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ALMOST-HERMITIAN STRUCTURES, TWISTOR SPACES AND HARMONIC MAPS

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In the first part of the paper, we review the notion of an almost-complex structure and some topological obstructions for existence of such a structure. Also, we present basic facts about twistor spaces as parametrization spaces of the almost-Hermitian structures, i.e. orthogonal almost-complex structures, on a Riemannian (or conformal) manifold. In the second part, we consider the problem of when an almost-Hermitian structure on a Riemannian or conformal manifold determines a harmonic or pseudo-harmonic map from the manifold into its twistor space. Recent results on this problem by the author (and collaborators) are discussed mainly in the case of a four-dimensional base manifold.

Since this survey is intended for a wide audience, including students, in both parts, we emphasize mostly on elucidating examples rather than technicalities of the proofs.

1. Introduction. Recall that an almost-complex structure on a Riemannian manifold is called almost-Hermitian or compatible if it is an orthogonal endomorphism of the tangent bundle of the manifold. It is well-known that if a Riemannian manifold admits an almost-Hermitian structure, it possesses many such structures, see, for example, [22]. Thus, it is natural to look for "reasonable" criteria that distinguish some of these structures. One way to obtain such criteria is to consider the almost-Hermitian structures on a Riemannian manifold (M, g) as sections of its twistor bundle that parametrizes the almost-Hermitian structures on (M, g). This is the bundle $\pi : \mathcal{Z} \to M$ whose fibre at a point $p \in M$ consists of all g-orthogonal complex structures $I_p : T_pM \to T_pM$ ($I_p^2 = -Id$) on the tangent space of M at p. The fibre of the bundle $\mathcal{Z}$ is the compact Hermitian symmetric space $O(2m)/U(m)$, $2m = \dim M$, and its standard metric $G = -\dfrac{1}{2}\,\mathrm{Trace}(I_1 \circ I_2)$ is Kähler-Einstein. The Levi-Civita connection of (M, g), as well as any metric connection, gives rise to a splitting $T\mathcal{Z} = \mathcal{H} \oplus \mathcal{V}$ of the tangent bundle of $\mathcal{Z}$ into horizontal and vertical subbundles. This decomposition allows one to define a 1-parameter family of Riemannian metrics $h_t = \pi^*g + tG$, $t > 0$, for which the projection map $\pi : (\mathcal{Z}, h_t) \to (M, g)$ is a Riemannian submersion with totally geodesic fibres. In the

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terminology of Besse's work[1] [6, Definition 9.7], the family h_t is the canonical variation of the metric g. Note also that each metric h_t is compatible with the natural almost-complex structures on $\mathcal{Z}$, which have been introduced by Atiyah-Hitchin-Singer [4] and Eells-Salamon [34] in the case when the dimension of the base manifold M is four.

Motivated by the theory of harmonic maps, C. M. Wood [71, 72] has suggested to consider as "optimal" those almost-Hermitian structures $J : (M, g) \to (\mathcal{Z}, h_1)$ that are critical points of the energy functional under variations through sections of $\mathcal{Z}$. In general, these critical points are not harmonic maps, but, by analogy, they are referred to as "harmonic almost-complex structures" in [72]; they are also called "harmonic sections" in [71], a term which seems more appropriate. In this context, a natural problem is when the Atiyah-Hitchin-Singer or the Eells-Salamon almost-complex structure on the twistor space of a Riemannian manifold (M, g) is a harmonic section of the twistor space. This problem has been solved in [20] in the case when the base manifold M is of dimension four.

Forgetting the bundle structure of $\mathcal{Z}$, we can consider the almost-Hermitian structures that are critical points of the energy functional under variations through arbitrary maps $M \to \mathcal{Z}$, not just sections. These structures are genuine harmonic maps from (M, g) into $(\mathcal{Z}, h_t)$, and we refer to [33] for basic facts about harmonic maps. This point of view is taken in [24, 26] where the problem of when an almost-Hermitian structure on a Riemannian four-manifold is a harmonic map from the manifold into its twistor space is discussed. In [24], the metric h_t is defined via the Levi-Civita connection; it is defined by means of a metric connection with totally skew-symmetric torsion in [26]. In [23] (cf. also [22]), the Riemannian 4-manifolds (M, g) for which the Atiyah-Hitchin-Singer or the Eells-Salamon almost-complex structure is a harmonic map from the twistor space $(\mathcal{Z}, h_t)$ of (M, g) into the twistor space of $(\mathcal{Z}, h_t)$ are described.

If g and $g^1 = e^f g$ are conformal metrics, the Riemannian manifolds (M, g) and (M, g^1) have the same twistor space, i.e. the twistor space depends on the conformal class of g rather than the metric g itself. Thus, it is natural to consider the twistor spaces in the context of conformal geometry, see, for example, [4, 37]. The harmonic map techniques has a useful extension in the conformal geometry introduced by G. Kokarev [49]. The problem of when a Hermitian structure on a conformal manifold endowed with a Weyl connection is a pseudo-harmonic map in the sense of Kokarev from the manifold into its twistor space is discussed in [27].

The current paper is an extended version of the author's talk at the 51-st Spring Conference of UBM, April, 2022. In the first part of the paper, the notion of an almost-complex structure and some topological obstructions for existence of such a structure are reviewd. Also, basic facts about twistor spaces are given. In the second part, recent results by author and collaborators on the harmonicity of an almost-Hermitian structure on a Riemannian or conformal manifold are surveyed. Since the paper is intended for a wide audience, the emphasis is on elucidating examples rather than the proofs.

[1] Arthur Besse is a pseudonym chosen by a group of French differential geometers, led by Marcel Berger, following the model of Nicolas Bourbaki (editor's note).

I. Almost-complex and almost-Hermitian structures. Twistor spaces

2. Almost-complex structures.

2.1. Almost complex manifolds. We begin with some basics facts about almost-complex manifolds referring to [47] for more details (see also [6]).

