeomorphic to the cartesian product of the neighbourhoods of 0 *of* $\Pi(m_i)$, $i = 1, \ldots, l$.

This is proved in [6], Annexe 1, p. 52–53. Note that a neighbourhood of $0 \in \Pi(1)$ is by definition $(\mathbb{R}, 0)$.

If $i \neq j$ (if $i = j$), then the strata S_2, S_3 from the lemma correspond to different (to one and the same) direct factor(s) $\Pi(m_i)$. Hence, if $i \neq j$, then the limits at a point from S_1 of their tangent spaces are transversal, if $i = j$ they coincide by Corollary 4.

The last statement of the lemma (which is the Whitney property (a)) follows from the above structure of cartesian product and from Corollary 4 —

the space $\{0\}$ which is the tangent space to the stratum (n) of Π, is contained in each of the spaces $Oa_1 \dots a_k$. Hence, the tangent space to S_1 is contained in the limits of the tangent spaces to S_2 and S_3.

The lemma is proved. $\square$

The reply to the first question is given by

Lemma 18. *If i is even, then $S_2 \cap L$ is to the right of $S_1 \cap L$, if it is odd, then $S_2 \cap L$ is to the left of $S_1 \cap L$.*

Proof. Construct after the multiplicity vectors $\vec{v}_1$, $\vec{v}_2$ of S_1, S_2 the vectors $\vec{w}_1$, $\vec{w}_2$ as it was done before Theorem 8; one has $\vec{w}_2 =(1,\dots,1)$ ($l+1$ times) and $\vec{w}_1 =(1,\dots,1,2,1,\dots,1)$ (l positions, “2” in position i). Hence, if i is odd, then $\vec{w}_1$ is of the form $(r,1,s,1,\dots)$ and according to Theorems 8 and 2, the projection S_1^{l+1} of S_1 in $Oa_1\dots a_{l+1}$ belongs to R^+_{l+1}, i.e. $S_2^{l+2} \cap L$ is to the left of $S_1^{l+2} \cap L$. And vice versa if i is even.

The lemma is proved. $\square$

The answer to the second question is given by the following two lemmas:

Lemma 19. *Consider two strata from Y_{l+1} — S_2^{l+2} and S_3^{l+2} — both to the left of S_1^{l+2} (in the intersection with L). Denote their multiplicity vectors by $\vec{v}_2$ and $\vec{v}_3$ (we use the notation from Lemma 16; hence, both i and j are odd). Suppose that $i \neq j$. Then S_2^{l+2} is above S_3^{l+2} iff $i > j$.*

If $S_2^{l+2} \cap L$ and $S_3^{l+2} \cap L$ are both to the right of $S_1^{l+2} \cap L$, then S_2^{l+2} is above S_3^{l+2} iff $i < j$.

Proof. We consider the case of both i and j odd. The case when they are even is considered by analogy and we leave it for the reader. To simplify the notation we consider the case $i=1$, $j=3$.

So consider the strata S_2, S_3 and S_4 with multiplicity vectors respectively $\vec{v}_2= (m_1', m_1'', m_2,\ m_3, \dots)$, $\vec{v}_3 = (m_1, m_2, m_3', m_3'', m_4, \dots)$ and $\vec{v}_4 = (m_1', m_1'', m_2, m_3', m_3'', m_4, \dots)$. Hence, the stratum $S_4^{l+2} \cap L$ is represented by the sector delimited by $S_2^{l+2} \cap L$ and $S_3^{l+2} \cap L$.

Let the strata T_2, T_3, T_4 of $\Pi(l+2)$ correspond to the multiplicity vectors $\vec{w}_2 = (1,1,1,2,1,\dots)$, $\vec{w}_3 = (2,1,1,1,1,\dots)$ and $\vec{w}_4 = (1,1,\dots)$. According to Theorem 2, the stratum T_2 is “below” the stratum T_4 (because “2” stands in an even position, i.e. the multiplicity vector is of the form $(1,r,1,s,\dots)$), while the stratum T_3 is “above” it (i.e. $T_2 \subset H^-_{l+2}$ and $T_3 \subset H^+_{l+2}$).

Theorem 8 implies that then the stratum $S_2^{l+2} \cap L$ is “below” $S_4^{l+2} \cap L$ and $S_3^{l+2} \cap L$ is “above” it, hence, $S_3^{l+2} \cap L$ is “above” $S_2^{l+2} \cap L$. Indeed, Theorem 8 has to be applied twice — once with the roles of $\vec{v}_1$, $\vec{v}_2$, $\vec{w}_1$, $\vec{w}_2$ from the theorem being played respectively by $\vec{v}_2$, $\vec{v}_4$, $\vec{w}_2$, $\vec{w}_4$ and once by $\vec{v}_3$, $\vec{v}_4$, $\vec{w}_3$, $\vec{w}_4$.

The lemma is proved. □

Lemma 20. *In the notation of the previous lemma, suppose that $i = j$ is odd. Then the stratum $S_2^{l+2} \cap L$ is "above" the stratum $S_3^{l+2} \cap L$ iff $m'_i > m'_j$. If $i = j$ is even, then the stratum $S_2^{l+2} \cap L$ is "above" the stratum $S_3^{l+2} \cap L$ iff $m'_i < m'_j$.*

P r o o f. Consider the case of $i = j$ odd (the case of $i = j$ even is considered by analogy). Simplify the notation by setting $i = 1$. To avoid ambiguity (because $i = j$), write p', p'' instead of m'_i, m''_i and q', q'' instead of m'_j, m''_j. Recall that $p' + p'' = q' + q''$.

Let $p' > q'$. Set $r = p' - q'$. Define the stratum S_4 by the multiplicity vector $\vec{v}_4 = (q', r, p'', m_2, m_3, \ldots)$; recall that $\vec{v}_2 = (p', p'', m_2, m_3, \ldots)$ and $\vec{v}_3 = (q', q'', m_2, m_3, \ldots)$. Define the strata F_2, F_3 and F_4 of $\Pi(l+2)$ by the multiplicity vectors $\vec{f}_2 = (2, 1, \ldots)$, $\vec{f}_3 = (1, 2, 1, \ldots)$, both with $l + 1$ positions, and $\vec{f}_4 = (1, 1, \ldots)$ ($l + 2$ units).

Applying Theorem 8 (like in the proof of the previous lemma), with the roles of $\vec{v}_1$, $\vec{v}_2$, $\vec{w}_1$, $\vec{w}_2$ played respectively by $\vec{v}_2$, $\vec{v}_4$, $\vec{f}_2$, $\vec{f}_4$, we find out that $S_2^{l+2} \cap L$ is above $S_4^{l+2} \cap L$ (because F_2 is above F_4). Similarly, by replacing S_2 by S_3 and F_2 by F_3, one finds that $S_3^{l+2} \cap L$ is below $S_4^{l+2} \cap L$. Hence, $S_2^{l+2} \cap L$ is above $S_3^{l+2} \cap L$.