Recall that a linear complex structure on a real vector space V is an endomorphism J of V such that $J^2 = -Id$. Such an endomorphism allows one to render V a complex vector space setting $i.v = Jv$ for $v \in V$. Conversely, this identity on a complex vector space defines a linear complex structure J on the underlying real vector space.

An almost-complex structure on a smooth (paracompact) manifold M is an endomorphism J of the tangent bundle TM of M with $J^2 = -Id$, i.e. a collection of linear complex structures J_p on the tangent spaces T_pM of M smoothly depending on the point p when it varies on the manifold. A smooth manifold endowed with a complex structure is called almost-complex, a notion due to C. Ehresmann and H. Hopf in the late 40's of the last century (see, for example, [2, 67]).

Examples of almost-complex manifolds.

Example 1. Let M be a complex manifold and let $(z_1, \ldots, z_n)$ be complex-analytic coordinates in a neighbourhood of a point p. Consider M as a smooth manifold. If $z_k = x_k + iy_k$, we can use the smooth coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ on M to define an almost-complex structure J_p of the tangent space T_pM of M at p setting

$$J_p\frac{\partial}{\partial x_k}(p) = \frac{\partial}{\partial y_k}(p), \quad J_p\frac{\partial}{\partial y_k}(p) = -\frac{\partial}{\partial x_k}(p), \quad k = 1, \ldots, n.$$

It is easy to see by means of the Cauchy-Riemann equations that J_p does not depend on the choice of the complex coordinates $(z_1, \dots, z_n)$. In this way we obtain an almost-complex structure on M.

The conjugate functions $(\bar{z}_1, \dots, \bar{z}_n)$ determine another structure of a complex manifold on the set M. With this structure, M is usually denoted by $\overline{M}$. It is clear that the almost-complex structure of $\overline{M}$ is equal to $-J$.

This justify the following terminology. If J is an almost-complex structure on a smooth manifold, $-J$ is also an almost-complex structure, called conjugate of J.

When an almost-complex structure is defined by means of complex coordinates, we drop the adjective "almost".

Example 2. Let M be a parallelizable even-dimensional manifold and $E_1, \dots, E_{2m}$ a global frame of vector fields on M. Then setting $JE_{2k-1} = E_{2k}$, $JE_{2k} = -E_{2k-1}$, $k = 1, \dots, m$ defines an almost-complex structure on M.

Example 3. Every orientable two-dimensional smooth manifold M admits an almost-complex structure. In order to define such a structure, fix an orientation on M and take any Riemannian metric on it. For each point $p \in M$, choose an oriented orthogonal basis (e_1, e_2) with $\|e_1\| = \|e_2\|$ of the tangent space T_pM. Set $J_pe_1 = e_2$, $J_pe_2 = -e_1$. The endomorphism J_p of T_pM is a complex structure on T_pM which does not depend on the choice of the basis (e_1, e_2). For each point p, we can find an oriented orthogonal frame of vector fields (X_1, X_2) with $\|X_1\| = \|X_2\|$ in a neighbourhood U of p. A straightforward consequence of this remark is that J_p depends smoothly on p. Thus, the endomorphisms J_p determine an almost-complex structure J on M.

Clearly, if we take a metric $g' = e^f g$ conformal to g, f being a smooth function, we get the same almost-complex structure. If we reverse the orientation of M, we obtain the conjugate almost-complex structure $-J$.

Note also that it can be shown that any almost-complex structure J on a two-dimensional manifold M is a complex structure, i.e. one can introduce complex coordinates on M which yield J as in the preceding example (for this fact see Sec. 2.3).

Example 4. Consider the unit sphere S^2 with its standard metric and orientation. Denote by $\times$ the usual vector-cross product on $\mathbb{R}^3$. Then the almost-complex structure on S^2, described in the preceding example, is given by $J_a(v) = a \times v$ for $a \in S^2$ and $v \in T_aS^2$. This almost-complex structure comes from the well-known complex-analytic atlas defined by means of the stereographic projection rendering S^2 a complex manifold biholomorphic to the complex projective line $\mathbb{CP}^1$. Recall that identifying $\mathbb{R}^3$ with the imaginary quaternions, the vector-cross product is $a \times b = Im\,(ab) = \frac{1}{2}(ab - ba)$ where the multiplication is in the algebra of quaternions. Similarly, identifying $\mathbb{R}^7$ with the imaginary octonions, one can define a vector-cross product in $\mathbb{R}^7$, the multiplication being this time in the algebra of octonions. Then the formula $J_a(v) = a \times v$ for $a \in S^6$, $v \in T_aS^6$, yields an almost-complex structure on S^6. This structure was defined by A. Kirchhoff in 1947 (as well by C. Ehresmann in 1950). Shortly after that in 1951, B. Eckmann and A. Frölicher as well as C. Ehresmann and P. Libermann showed that this structure does not come from complex coordinates. We refer to [2, 3] for more information about this story.

It is a result of A. Borel and J.-P. Serre (1953) that S^2 and S^6 are the only spheres that admit an almost-complex structure. As we have mentioned, every almost-complex structure on S^2 is complex, and it is a long-standing problem whether S^6 admits any complex structure. We refer to [2] for a nice overview on the history of this problem

Example 5. E. Calabi [14] has generalized the construction of the almost-complex structure on S^6 to every orientable hypersurface M in $\mathbb{R}^7$. If $\mathfrak{n}$ is a normal field on M, he studied the almost-complex structure defined by $Jv = \mathfrak{n} \times v$, $v \in TM$. He found necessary and sufficient conditions for J to be induced by complex coordinates, and showed that these conditions are not satisfied if M is compact.

Example 6. E. Calabi and B. Eckmann [13] have constructed complex structures on the product of two odd-dimensional spheres $S^{2n+1} \times S^{2m+1}$. Consider the action of the additive group $\mathbb{C}$ on $\mathbb{C}^n \setminus \{0\}$ and $\mathbb{C}^m \setminus \{0\}$, $n, m > 1$, defined by $\zeta.(z, w) = (e^{\zeta} z, e^{\tau\zeta} w)$ where τ is a fixed complex number with $Im\, \tau \neq 0$. The quotient space is a complex manifold diffeomorphic to $S^{2n+1} \times S^{2m+1}$.

Example 7. Any almost symplectic manifold M, i.e. admitting a non-degenerate skew-symmetric 2-form Ω, possesses an almost-complex structure. To define one, fix a Riemannian metric g on M. Then, as is well-known, the identity $g(JX, Y) = \Omega(X, Y)$, $X, Y \in TM$, defines a unique almost-complex structure J, see, for example, [8]. Recall that M is called symplectic if the form Ω is closed, $d\Omega = 0$.

2.2. Topological obstructions for existence of almost-complex structures. Every almost-complex structure on a manifold M induces on each tangent space of M the structure of a complex vector space, so the tangent bundle of TM becomes a (smooth) complex bundle. Hence M is of even dimension and orientable. An immediate consequence of this trivial remark is that no real projective space admits an almost-complex structure. So, the vanishing of the first Stiefel-Whitney class $w_1(M) \in H^1(M, \mathbb{Z}_2)$ is a topological obstruction for existence of an almost-complex structure on an even-dimensional manifold. We briefly discuss other topological obstructions in cohomological terms on oriented compact manifolds. As we have noted, if $\dim M = 2$, there is no other obstructions. Suppose that $\dim M = 4$. Let $\sigma(M)$ and $\chi(M)$ be the signature and the Euler characteristic of M. By the classical W.-T. Wu theorem [73] (as stated in [42]), M admits an almost-complex structure if and only if there is $h \in H^2(M, \mathbb{Z})$ such that $h^2[M] = 3\sigma(M) + 2\chi(M)$ and $h \equiv w_2(M) \mod 2$. In this case h is the first Chern class of the complex bundle TM. In order to state another criterion for existence of an almost-complex structure in dimension four, denote the intersection form on $H^2(M, \mathbb{Z})$ by Q. Then by a result of A. Dessai [30, Theorem 1.4] in the reformulation of [53], M admits an almost-complex structure if and only if $\sigma(M) + \chi(M) \equiv 0\, mod\, 4$ and one of the following conditions is satisfied: (i) Q is indefinite; (ii) Q is positive definite and the Betti numbers b_1 and b_2 satisfy the inequality $b_1 - b_2 \leq 1$; (iii) Q is negative definite and, in case $b_2 \leq 2$, the number $4(b_1 - 1) + b_2$ is the sum of b_2 integer squares. Now, let $\dim M = 6$. According to a result by C. T. C. Wall [69, Theorem 9], the only obstruction for existence of an almost-complex structure in dimension 6 is the vanishing of the third integer Stiefel-Whitney class, i.e. $w_2(M)$ to be mapped to zero by the Bockstein homomorphism $\partial : H^2(M, \mathbb{Z}_2) \to H^2(M, \mathbb{Z})$.

Examples of almost-complex manifolds not admitting any complex structure.

Example 8. The first examples of this type have been constructed by A. Van de Ven [67]. Consider the complex projective plane $\mathbb{CP}^2$ as a smooth manifold with the orientation induced by the complex structure. Denote this manifold by $\mathbb{P}^2$. It has been shown in [67] that the connected sum $\mathbb{P}^2 \sharp (S^1 \times S^3) \sharp (S^1 \times S^3)$ admits an almost-complex structure but does not possess any complex structure. The connected sum $(S^1 \times S^3) \sharp (S^1 \times S^3) \sharp (S^2 \times S^2)$ has the same property.

Example 9. S.-T. Yau [74] has showed that $(T^3 \sharp \mathbb{RP}^3) \times S^1$, where T^3 is the 3-dimensional real torus, is a parallelizable 4-manifold, which does not admit a complex structure.

Example 10. A parallelizable 4-manifold without any complex structure has also been constructed by N. Brotherton [12].

No examples of almost-complex manifolds not admitting a complex structure are known in dimensions greater than four.

S.-T. Yau's conjecture [75, Problem 52]. Every compact almost-complex manifold of dimension at least 6 admits a complex structure.

2.3. Integrable almost-complex structures. For a smooth manifold M, we denote the complexification of a tangent space T_pM by $T_p^{\mathbb{C}}M$. Let J be an almost-complex structure on M. Extend J by complex linearity to the complexification $T^{\mathbb{C}}M$ of the tangent bundle TM. The extension also satisfies the identity $J^2 = -Id$, thus it has eigenvalues $\pm i$. As usual, denote the eigen-subbundles of $T^{\mathbb{C}}M$ corresponding to $(+i)$ and $(-i)$ eigenvalues of J by $T^{(1,0)}M$ and $T^{(0,1)}M$.

An almost-complex structure J is called integrable if the subbundle $T^{(1,0)}M$ of $T^{\mathbb{C}}M$ is integrable (i.e. the Lie bracket of two vector fields with values in $T^{(1,0)}M$ is again such a vector field). This is equivalent to $T^{(0,1)}M$ being integrable.

Denote by N_J the Nijenhuis tensor of J defined by

$$N_J(X,Y) = -[X,Y] + J[JX,JY] - J[JX,Y] - J[X,JY]$$

where X, Y stand for vector fields on the manifold. It is a simple exercise that the structure J is integrable if and only if its Nijehuis tensor N_J vanishes.

The following famous theorem is due to A. Newlander and L. Nirenberg

Theorem 1 ([55]). *An almost-complex structure is complex, i.e. it is induced by a complex-analytic atlas, if and only if it is integrable.*

It is easy to see that in real dimension 2, every almost-complex structure is integrable.