The lemma is proved. □

To answer the forth question we need some

Notation 21. Denote by $C_1, \ldots, C_l$ curves which are intersections of strata from Y_{l+1} with L where the multiplicity vector of C_i is like $\vec{v}_2$ from Lemma 16. If for i fixed several choices are possible, then we make no matter which of them (according to Lemma 16, the limits of the tangent lines to the curves when the point of the curve tends to $S_1 \cap L$ are the same if i is the same). If $m_i = 1$, then we do not define C_i, see Remark 15. Denote by k_i the slope of the limit at $S_1 \cap L$ of the tangent line to C_i (its equation is of the form $a_{l+2} = k_i a_{l+1} + \theta_i$, $k_i, \theta_i \in \mathbb{R}$).

Lemma 22. *One has $k_1 > \cdots > k_l$.*

Notice that Lemma 19 can be deduces from Lemma 22. However, we prove Lemma 19 first and then use it to prove Lemma 22. To prove Lemma 22 we need another lemma as well which is of independent interest:

Lemma 23. *Set $\vec{v}_* = (m_1, \ldots, m_{i-1}, m'_i, m''_i, m_{i+1}, \ldots, m_{j-1}, m'_j, m''_j, m_{j+1}, \ldots, m_l)$ and denote by S_* the stratum of Π defined by $\vec{v}_*$. The curves $C_i = S_2 \cap L$, $C_j = S_3 \cap L$ define in the neighbourhood of $S_1 \cap L$ two sectors. The* *set $S_* \cap L$ is the smaller of the two (i.e. of opening $< \pi$; the opening is defined by the limits at $S_1 \cap L$ of the tangent lines to $S_2 \cap L$ and $S_3 \cap L$).*

P r o o f. 1^0. Denote the roots of a polynomial with multiplicity vector $\vec{v}_1 = (m_1, \dots, m_l)$ by $\xi_1, \dots, \xi_l$. Set

$$f = \prod_{\nu=1}^{l} ((x-\xi_\nu)^{m_\nu} + a_{1,\nu}(x-\xi_\nu)^{m_\nu - 1} + \cdots + a_{m_\nu,\nu}), \quad (a_{1,1}, \dots, a_{m_l,l}) \in \mathbb{R}^n.$$

For each ν such that $m_\nu > 1$ the half-line

$$\Lambda_\nu \subset \mathbb{R}^{m_\nu}, \quad \Lambda_\nu = \{a_{j,\nu} = 0 \text{ for } j \neq 2,\ a_{2,\nu} = t,\ t \leq 0\},$$

belongs to $\Pi(m_\nu)$. Hence, the half-line $\Lambda = \Lambda(\alpha_1, \dots, \alpha_l) \subset \mathbb{R}^n$,

$$\Lambda = \{a_{j,\nu} = 0 \text{ for } j \neq 2, a_{2,\nu} = a_\nu t, \nu = 1, \dots, l; t \leq 0; \alpha_\nu \geq 0; \alpha_1^2 + \cdots + \alpha_l^2 \neq 0\}$$

belongs to $\Pi^* = \Pi(m_1) \times \cdots \times \Pi(m_l)$ (if $m_\nu = 1$ for some ν, then we do not define $a_{2,\nu}$ and α_ν). In particular, $\Lambda^* \subset \Pi^*$ where Λ^* is the curve Λ obtained for $\alpha_i = \alpha_j = 1$, $\alpha_\nu = 0$ for $\nu \neq i, j$.

2^0. Recall that the differmorphism ϕ was defined in Proposition 17. By this proposition, $\phi(\Lambda^*)$ is a curve beginning at $S_1 \cap L$. Its tangent vector at $S_1 \cap L$ equals $\vec{\eta}_i + \vec{\eta}_j$ where $\vec{\eta}_i, \vec{\eta}_j$ are the tangent vectors to $\phi(\Lambda(0, \dots, 0, 1, 0, \dots, 0))$ (units respectively in position i and j).

These tangent vectors are ones to the closures of $S_2 \cap L$, $S_3 \cap L$ at $S_1 \cap L$ (directed so that the point escape from $S_1 \cap L$). Indeed, the half-line Λ_ν is tangent to all strata of $\Pi(m_\nu)$ except (m_ν).

As $\alpha_i > 0$ and $\alpha_j > 0$, $\phi(\Lambda^*)$ must lie in the sector (defined by $S_2 \cap L$, $S_3 \cap L$) of opering $< \pi$.

The lemma is proved. □

P r o o f o f L e m m a 22. 1^0 If i and j have the same parity, then the inequality $k_i > k_j$ follows from Lemma 19. Suppose that i is odd, j is even and $i < j$. Set $\vec{\xi}' = (1, \dots, 1)$ ($l + 2$ times), $\vec{\xi}'' = (1, \dots, 1, 2, 1, \dots, 1)$ (l units and "2" in the i-th position), and $\vec{\xi}''' = (1, \dots, 1, 2, 1, \dots, 1)$ (l units and "2" in the $(j+1)$-st position). The multiplicity vectors $\vec{\xi}'$, $\vec{\xi}''$ and $\vec{\xi}'''$ define three strata of $\Pi(l+2) - S'$, S'' and S'''; S' is the interior of $\Pi(l+2)$.

2^0. According to Theorem 8, the strata S'' and S''' are "above" the stratum S' along the axis Oa_{l+2} (because $i - 1$ and j are both even, the strata S'' and S''' are even and $s_{S''} = s_{S'''} = 0$, see Definition 10 for $s_{S''}$ and $s_{S'''}$). In the notation from the lines preceding Theorem 8, one has S'', $S''' \in G^+_{l+2}$.

3^0. The vectors $\vec{\xi}'$, $\vec{\xi}''$ (respectively, $\vec{\xi}'$, $\vec{\xi}'''$) are obtained from the vectors $\vec{v}_*$, $\vec{v}_2$ (respectively, $\vec{v}_*$, $\vec{v}_3$) in the same way as the vectors $\vec{w}_1$, $\vec{w}_2$ were obtained from the vectors $\vec{v}_1$, $\vec{v}_2$ before Theorem 8. According to Theorems 8 and 2, the strata S_2 and S_3 are "above" the stratum S_* (along the axis Oa_{l+2}).

4^0. Hence, $S_2 \cap L$ is to the left and $S_3 \cap L$ is to the right of $S_1 \cap L$ (Lemma 18) and they both delimit from above the sector Φ which is $S_* \cap L$. The limits at $S_1 \cap L$ of the tangent lines to C_i and C_j are non-vertical and by Lemma 23, the sector Φ is of opening $< \pi$. Hence, $k_i > k_j$.

5^0. If i is even, j is odd and $i < j$, then one shows in the same way that $S_2 \cap L$ and $S_3 \cap L$ are "below" $S_* \cap L$ along the axis Oa_{l+2} and that $k_i > k_j$; in this case $S_2 \cap L$ is to the right and $S_3 \cap L$ is to the left w.r.t. $S_1 \cap L$ along the axis Oa_{l+1} (Lemma 18).