3. Almost-Hermitian structures. For any almost-complex structure J, there is a Riemannian metric g such that J is g-orthogonal: $g(JX, JY) = g(X,Y)$ for every tangent vectors X, Y at a point p. Just take any Riemannian metric h and set $g(X,Y) = h(X,Y) + h(JX,JY)$. If (M,g) is a Riemannian manifold, a g-orthogonal almost-complex structure is called almost-Hermitian or compatible with g. An important role in the study of almost-Hermitian manifolds (M,g,J) is played by their fundamental, or Kähler, 2-form $\Omega(X,Y) = g(JX,Y)$, $X, Y \in TM$. Recall that if Ω is closed, the almost-Hermitian manifold is called almost-Kähler or symplectic; if in addition J is integrable, it is called Kähler. The latter condition is equivalent to $\nabla J = 0$, where ∇ is the Levi-Civita connection of the Riemannian metric g.

Examples of almost-Hermitian manifolds.

Example 11. Let M be a parallelizable even-dimensional manifold with a global frame $E_1, \dots, E_{2m}$ of vector fields on it. Denote by g the Riemannian metric for which the frame $E_1, \dots, E_{2m}$ is orthonormal. Then the natural almost-complex structure J defined above by $JE_{2k-1} = E_{2k}$, $JE_{2k} = -E_{2k-1}$ is g-orthogonal.

Example 12. The complex structure on an oriented 2-dimensional manifold defined by means of a Riemannian metric g on it is g-orthogonal. In particular, the standard complex structure of S^2 is Hermitian with respect to the standard metric g_{S^2} of the sphere.

Example 13. The complex projective space $\mathbb{CP}^n$ endowed with Fubini-Study metric is a Hermitian manifold. The standard biholomorphism $S^2 \cong \mathbb{CP}^1$ defined by means of the stereographic projection sends the metric $4g_{S^2}$ on S^2 to the Fubini-Study metric of $\mathbb{CP}^1$.

Example 14. The almost-complex structure on S^6 defined above is compatible with the standard metric on S^6.

Example 15. The Calabi almost-complex structure on an oriented hypersurface in $\mathbb{R}^7$ is compatible with the metric on the hypersurface induced by the standard metric of $\mathbb{R}^7$.

Example 16. The Calabi-Eckmann complex structures on $S^{2n+1} \times S^{2m+1}$ are compatible with the product metric of the standard metrics of the spheres. The Calabi-Eckmann Hermitian structure is Kähler exactly when $n = 0$ or $m = 0$; it is locally conformally Kähler (but non-Kähler) if and only if $n = 0, m \geq 1$ or $n \geq 1, m = 0$ [39, Theorem 4.4] (for the latter claim, see also [32, 66]).

Example 17. It is obvious that any Kähler manifold is symplectic. The first example of a compact symplectic manifold which does not admit any Kähler structure has been given by W. Thurston [65]. A second version of this example has been constructed by E. Abbena [1] who also gave a natural associated metric for the symplectic structure. Let

$$H = \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, \quad x, y, z \in \mathbb{R}, \tag{1}$$

be the real Heisenberg group, and let $G = H \times S^1$ be the direct product of the groups H and S^1. Let Γ be the discrete subgroup of G of the matrices (1) with integer entries. The quotient space $M = G/\Gamma$ is a compact smooth manifold. Let $e^{i2\pi t} \to t$ be the standard local coordinate on S^1. The form $\widetilde{\Omega} = dt \wedge dx + dy \wedge (dz - xdy)$ is a left invariant symplectic form on G, hence it descents to a symplectic form Ω on M. The vector fields

$$E_1 = \frac{\partial}{\partial x}, \quad E_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad E_3 = \frac{\partial}{\partial z}, \quad E_4 = \frac{\partial}{\partial t} \tag{2}$$

dual to dx, dy, $dz - xdy$, dt are left invariant and constitute a global frame on G. Let $\widetilde{g}$ be the metric on G for which this frame is orthonormal. Also, let $\widetilde{J}$ is the almost-complex structure on G for which $\widetilde{J}E_1 = E_4$, $\widetilde{J}E_2 = E_3$. Then $\widetilde{\Omega}(E_k, E_l) = \widetilde{g}(\widetilde{J}E_k, E_l)$. The left-invariant almost-Hermitian structure $(\widetilde{g}, \widetilde{J})$ on G, as well as $\widetilde{\Omega}$, descends to such a structure (g, J) on M satisfying the identity $\Omega(X, Y) = g(JX, Y)$, $X, Y \in TM$. The almost-complex structure $\widetilde{J}$ is not integrable, hence neither is J. Thus, (g, J) is an almost-Kähler structure which is not Kähler. Moreover, M does not admit any Kähler structure for topological reasons; it has an odd first Betti number, namely $b_1 = 3$ ([1, 65]), while, as is well-known, the odd Betti numbers of a Kähler manifold are even.

Example 18. Let $H^{\mathbb{C}}$ be the complex Heisenberg group consisting of all matrices of the form (1) with complex entries $x, y, z \in \mathbb{C}$. Let Γ be the discrete subgroup for which x, y, z have integer real and imaginary parts. Then the quotient space $M = H^{\mathbb{C}}/\Gamma$ is a compact complex 3-manifold known as the Iwasawa manifold. The complex structure of M is compatible with the Riemannian metric g for which the real and imaginary parts of the complex vector fields defined by (2) constitute an orthonormal frame. The Betti numbers of the Iwasawa manifold are $b_1 = 4, b_3 = 10, b_5 = 4$ and $b_0 = 1, b_2 = 8, b_4 = 8, b_6 = 1$, see, for example, [35]. Although all odd Betti numbers are even, the Iwasawa manifold has no complex structure admitting a Kähler metric [ibid]. But the Iwasawa manifold admits a symplectic form, hence an almost Kähler structure [16] (which, of course, is non-integrable).

Remark 1. As is well-known, in complex dimension two, every compact complex surfaces with even first Betti number admits a Kähler metric, see, for example, [5, Sec. IV.3].

Example 19. Let (M, g) be a Riemannian manifold and $\pi : TM \to M$ its tangent bundle. The Levi-Civita connection of (M, g) gives rise to a splitting $T(TM) = \mathcal{H} \oplus \mathcal{V}$ of the tangent bundle of the manifold TM into horizontal and vertical subbundles. For any $a \in TM$, the restriction of the differential $\pi_{*a} : T_a(TM) \to T_{\pi(a)}M$ to the horizontal space $\mathcal{H}_a$ is an isomorphism of $\mathcal{H}_a$ onto $T_{\pi(a)}M$. As usual, if $X \in T_{\pi(a)}M$, we denote the horizontal lift $(\pi_{*a}|\mathcal{H}_a)^{-1}(X)$ of X by X^h_a. The vertical space $\mathcal{V}_a$ is the tangent space at a of the fibre $T_{\pi(a)}M$ of TM through the point a. We denote by X^v_a the image of X under the standard identification $T_{\pi(a)}M \cong T_a(T_{\pi(a)}M) = \mathcal{V}_a$. Following T. Nagano [54] (see also [44, 31]), define an almost-complex structure J on the manifold TM setting

$$J(X^h, Y^v) = (-Y^h, X^v), \quad X, Y \in TM.$$

This structure is integrable if and only if the Riemannian manifold (M, g) is flat ([54, 44, 31]). The almost-complex structure J is compatible with the Sasaki metric on TM. Recall that this metric is defined by

$$h(X^h_1 + Y^v_1, X^h_2 + Y^v_2) = g(X_1, X_2) + g(Y_1, Y_2).$$

Thus, we have an almost-Hermitian structure (h, J) on the manifold TM. It is almost-Kähler [63] and is Kähler exactly when (M, g) is flat.

4. Basics about twistor spaces. The twistor theory has been originated by the English physicist R. Penrose [56, 57] as a tool for transferring physical information from the space-time into geometric data on a projective space with the aim to propose an approach to the quantum gravity. The ideas of Penrose have been developed in the context of Riemannian geometry by Atiyah, Hitchin and Singer [4] in the case of manifolds of dimension four. From mathematical point of view, the basic idea is that the geometry of a Riemannian manifold can be encoded in holomorphic terms of the so-called twistor space associated to the manifold.

4.1. The manifold of compatible linear complex structures. Let V be a real vector space of even dimension $n = 2m$ endowed with an Euclidean metric g. Denote by $F(V)$ the set of all complex structures on V compatible with the metric g, i.e. g-

orthogonal. This set has the structure of an imbedded smooth submanifold of the vector space $\mathfrak{so}(V)$ of skew-symmetric endomorphisms of (V,g). The tangent space of $F(V)$ at a point J consists of all endomorphisms $Q\in\mathfrak{so}(V)$ anti-commuting with J, and we have the decomposition $\mathfrak{so}(V) = T_JF(V)\oplus\{S\in\mathfrak{so}(V) : \ SJ-JS=0\}$ which is orthogonal with respect to the metric $G(S,T) = -\frac{1}{2}\operatorname{Trace}(S\circ T)$ on $\mathfrak{so}(V)$. The smooth manifold $F(V)$ admits a natural almost-complex structure $\mathcal{J}$ defined by $\mathcal{J}Q = J\circ Q$ for $Q\in T_JF(V)$. This structure is compatible with the metric G. It is not hard to find an explicit formula for the Levi-Civita connection ∇ of G and to see that $\nabla\mathcal{J}=0$, so $(G,\mathcal{J})$ is a Kähler structure on $F(V)$. In particular, $\mathcal{J}$ is an integrable structure, hence it is induced by complex coordinates. In fact, such coordinates can explicitly be defined by means of the Cayley transformation of skew-symmetric endomorphisms. One can also show that the metric G is Einstein.

The group $O(V)\cong O(2m)$ of orthogonal transformations of V acts on $F(V)$ by conjugation and the isotropy subgroup at a fixed complex structure J_0 is isomorphic to the unitary group $U(m)$. Therefore $F(V)$ can be identified with the homogeneous space $O(2m)/U(m)$. In particular, $\dim F(V) = m^2-m$. Note also that the manifold $F(V)$ has two connected components $F_\pm(V)$: if we fix an orientation on V, these components consist of all complex structures on V compatible with the metric g and inducing the positive/negative orientation of V; each of them has the homogeneous representation $SO(2m)/U(m)$.

The metric g of V induces a metric on Λ^2V given by

$$g(v_1\wedge v_2, v_3\wedge v_4) = \frac{1}{2}[g(v_1,v_3)g(v_2,v_4) - g(v_1,v_4)g(v_2,v_3)]. \tag{3}$$

Then we have an isomorphism $\mathfrak{so}(V)\cong\Lambda^2V$, which sends $S\in\mathfrak{so}(V)$ to the 2-vector $S^\wedge$ for which

$$2g(S^\wedge, u\wedge v) = g(Su,v), \quad u,v\in V. \tag{4}$$

This isomorphism is an isometry with respect to the metric $\frac{1}{2}G$ on $\mathfrak{so}(V)$ and the metric g on Λ^2V.

Suppose that V is of dimension four. Then the Hodge star operator defines an endomorphism $*$ of Λ^2V with $*^2 = Id$. Hence we have the orthogonal decomposition

$$\Lambda^2V = \Lambda^2_+V\oplus\Lambda^2_-V,$$

where $\Lambda^2_\pm V$ are the subspaces of Λ^2V corresponding to the (±1)-eigenvalues of the operator $*$. Let (e_1,e_2,e_3,e_4) be an oriented orthonormal basis of V. Set

$$s_1^\pm = e_1\wedge e_2\pm e_3\wedge e_4, \quad s_2^\pm = e_1\wedge e_3\pm e_4\wedge e_2, \quad s_3^\pm = e_1\wedge e_4\pm e_2\wedge e_3. \tag{5}$$

Then $(s_1^\pm, s_2^\pm, s_3^\pm)$ is an orthonormal basis of $\Lambda^2_\pm V$. The orientation of $\Lambda^2_\pm V$ determined by this basis does not depend on the choice of the basis (e_1,e_2,e_3,e_4) (see, for example, [22]).