The lemma is proved. □

Example 24. Let $n = 10$. Consider the stratum S_1 with multiplicity vector $(2,3,1,1,3)$ (see Fig. 1). In this example $l = 5$. There are 5 strata from Y_6: $K_1 : (1,1,3,1,1,3)$, $K_2 : (2,2,1,1,1,3)$, $K_3 : (2,1,2,1,1,3)$, $K_4 : (2,3,1,1,2,1)$ and $K_5 : (2,3,1,1,1,2)$. Lemma 18 implies that $K_1^7 \cap L$, $K_4^7 \cap L$ and $K_5^7 \cap L$ are to the left while $K_2^7 \cap L$, $K_3^7 \cap L$ are to the right of $S_1^7 \cap L$.

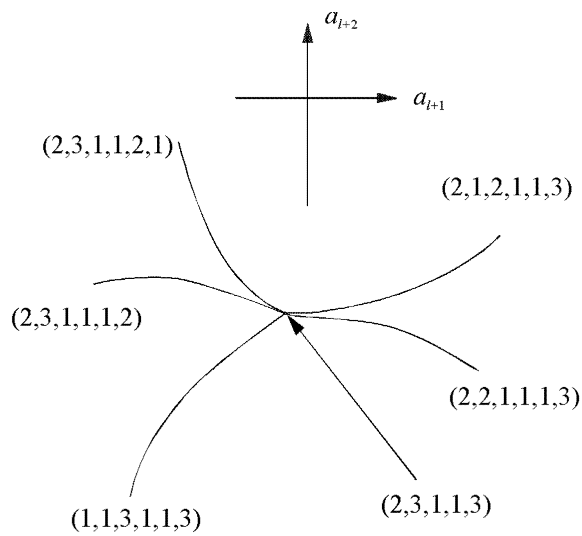


Fig. 1

Lemma 19 implies that $K_4^7 \cap L$ and $K_5^7 \cap L$ are above $K_1^7 \cap L$. Lemma 20 implies that $K_4^7 \cap L$ is above $K_5^7 \cap L$ and that $K_2^7 \cap L$ is below $K_3^7 \cap L$.

Lemma 16 implies that

1. the limits of the tangent spaces at S_1^7 to the strata $K_4^7\cap L$ and $K_5^7\cap L$ are the same, idem for the strata $K_2^7\cap L$ and $K_3^7\cap L$;

2. these limits are different and different from the corresponding limit defined for $K_1^7\cap L$ (all limits are non-vertical by 4) of Theorem 2).

The sector delimited by $K_4^7\cap L$ and $K_5^7\cap L$ represents the stratum J_1 : $(2,3,1,1,1,1,1)$, the one delimited by $K_1^7\cap L$ and $K_4^7\cap L$ represents the stratum $J_2:(1,1,3,1,1,2,1)$, both from Y_7 (note that the second sector contains the first one).

1.3. Proof of Theorems 7 and 8. We prove the theorems together. The proof of Theorem 7 is finished in 3^0.

Definition 25. For fixed *weights* $p_i>0$, $i=1,\ldots,n$ define *Vandermonde's mapping* $W:\mathbb{R}^n\to\mathbb{R}^n$ as

$$W:(x_1,\ldots x_n)\mapsto(w_1(x),\ldots,w_n(x))\ ,\ w_j(x)=\sum_{k=1}^n p_kx_k^j$$

(one often adds the condition $p_1+\ldots+p_n=1$ so that the weights have the meaning of probabilities; we do not need this restriction). Denote by $K_n\subset\mathbb{R}^n$ the "chamber" $\{x_1\geq\ldots\geq x_n\}$ and by $\tilde{\Pi}$, $\tilde{\Pi}^k$ its image $W(K_n)$ and the projection $W^k(K_n)$ of $W(K_n)$ in $Ow_1\ldots w_k$.

Define *Vieta's mapping* $\Lambda_n:K_n\to\mathbb{R}^n$ as $(x_1,\ldots,x_n)\mapsto(a_1(x),\ldots,a_n(x))$ where $a_j=(-1)^j\sum_{1\leq i_1<\ldots<i_j\leq n}x_{i_1}\ldots x_{i_j}$.

Remark 26. It follows from [4, Corollary 1.5], that W is a homeomorphism $K_n\to\tilde{\Pi}$ and that its restriction to any stratum S of K_n of dimension q defines a homeomorphism $S\to W^q(S)$. The "chamber" K_n has the structure of a stratified manifold and its strata are defined by multiplicity vectors.

1^0. Consider Vandermonde's mapping in two situations — as defined above and when n is replaced by l. Denote them by W and V. Denote by T_1, T_2 (by Q_1, Q_2) the strata of K_n (of K_l) with multiplicity vectors respectively $\vec{v}_1$, $\vec{v}_2$ (respectively $\vec{w}_1$, $\vec{w}_2$); these vectors were definded before Theorem 8. In the definition of W we set $p_i=1$, $i=1,\ldots,n$ and in the one of V we set $p_i=a_i$, $i=1,\ldots,l$. Then for $\nu\leq l$ one has $W^\nu(T_s)=V^\nu(Q_s)$, $s=1,2$ (by definition).

Note that by definition, $S_i=\Lambda_n(T_i)$, $i=1,2$.

2^0. There exist the following formulas connecting the classical symmetric functions $\sigma_j = (-1)^j a_j$ and Newton's symmetric functions $s_j = \sum_{i=1}^{n} x_i^j$:

$$s_j = (-1)^{j-1} j\sigma_j + M_j(\sigma_1, \ldots, \sigma_{j-1}),\ (-1)^{j-1} j\sigma_j = s_j + M_j^*(s_1, \ldots, s_{j-1})$$

where M_j, M_j^* are polynomials. These formulas result from the well-known relations between the classical and Newton's symmetric functions

$$s_j - s_{j-1}\sigma_1 + \cdots + (-1)^{j-1} s_1 \sigma_{j-1} = (-1)^{j-1} j\sigma_j.$$

One has $(-1)^j \sigma_j = a_j$. Hence,

$$(3)\ \ s_j = -ja_j + M_j(-a_1, \ldots, (-1)^{j-1} a_{j-1}) \ ,\ -ja_j = s_j + M_j^*(s_1, \ldots, s_{j-1}).$$

Formulas (3) express analytically the mappings $W \circ \Lambda_n^{-1}$ and $\Lambda_n \circ W^{-1}$ when $p_1 = \ldots = p_n = 1$.

3^0.

Remark 27. Parts 1), 2) and 4) of Theorem 2 and Theorem 7 remain true if Π, Π^k are replaced by $\tilde{\Pi}$, $\tilde{\Pi}^k$ and for any weights $p_i > 0$, see [1], [5] and [4]; in fact, the version with Π, Π^k is deduced from the one with $\tilde{\Pi}$, $\tilde{\Pi}^k$. It is deduced with the help of formulas (3) which are "triangular" and show that when passing from Vieta's to Vandermonde's mapping the couples of graphs of functions $H_k^{\pm}$ (for all k) are mapped onto such couples of graphs of functions (continuous, piecewise smooth and locally Lipschitz); the negative sign before a_j in (3) shows that $H_k^{\pm}$ exchange their roles, i.e. the image $\tilde{H}_k^-$ of H_k^+ is lower and the image $\tilde{H}_k^+$ of H_k^- is upper graph.