It is easy to see that the isomorphism $\varphi\to\varphi^\wedge$ identifies $F_\pm(V)$ with the unit sphere $S(\Lambda^2_\pm V)$ of the Euclidean vector space $(\Lambda^2_\pm V, g)$ (so, the factor $1/2$ in (3) is chosen in order to have spheres with radius 1). Under this isomorphism, if $J\in F_\pm(V)$, the tangent space $T_JF(V) = T_JF_\pm(V)$ is identified with the orthogonal complement $(\mathbb{R}J)^\perp$ of the space $\mathbb{R}J$ in $\Lambda^2_\pm V$. Moreover, the restriction of the complex structure $\mathcal{J}$ to $F_\pm(V)$ is identified with the complex structure of the unit sphere in $\Lambda^2_{\pm}V$. Thus $(\mathcal{J}Q)^{\wedge} = \pm(J^{\wedge} \times Q^{\wedge})$ where $\times$ is the vector-cross product on the 3-dimensional Euclidean space $(\Lambda^2_{\pm}V, g)$ endowed with the orientation defined above.

4.2. The twistor space of a Riemannian manifold. Let (M, g) be a Riemannian manifold of even dimension. The twistor space of (M, g) is the bundle $\pi : \mathcal{Z} \to M$ whose fibre at a point $p \in M$ is the manifold $F(T_pM)$ of all complex structures of the tangent space T_pM compatible with the metric g_p. If M is oriented, we can consider the bundles over M whose fibre at p is the manifold of compatible complex structures yielding the positive and, respectively, the negative orientation of T_pM. The latter bundles are disjoint open subsets of $\mathcal{Z}$ and are frequently called the positive and the negative twistor spaces of (M, g), respectively. They are usually denoted by $\mathcal{Z}_+$ and $\mathcal{Z}_-$. If M is connected and oriented, they are the connected components of the manifold $\mathcal{Z}$. Clearly, changing the orientation interchanges the role of the positive and negative twistor spaces. If M is oriented and of dimension four, $\mathcal{Z}_{\pm}$ can be identified with the unit sphere subbundle of $\Lambda^2_{\pm}TM$, the latter vector bundle being the the subbundle of Λ^2TM corresponding to the (±1)-eigenvalue of the Hodge star operator $*$.

4.3. The Atiyah-Hitchin-Singer and Eells-Salamon almost-complex structures on a twistor space. The manifold $\mathcal{Z}$ admits two almost-complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ defined, respectively, by Atiyah-Hitchin-Singer [4] and Eells-Salamon [34] in the case when the base manifold M is of dimension four.

The bundle $\mathcal{Z}$ can be considered as a subbundle of the bundle $A(TM)$ of skew-symmetric endomorphisms of the tangent bundle TM. The Levi-Civita connection of (M, g) determines a connection on the bundle $A(TM)$, both denoted by ∇. The horizontal space of the vector bundle $A(TM)$ with respect to ∇ at a point $J \in \mathcal{Z}$ is tangent to $\mathcal{Z}$. Hence we have the splitting $T\mathcal{Z} = \mathcal{H} \oplus \mathcal{V}$ of the tangent bundle of $\mathcal{Z}$ into horizontal and vertical subbundles. For every $J \in \mathcal{Z}$, the horizontal subspace $\mathcal{H}_J$ of $T_J\mathcal{Z}$ is isomorphic to the tangent space $T_{\pi(J)}M$ via the differential $\pi_{*\,J}$, and we set

$$\mathcal{J}_k|\mathcal{H}_J = (\pi_*|\mathcal{H}_J)^{-1} \circ J \circ (\pi_*|\mathcal{H}_J), \quad k = 1, 2.$$

The vertical subspace $\mathcal{V}_J$ of $T_J\mathcal{Z}$ is the tangent space at J to the fibre of the bundle $\mathcal{Z}$ through J. This fibre is the complex manifold $F(T_{\pi(J)}M)$ we have discussed. Denote its complex structure by $\mathcal{J}_{\mathcal{V}_J}$. Then the almost-complex structures $\mathcal{J}_k$ are defined on the vertical space $\mathcal{V}_J$ by

$$\mathcal{J}_1|\mathcal{V}_J = \mathcal{J}_{\mathcal{V}_J}, \quad \mathcal{J}_2|\mathcal{V}_J = -\mathcal{J}_{\mathcal{V}_J}.$$

The decomposition $T\mathcal{Z} = \mathcal{H} \oplus \mathcal{V}$ allows one to define a natural 1-parameter family of Riemannian metrics h_t, $t > 0$, on the manifold $\mathcal{Z}$ in the following (standard) way. The metric h_t on $\mathcal{H}_J$ is the lift of the metric g of $T_{\pi(J)}M$, $h_t|\mathcal{H}_J = \pi^*g$. On the vertical space $\mathcal{V}_J$, the metric h_t is defined as t times the metric G of the fibre through J. Finally, the horizontal space $\mathcal{H}_J$ and the vertical space $\mathcal{V}_J$ are declared to be orthogonal. For $J \in \mathcal{Z}$ and $X \in T_{\pi(J)}M$, we denote the horizontal lift $(\pi_*|\mathcal{H}_J)^{-1}(X)$ of X by X^h_J. Then for $X, Y \in T_{\pi(J)}M$ and $V, W \in \mathcal{V}_J$

$$h_t(X^h_J + V, Y^h_J + W) = g(X, Y) + tG(V, W). \tag{6}$$

The almost-complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ are both compatible with each metric h_t. Thus, for a fixed t, we have two almost-Hermitian structures on the twistor space which have intensively been studied by many authors.

4.4. Integrability of the Atiyah-Hitchin-Singer and Eells-Salamon almost-complex structures. It is a result of Eells and Salamon [34] that the almost-complex structure $\mathcal{J}_2$ is never integrable, so it does not come from complex coordinates. Nevertheless, $\mathcal{J}_2$ is very useful for constructing harmonic maps. The integrability condition for $\mathcal{J}_1$ has been found by Atiyah, Hitchin, and Singer [4]. To state their result, we first recall a well-known curvature decomposition.

Let $R(X,Y,Z) = \nabla_{[X,Y]}Z - \nabla_X\nabla_Y Z + \nabla_Y\nabla_Y Z$ be the curvature tensor of the Levi-Civita connection ∇ of (M,g). The curvature operator $\mathcal{R}$ corresponding to the curvature tensor is the endomorphism of $\Lambda^2 TM$ defined by

$$g(\mathcal{R}(X\wedge Y), Z\wedge U) = g(R(X,Y)Z,U), \quad X,Y,Z,U \in TM.$$

Denote by s the scalar curvature of the manifold (M,g) and let $\rho : T_pM \to T_pM$ be its Ricci operator defined by $g(\rho(X),Y) = \mathrm{Ricci}(X,Y)$, $X,Y \in TM$. It is well-known that the curvature operator $\mathcal{R}$ decomposes as (cf. e.g. [6, Sec. 1 G], [61])

$$\mathcal{R} = \frac{2s}{n(n-1)} Id + \mathcal{B} + \mathcal{W}$$

where the operator $\mathcal{W}$ corresponds to the Weyl conformal tensor W, while $\mathcal{B}$ corresponds to the traceless Ricci tensor and is defined by

$$\mathcal{B}(X\wedge Y) = \frac{2}{n-2}[X\wedge\rho(Y) + \rho(X)\wedge Y - \frac{2s}{2n+1}X\wedge Y], \quad X,Y \in T_pM.$$

Note that this differs from [6] by a factor $\frac{1}{2}$ (the curvature operator therein is $\frac{1}{2}\mathcal{R}$) because of the factor $1/2$ in our definition of the induced metric on $\Lambda^2 TM$. The decomposition of $\mathcal{R}$ is known as the Ricci decomposition.

The Riemannian manifold (M,g) is Einstein exactly when $\mathcal{B} = 0$. It is locally conformally flat if and only if $\mathcal{W} = 0$. The manifold is of constant sectional curvature c exactly when $\mathcal{B} = \mathcal{W} = 0$ and $s = n(n-1)c$.

Suppose that M is oriented and of dimension 4. In this case, the operator $\mathcal{B}$ sends $\Lambda^2_{\pm}TM$ into $\Lambda^2_{\mp}TM$. Moreover, the Weyl operator has an additional decomposition $\mathcal{W} = \mathcal{W}_+ + \mathcal{W}_-$, where $W_{\pm}$ is the restriction of $\mathcal{W}$ to $\Lambda^2_{\pm}TM$ and zero on $\Lambda^2_{\mp}TM$, i.e. $\mathcal{W}_{\pm} = \frac{1}{2}(\mathcal{W} \pm *\mathcal{W})$ where $*$ is the Hodge star operator on $\Lambda^2 TM$. The manifold (M,g) is called self-dual (anti-self-dual) if $\mathcal{W}_- = 0$ (resp. $\mathcal{W}_+ = 0$). Let $\widetilde{g} = e^f g$ be a Riemannian metric conformal to g and $\widetilde{W}$ its Weyl tensor. Then we have $\widetilde{g} = e^{2f}g$ for the induced metrics on $\Lambda^2 TM$. Hence $\widetilde{\mathcal{W}} = e^{-f}\mathcal{W}$ since $\widetilde{W}(X,Y)Z = W(X,Y)Z$ for all $X,Y,Z \in TM$. Note also that the Hodge star operator on $\Lambda^2 TM$ does not change when the metric on M is replaced by a conformal metric. Therefore the self-duality (anti-self-duality) condition is invariant under conformal changes of the metric .

In connection with the last observation, note that the Atiyah-Hitchin-Singer almost complex structure $\mathcal{J}_1$ is invariant under conformal changes of the metric of M, while the Eells-Salamon one $\mathcal{J}_2$ is not, see Sec. 9.

Note also that changing the orientation of M interchanges the roles of $\Lambda^2_- TM$ and $\Lambda^2_+ TM$, hence the roles of $\mathcal{W}_-$ and $\mathcal{W}_+$.

The integrability condition for the almost-complex structure $\mathcal{J}_1$ on $\mathcal{Z}_{\pm}$ has been found by Atiya, Hitchin and Singer.

Theorem 2 ([4]). *Let (M,g) be an oriented Riemannian 4-manifold. The almost-*

complex structure $\mathcal{J}_1$ on the negative twistor space $\mathcal{Z}_-$ (resp. positive twistor space $\mathcal{Z}_+$) of (M,g) is integrable if and only if (M,g) is self-dual (resp. anti-self-dual).

If $\dim M \geq 6$, the almost-complex structure $\mathcal{J}_1$ is integrable if and only if (M,g) is locally conformally flat (see, e.g., [59]). Thus the case of a 4-dimensional base manifold is more interesting in this context.

Examples. Let (M,g) be an oriented Riemannian 4-manifold and ∇ its Levi-Civita connection. In the following examples, $\mathcal{Z}_+$ and $\mathcal{Z}_-$ stand for the positive and negative twistor spaces, respectively. The first five examples are taken from [4] (see also [6, Sec. 13 D]), for the last two we refer to [68].