Due to this exchange of roles part 3) of Theorem 2 formulated for $\tilde{\Pi}$ looks like this:

Theorem 28. *The strata of* $\tilde{\Pi}$ *of maximal dimension whose projections in* $Oa_1 \ldots a_k$ *constitute* $\tilde{H}_k^+$ *have multiplicity vectors of the form* $(1, r, 1, s, \ldots)$ *and the ones constituting* $\tilde{H}_k^-$ *have multiplicity vectors of the form* $(r, 1, s, 1, \ldots)$, $r, s \in \mathbb{N}^+$. *The other strata of* $\tilde{H}_k^+$ *and* $\tilde{H}_k^-$ *(i.e. of non-maximal dimension) belong to the closure of these ones.*

Remark 29. The analogs of Theorems 2 and 7 with $\tilde{\Pi}$ instead of Π hold for *any* weights $p_i > 0$. Thus for any positive weights the strata of K_n whose images form the graphs $\tilde{H}_k^{\pm}$ are the same. Indeed, the strata of maximal dimension are explicitly defined by their multiplicity vectors. They are the same for any weights. The strata of non-maximal dimension are all strata from their closure. Hence, they are also the same. The strata of K_n whose images under Vieta's mapping form $H_k^{\pm}$ are the same as the ones whose images under Vandermonde's mapping form $\tilde{H}_k^{\mp}$.

Corollary 30. *For any weights $p_i > 0$ the mappings $W \circ \Lambda_n^{-1}$ and $\Lambda_n \circ W^{-1}$ commute with the projections $Proj(k+1) \circ \cdots \circ Proj(n)$, see* (1).

Indeed, for $p_i = 1$ the corollary follows from formulas (3) and their "triangular" form. To prove it in the general case it suffices to show that the mappings $W_1 \circ W_2^{-1}$ commute with the projections $Proj(k+1) \circ \cdots \circ Proj(n)$ where W_1, W_2 are two Vandermonde's mappings, with different weights p_i. This is really true because for any weights $p_i > 0$ the restriction of the mapping W^k to a stratum of K_n of dimension k is a homeomorphism onto its image [4, Corollary 1.5]; on the other hand, for any weights $p_i > 0$ the sets $\tilde{H}_k^{\pm}$ (resp. the set $\tilde{\Pi}^k$) are unions of images of one and the same strata of K_n of dimension $\leq k-1$ (resp. k) under the mapping W^k.

Thus one can forget the last $n-k$ coordinates w_i either before or after applying the mapping $W_1 \circ W_2^{-1}$ which proves the corollary.

Prove Theorem 7. The sets $W^\nu(T_2)$, $\nu \leq l$ satisfy the conclusions of the analog of Theorem 2 for Π replaced by $\tilde{\Pi}$, see Remarks 27. Indeed, one has $W^\nu(T_2) = V^\nu(Q_2)$ and one applies the theorem to the mapping V which is just another Vandermonde's mapping (one has $\tilde{\Pi} = V(Q_2)$ if $\tilde{\Pi}$ is defined for l instead of n); see the last remarks. One has $S_2 = (\Lambda_n \circ W^{-1})(W(T_2))$. It follows from formulas (3) (i.e. from their "triangular" form) that the projections S_2^ν of the stratum S_2 for $\nu \leq l$ satisfy the conclusions of Theorem 7, see Remarks 27.

Theorem 7 is proved. □

4^0. Prove Theorem 8. Set $\tilde{S}_i = W(T_i)$, $\tilde{\Sigma}_i = V(Q_i)$, $i = 1, 2$. Denote by $\tilde{R}_\nu^{\pm}$ the graphs $R_\nu^{\pm}$ from Theorem 7 when formulated not for $S_2 = \Lambda_n(T_2)$, but for $\tilde{S}_2 = W(T_2)$. Denote by $\tilde{G}_\nu^{\pm}$ the graphs $\tilde{H}_\nu^{\pm}$ when in the definition of Vandermonde's mapping n is replaced by l. Note that this notation further will be used for different choices of the weights $p_i > 0$.

One has $\tilde{S}_1^\nu \subset \tilde{R}_\nu^+$ (or $\tilde{S}_1^\nu \subset \tilde{R}_\nu^-$) if and only if $\tilde{\Sigma}_1^\nu \subset \tilde{G}_\nu^+$ (resp. $\tilde{\Sigma}_1^\nu \subset \tilde{G}_\nu^-$).

This follows from $S_s^\nu = W^\nu(T_s) = V^\nu(Q_s)$, $s = 1, 2$, $\nu \leq l$.

5^0. Denote by V_1 Vandermonde's mapping for n replaced by l and for $p_1 = \ldots = p_l = 1$. Set $\tilde{\sigma}_i = V_1(Q_i)$, $i = 1, 2$. It follows from Remark 29 that

$\tilde{S}_1^\nu \subset \tilde{R}_\nu^\pm$ *if and only if* $\tilde{\sigma}_1^\nu \subset \tilde{G}_\nu^\pm$.

But $\tilde{\sigma}_1^\nu = V_1(Q_1) \subset \tilde{G}_\nu^\pm$ iff $\Sigma_1^\nu = (\Lambda_l \circ V_1^{-1})(V_1^\nu(Q_1)) \subset G_\nu^\mp$ (it suffices to apply $\Lambda_l \circ V_1^{-1}$ to both sides of last but one inclusion and then use Remark 29 and Corollary 30; note the change of sign).

6^0. On the other hand, $S_1^\nu = \Lambda_n^\nu(T_1) \subset R_\nu^\mp$ iff $W^\nu(T_1) = \tilde{S}_1^\nu \subset \tilde{R}_\nu^\pm$ (apply $W \circ \Lambda_n^{-1}$ to both sides of last but one inclusion and use Corollary 30).

Thus

$S_1^\nu \subset R_\nu^\mp$ iff $\tilde{S}_1^\nu \subset \tilde{R}_\nu^\pm$ ...

iff $\tilde{\Sigma}_1^\nu \subset \tilde{G}_\nu^\pm$, see 4^0 ...

iff $\tilde{\sigma}_1^\nu \subset \tilde{G}_\nu^\pm$, see Remark 29 ...

iff $\Sigma_1^\nu \subset G_\nu^\mp$, see 5^0.

Theorem 8 is proved. □

2. The projections Π^3, Π^4, Π^5 and Π^6

2.1. Preliminaries. It suffices to show the intersections of Π^3, Π^4, Π^5 and Π^6 with $\{a_1 = 0\}$. Indeed, the shift $x \mapsto x + a$, $a \in \mathbb{R}$, induces a linear isomorphism in $Oa_1 \ldots a_n$ of the form $a_i \mapsto a_i + \sum_{j=1}^{i-1} \psi_j a_j$, $\psi_j \in \mathbb{R}$, $i = 1, \ldots, n$ (i.e. with a triangular matrix) and, hence, a diffeomorphism between any two sets of the form $\Pi^k \cap \{a_1 = \text{const}\}$ (with the same k).

Next, we make use of the quasi-homogeneity of Π (i.e. invariance under the mappings (2)). This makes it sufficient to show only the intersections of Π^3, Π^4, Π^5 and Π^6 with $\{a_1 = 0, a_2 = -1\}$. The only stratum, whose intersection with $\{a_1 = 0, a_2 = -1\}$ is empty, is 0 (whose multiplicity vector is (n)).

Notation 31. Further the lower index 0 means intersection with $\{a_1 = 0, a_2 = -1\}$: $\Pi_0 = \Pi \cap \{a_1 = 0, a_2 = -1\}$, $\Pi_0^k = \Pi^k \cap \{a_1 = 0, a_2 = -1\}$, $H_{0k}^\pm = H_k^\pm \cap \{a_1 = 0, a_2 = -1\}$.

Definition 32. For the sake of convenience we call *strata* the projections in $Oa_2 \ldots a_s$, $s = 3$, 4, 5 or 6 of the corresponding strata of Π and also the intersections of these projections with $\{a_2 = -1\}$.

Lemma 33. *1) For k odd, the sets H_k^+ and H_k^- are diffeomorphic to one another which is not true for k even. The diffeomorphism is an involution induced by the change $x \mapsto -x$.*

2) For k even, each set H_k^+ and H_k^- is invariant under an involution induced by the change $x \mapsto -x$.

P r o o f. The set Π and its projections are invariant under the mapping $a_j \mapsto (-1)^j a_j$ (resulting from $x_i \mapsto -x_i$). In particular, Π_0^4 is symmetric w.r.t. Oa_4.

Hence, for k odd the above mapping induces a diffeomorphism of each stratum of dimension $k-1$ with multiplicity vector $(r, 1, s, 1, \ldots, t, 1)$ onto the stratum with multiplicity vector $(1, t, \ldots, 1, s, 1, r)$. The former is a stratum of maximal dimension from H_k^+, the second — from H_k^-.

This diffeomorphism is extended in a natural way to strata of lower dimension. Evidently, it is an involution.

For k even, $k > 2$, there are more strata of maximal dimension in H_k^+ (i.e. with multiplicity vectors $(r, 1, s, 1, \ldots 1, t)$) than in H_k^-. Hence, H_k^+ and H_k^- are not diffeomorphic (the sets where they are not smooth, i.e. their strata of non-maximal dimension, have different topologies).

For k even, the above mapping maps strata from H_k^+ (i.e. with multiplicity vectors $(r, 1, s, 1, \ldots)$) onto such strata (idem for H_k^-); the multiplicity vectors are "read from the back". Evidently, this is an involution.

The lemma is proved. □

2.2. The projections Π^3, Π^4 and Π^5. Part 3) of Theorem 2 implies that the boundary of Π_0^3 consists of the strata $(1, n-1) \equiv H_{03}^-$ and $(n-1, 1) \equiv H_{03}^+$, see Fig. 2. The order of the projections of the strata $(l, n-l)$, $l = 2, \ldots, n-2$ on Oa_3 can be found by direct computation (one sets $lx_1 + (n-l)x_2 = 0$, $a_2 = C_l^2(x_1)^2 + l(n-l)x_1x_2 + C_{n-l}^2(x_2)^2 = -1$, then finds x_1, x_2 from this system and then computes a_3).

Rule. Each stratum $(s, 1, r)$, $s + r = n - 1$ contains in its closure the strata $(s+1, r)$ and $(s, r+1)$. More generally, the strata (a, b) and (c, d) are contained in the closure of the stratum $(\min(a, c), n - \min(a, c) - \min(b, d), \min(b, d))$.

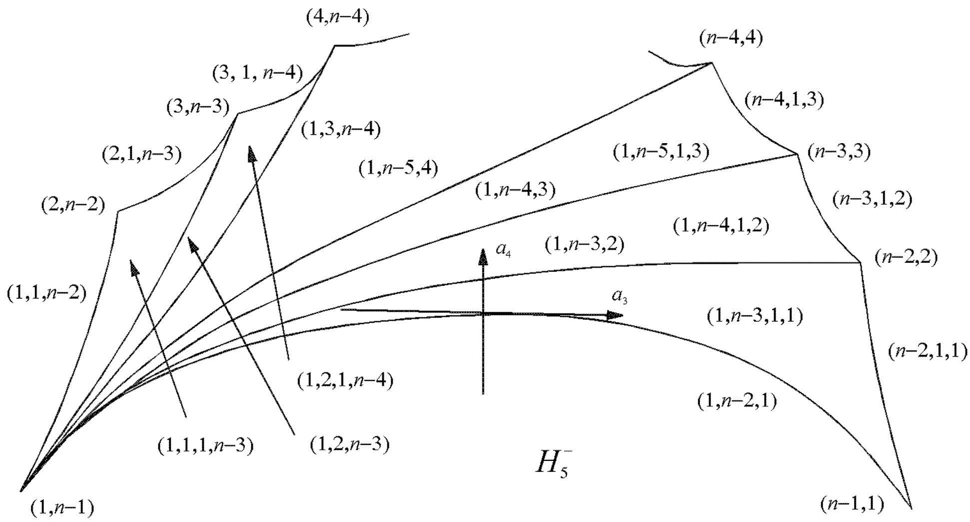


Fig. 2

By 3) of Theorem 2, the closure of the stratum $(1, n-2, 1)$ is H_{04}^{-}. This closure contains the strata $(1, n-1)$, $(n-1, 1)$. Using 2) of Theorem 2 (with $k = 4$) and also 4), we conclude that Π_0^4 looks like shown on Fig. 2; in particular, part 4) implies that there is nowhere a vertical tangent line to the strata of maximal dimension forming the boundary of Π_0^4 (or a vertical tangent line to the left or to the right). (Properly speaking, Fig. 2 shows not only Π_0^4, but H_{05}^{-}.)

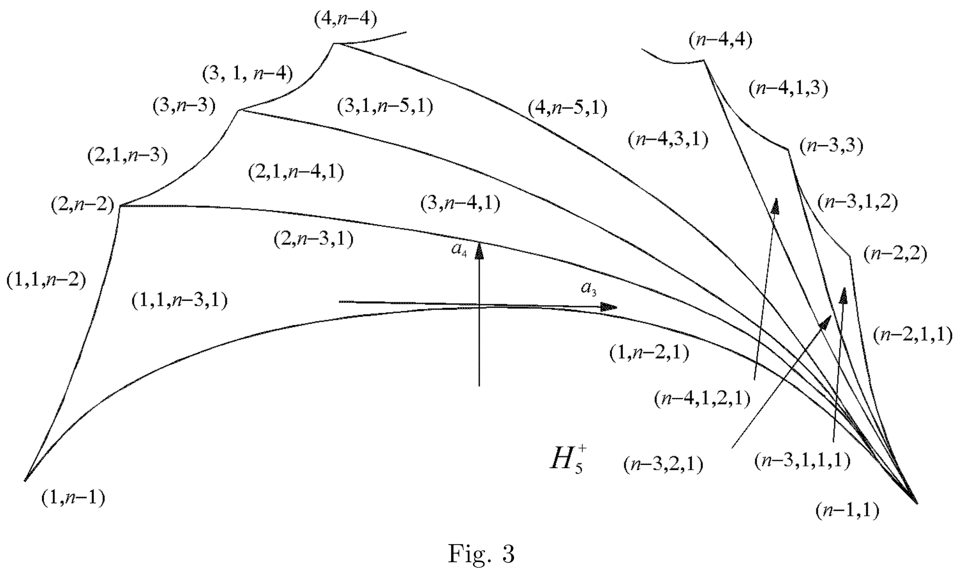


Fig. 3

Note that the convexity of the strata forming H_{04}^+ and H_{04}^- is like the one shown on the figure (i.e. the convexity is towards the domain with more real roots). This follows from the results of [6, p. 24-25].

Using again 3) of Theorem 2 and the above rule, one obtains the disposition of the strata forming H_{05}^+ and H_{05}^-, see Fig. 2 for H_{05}^- and Fig. 3 for H_{05}^+. We show on these figures the projections of the strata in $Oa_1a_2a_3a_4 \cap \{a_1 = 0, a_2 = -1\}$.

2.3. The projection Π^6. We draw several figures to represent Π_0^6. These figures show the intersections of H_{06}^+ and H_{06}^- with affine subspaces of the type $a_3 = b_k$ where $b_k \in \mathbb{R}$ is chosen such that this affine space passes between the strata $(n-k, k)$ and $(n-k-1, k+1)$ of Π_0, $k = 1, \ldots, n-2$, see Fig. 4. The tangent space to each stratum of dimension ≥ 3 is transversal to the spaces $\{a_3 = b_k\}$. Hence, the intersections $H_{06}^{\pm} \cap \{a_3 = c\}$, $c \in \mathbb{R}$, form a locally trivial fibre bundle when $\{a_3 = c\}$ does not pass (in $\{a_1 = 0, a_2 = -1\}$) through a stratum (p, q). (One can use Thom's first isotopy lemma, see [2, p. 4].)

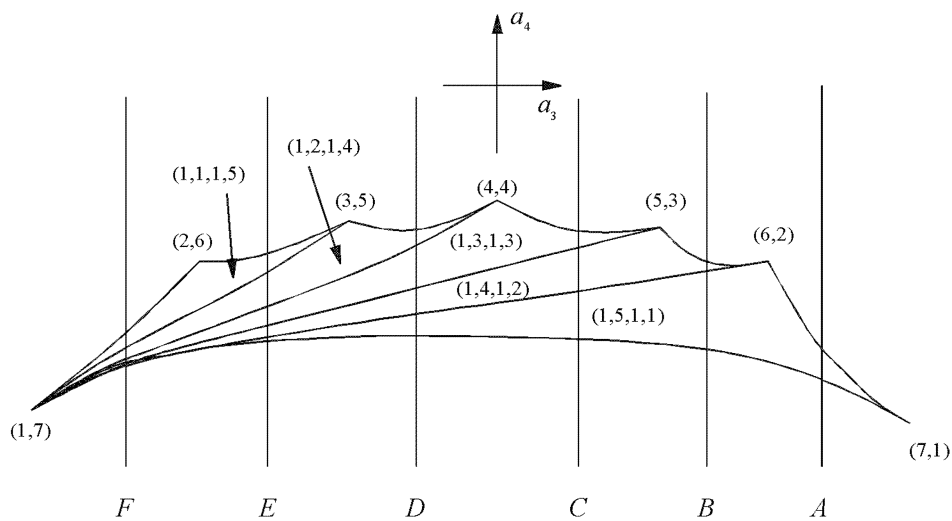
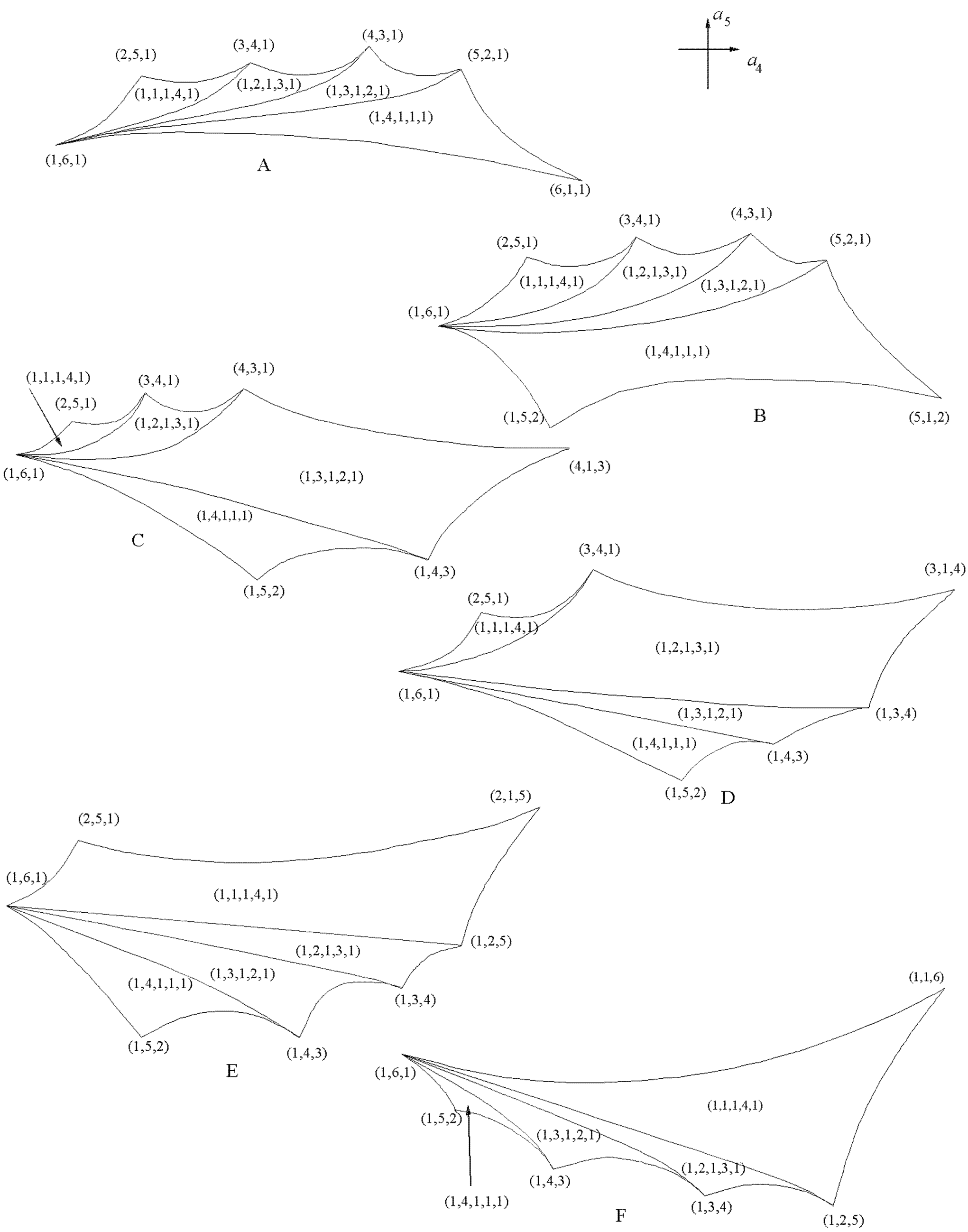


Fig. 4

Remark 34. We draw the figures for $n = 8$ but all explanations are given for any n. In order not to overload the figures, only the strata of dimensions 3 and 5 (but not 4) are indicated. We claim only that the convexity of the exterior strata of dimension 4 corresponds to the reality (the figures are not drawn as a result of exact computations). The strata of dimension 3 (4 or 5) are represented on the figures by points (curves or curvilinear polygons).



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(1,3,1,2,1)
(1,2,1,3,1)
(1,4,3)
(1,4,1,1,1)
(1,3,4)
F
(1,2,5)

Fig. 5. H_{06}^{-}

Example 35. If $k=1$, then the intersection $\Pi_0^6 \cap \{a_3 = b_k\}$ will be a stratified manifold and the disposition of the strata forming its boundary will be the same as the one of the strata forming the boundary of $(\Pi(n-1))_0^5$. (There is no analog of the symmetry w.r.t. Oa_4.) This follows from Proposition 17; note that the stratum $(1, n-1)$ represents an A_{n-2} singularity and near this stratum the boundary of Π (which is defined by the strata different from $(1,\dots,1)$) comprises only strata obtained from the bifurcation of the root of multiplicity $n-1$.

Explain first the evolution of the sets $H_{06}^- \cap \{a_3 = b_k\}$ *(see Fig. 5):*

1. For $k=1$ the affine subspace $\{a_3 = b_1\}$ corresponds to the vertical line A on Fig. 4. The disposition of the strata can be deduced from the above example. The set $\Pi_0^6 \cap \{a_3 = b_1\}$ is diffeomorphic to the set $(\Pi(n-1))_0^5$.

2. When we pass from $H_{06}^- \cap \{a_3 = b_k\}$ to $H_{06}^- \cap \{a_3 = b_{k+1}\}$, the stratum $(n-k-1, 1, k)$ disappears and there appears the stratum $(n-k-2, 1, k+1)$. Both are from H_{04}^+.

3. For $1 \leq k \leq n-3$, when passing from $H_{06}^- \cap \{a_3 = b_k\}$ to $H_{06}^- \cap \{a_3 = b_{k+1}\}$, there appears the stratum $(1, n-k-2, k+1)$ from H_{05}^-, and for $2 \leq k \leq n-2$ there disappears the stratum $(n-k-1, k, 1)$ from H_{05}^+, both of dimension 3 (as strata of Π).

4. The sets $H_{06}^- \cap \{a_3 = b_k\}$ contain the same strata of maximal dimension (i.e. equal to 5) — these are the strata $(1, l, 1, n-l-3, 1)$, $l = 1, \dots, n-4$. In $H_{06}^- \cap \{a_3 = b_k\}$, the closure of the stratum $(1, l, 1, n-l-3, 1)$ contains the following strata of dimension 3:

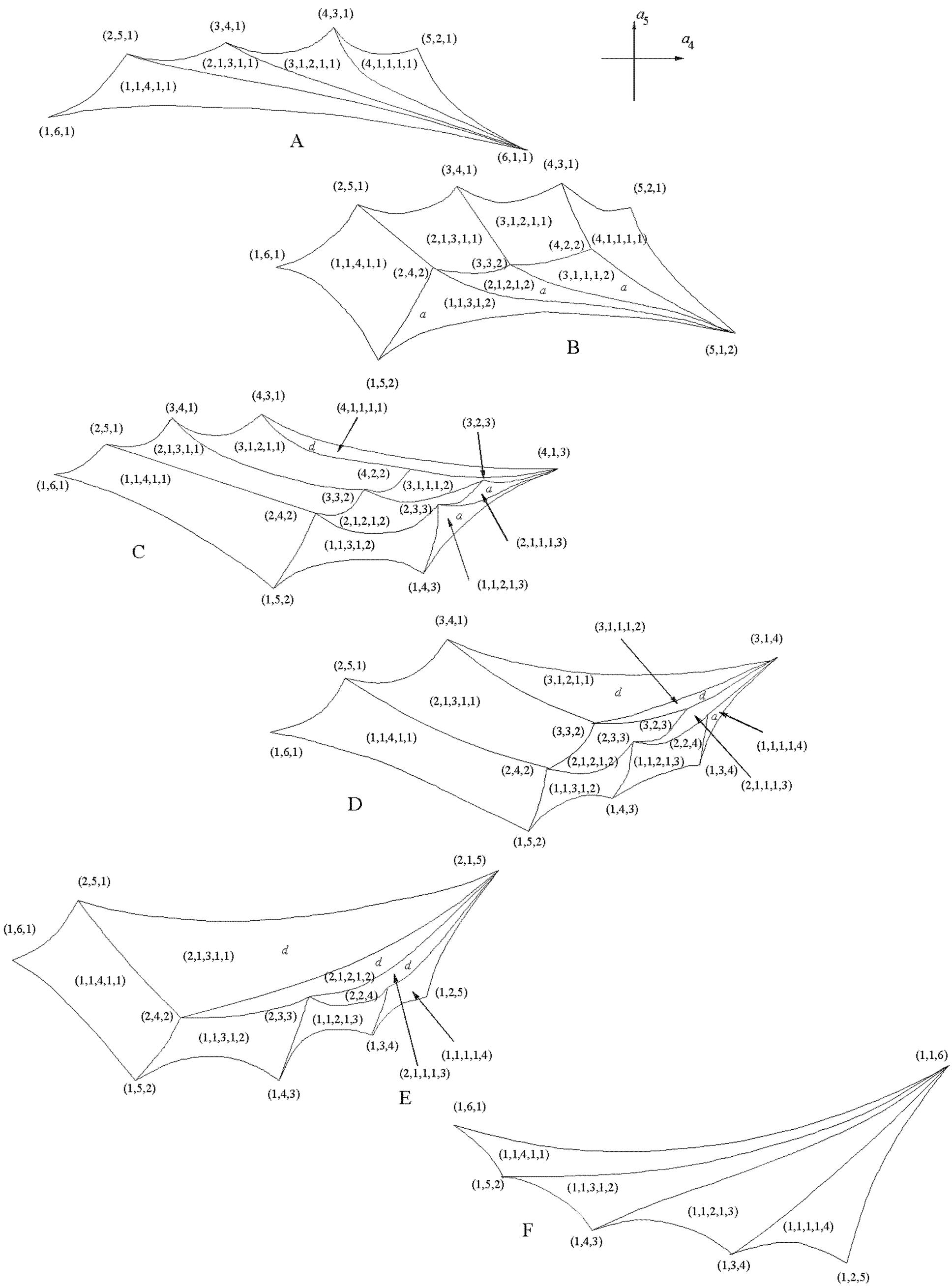
   $(1, n-2, 1)$, $(l+1, n-l-2, 1)$ and $(l+2, n-l-3, 1)$ if $l < n-k-2$;

   $(1, n-2, 1)$, $(l+1, n-l-2, 1)$, $(l+1, 1, n-l-2)$ and $(1, l+1, n-l-2)$ if $l = n-k-2$;

   $(1, n-2, 1)$, $(1, l+1, n-l-2)$ and $(1, l, n-l-1)$ if $l > n-k-2$.

From these statements only 4. needs to be proved, the others follow from the example or can be deduced from the figures. It follows from 1., 2. and 3. that the set $H_{06}^- \cap \{a_3 = b_k\}$ contains $n-2$ strata of dimension 3 if $k=1$ or $k = n-2$ and $n-1$ of them if $2 \leq k \leq n-3$.

Every set $\Pi_{06} \cap \{a_3 = b_k\}$ can be intersected with a subspace $\{a_4 = c\}$ passing close to the stratum $(1, n-2, 1)$. Reasoning like in the above example, one finds that the set $M = (\Pi_{06} \cap \{a_3 = b_k\}) \cap \{a_4 = c\}$ is diffeomorphic to $(\Pi(n-2))_0^4$.



A
B
C
D
E
F
a_5
a_4
(1,6,1)
(2,5,1)
(3,4,1)
(4,3,1)
(5,2,1)
(6,1,1)
(1,1,4,1,1)
(2,1,3,1,1)
(3,1,2,1,1)
(4,1,1,1,1)
(4,2,2)
(3,3,2)
(2,4,2)
(2,1,2,1,2)
(3,1,1,1,2)
(1,1,3,1,2)
(5,1,2)
(1,5,2)
(4,1,3)
(3,2,3)
(2,3,3)
(2,1,1,1,3)
(1,1,2,1,3)
(1,4,3)
(3,1,4)
(2,2,4)
(1,1,1,1,4)
(1,3,4)
(2,1,5)
(1,2,5)
(1,1,6)
a
d

Fig. 6. H_{06}^{+}

Hence, all strata $(1, s, n-s-2, 1)$, $s = 1, \ldots, n-3$ are present in M and so are the strata $(1, l, 1, n-l-3, 1)$, $l = 1, \ldots, n-4$, in the order indicated on the figures. One then deduces easily which strata of dimension 3 and 4 are adjacent to which strata of dimension 5.

The evolution of the sets $H_{06}^+ \cap \{a_3 = b_k\}$ is a bit more complicated, see Fig. 6:

1. When $k = 1$ (i.e. the affine space with which we intersect corresponds to the vertical line A), the disposition of the strata can be deduced from the above example. The set $\Pi_0^6 \cap \{a_3 = b_1\}$ is diffeomorphic to the set $(\Pi(n-1))_0^5$.
2. When passing to $k = 2$ (i.e. from line A to line B), then $n - 5$ new strata of dimension 5 appear. These are $(1, 1, n-5, 1, 2)$, $(2, 1, n-6, 1, 2)$, ..., $(n-5, 1, 1, 1, 2)$. No strata of dimension 5 disappear. There appear the following strata of dimension 3 — $(s, n-s-2, 2)$, $s = 2, \ldots, n-4$, recall Example 14. Some of the projections of such strata in $Oa_1 \ldots a_4 \cap \{a_1 = 0, a_2 = -1\}$ are shown (for $n = 8$) on Fig. 7.

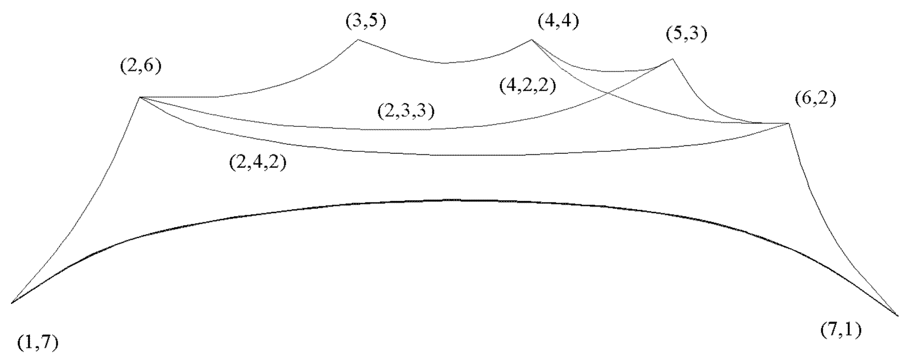


Fig. 7

3. For $1 \leq k \leq n-5$, when passing from $H_{06}^+ \cap \{a_3 = b_k\}$ to $H_{06}^+ \cap \{a_3 = b_{k+1}\}$, there appear the strata $(l, 1, n-k-l-3, 1, k+1)$, $l = 1, \ldots, n-k-4$ (of dimension 5). There also appear the strata $(s, n-s-k-1, k+1)$, $s = 2, \ldots, n-k-3$, of dimension 3.
4. For $3 \leq k \leq n-3$, when passing from $H_{06}^+ \cap \{a_3 = b_k\}$ to $H_{06}^+ \cap \{a_3 = b_{k+1}\}$, there disappear the strata $(n-k-1, 1, k-l-1, 1, l)$, $l = 1, \ldots, k-2$, of dimension 5 and the strata $(n-k-1, k-s, s+1)$, $s = 1, \ldots, k-2$, of dimension 3.

Remark 36. If one chooses $b_k = -b_{n-1-k}$, then $H_{06}^{\pm} \cap \{a_3 = b_k\}$ will be diffeomorphic to $H_{06}^{\pm} \cap \{a_3 = b_{n-1-k}\}$, the diffeomorphism being induced by $x \mapsto -x$, see 2) of Lemma 33.

To justify 2., 3. and 4., it suffices to observe that according to Proposition 17, in the neighbourhood of the stratum (p, q), the set Π is diffeomorphic to $\Pi(p) \times \Pi(q)$. It follows from Lemma 18 that the projections on Oa_3 of the strata (p', p'', q) (hence, of the strata (p', p'', p''', q) and $(p', p'', p''', p'''', q)$ as well) are to the left and the ones of the strata (p, q', q''), (p, q', q'', q''') and $(p, q', q'', q''', q'''')$ are to the right of the one of (p, q).

Remark 37. On the figures the strata of dimension 5 which have just appeared bear the letter "a", the ones which are due to disappear in the next figure bear the letter "d". We hope that the reader will be able to give the more detailed justification of the evolution of the sets $H_{06}^{+} \cap \{a_3 = b_k\}$ (if needed) oneself. To do this it would be helpful to write down explicitly the multiplicity vectors of the strata of dimension 4.

There arizes the natural question: why not continue like this with Π^7, Π^8 etc. by representing them with two-dimensional figures when the first 5, 6 etc. coordinates a_i are fixed? To do so one needs to know given the two couples of strata (p, q, r), (p', q', r') and (u, v, w), (u', v', w') with intersecting projections in $Oa_1a_2a_3a_4 \cap \{a_1 = 0, a_2 = -1\}$, the projection on Oa_3 of which intersection is to the left and of which to the right (see Fig. 7 and the strata $(2, 3, 3)$ and $(4, 2, 2)$ on it). The answer to such questions becomes more and more difficult to give (for all n) and it cannot be given by means only of the methods used in this paper.

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