Suppose that the bundle $\pi : \Lambda^2_+ TM \to M$ admits a global orthonormal frame s_1^+, s_2^+, s_3^+, so that it is isomorphic to the trivial bundle $M \times \mathbb{R}^3$ by the map $\sigma = \sum_{i=1}^{3} x_i s_i^+ \to (\pi(\sigma), x = (x_1, x_2, x_3))$. Clearly such a frame exists if M is parallelizable. The restriction f of the trivialization map $\Lambda^2_+ TM \cong M \times \mathbb{R}^3$ to $\mathcal{Z}_+$ is obviously a (smooth) isomorphism of the bundle $\mathcal{Z}_+$ onto the bundle $M \times S^2$. Denote by $\mathcal{I}_k$ the almost-complex structure on $M \times S^2$ corresponding under f to the structure $\mathcal{J}_k$ on $\mathcal{Z}_+$, $k = 1,2$. If we assume in addition that the frame s_1^+, s_2^+, s_3^+ is parallel, $\nabla s_i^+ = 0$, the structure $\mathcal{I}_k$ has a simple explicit description since each horizontal space $\mathcal{H}_\sigma$ is mapped by f_* onto the subspace $T_{\pi(\sigma)}M$ of $T_{f(\sigma)}(M \times S^2) \cong T_{\pi(\sigma)}M \times T_x S^2$. For $\sigma \in \mathcal{Z}_+$, denote by K_σ the complex structure on $T_{\pi(\sigma)}M$ defined by $g(K_\sigma X, Y) = 2g(\sigma, X \wedge Y)$, $X, Y \in TM$. Also, denote the standard complex structure of S^2 by $\mathcal{I}$. Let $(X,V) \in T_{(p,x)}(M \times S^2)$. Then $\mathcal{I}_k X = K_{f^{-1}(p,x)} X$ and $\mathcal{I}_k V = (-1)^{k+1} \mathcal{I} V$.

Similar considerations hold for $\mathcal{Z}_-$.

Example 20. Consider $M = \mathbb{R}^4$ with its standard flat metric. Set $E_j = \dfrac{\partial}{\partial x_j}$, $j = 1,2,3,4$, and define s_i^+, $i = 1,2,3$, by means of $E_1, \dots, E_4$ via (5). The sections s_1^+, s_2^+, s_3^+ of $\Lambda^2_+ \mathbb{R}^4$ are parallel, hence the twistor space $(\mathcal{Z}_+, \mathcal{J}_k)$ is biholomorphic to $(\mathbb{R}^4 \times S^2, \mathcal{I}_k)$. Note also that the Atiyah-Hitchin-Singer almost-complex structure $\mathcal{J}_1$ is integrable, i.e. it is a complex structure since the base manifold $\mathbb{R}^4$ is flat. Hence $\mathcal{I}_1$ is a complex structure on $\mathbb{R}^4 \times S^2$. Similarly $(\mathcal{Z}_-, \mathcal{J}_k) \cong (\mathbb{R}^4 \times S^2, \mathcal{I}_k)$. Identifying $\mathbb{R}^4$ with $\mathbb{C}^2$, we get another complex structure on $\mathbb{R}^4 \times S^2$, the product one. The complex structure $\mathcal{I}_1$ is different from the product complex structure on $\mathbb{R}^4 \times S^2$.

Example 21. Let $M = \mathbb{R}^4/\Gamma$ be a torus, Γ being a lattice. Consider the torus M with the flat metric induced by the metric of $\mathbb{R}^4$. The vector fields E_j on $\mathbb{R}^4$ descend to a global orthonormal frame on the torus. The corresponding sections s_1^+, s_2^+, s_3^+ of $\Lambda^2_+ TM$ are parallel, so the twistor space $(\mathcal{Z}_+, \mathcal{J}_k)$ of the torus is biholomorphic to $((\mathbb{R}^4/\Gamma) \times S^2, \mathcal{I}_k)$. The almost-complex structure $\mathcal{I}_1$ is integrable. Similarly for the negative twistor space. Note that the structure $\mathcal{I}_1$ has earlier been considered by A. Blanchard [9, Exemple de fibration au sense de la proposition I.2.2].

Example 22. The product metric of $S^1 \times S^3$ is locally conformally flat. The manifold $S^1 \times S^3$ is parallelizable, hence its positive and negative spaces are isomorphic to $S^1 \times S^3 \times S^2$. Other descriptions of the twistor space of $S^1 \times S^3$ can be seen in [19, 21] and in [43, Sec. 8].

Example 23. In order to deal with the twistor space of the unit 4-sphere $M = S^4$, we identify S^4 with the quaternionic projective line $\mathbb{HP}^1$. This can be done in the same way as the 2-sphere S^2 is identified with the complex projective line $\mathbb{CP}^1$. Write the quaternions as $z_1 + z_2 j$ with $z_1, z_2 \in \mathbb{C}$. The twistor space $\mathcal{Z}_+$ of S^4 can be identified with the complex projective space $\mathbb{CP}^3$, the projection map $\pi : \mathcal{Z}_+ \cong \mathbb{CP}^3 \to S^4 \cong \mathbb{HP}^1$ being given in homogeneous coordinates by $[z_1, z_2, z_3, z_4] \to [z_1 + z_2 j, z_3 + z_4 j]$. Similarly for the negative twistor space. The sphere S^4 is locally conformally flat, so the Atiyah-Hitchin-Singer almost-complex structure on both twistor spaces $\mathcal{Z}_+$ and $\mathcal{Z}_-$ of S^4 is integrable. It coincides with the complex structure of $\mathbb{CP}^3$. We refer to [70, Sec. 5.12] or [25] for details.

Example 24. Consider the complex projective space $M = \mathbb{CP}^2$ as a smooth manifold endowed with the Fubini-Study metric and the orientation induced by the complex structure. The twistor spaces $\mathcal{Z}_\pm$ of this manifold can be identified as smooth manifolds with the complex flag manifold $F_3 = F_{1,2}$. Recall that the points of F_3 are pairs (l, m) of a complex line l and a complex plane m in $\mathbb{C}^3$ such that $l \subset m$. In this setting, the projection map $\pi : \mathcal{Z}_\pm \cong F_3 \to \mathbb{CP}^2$ is $(l, m) \to l^\perp \cap m$, where $l^\perp$ is the orthogonal complement of l in $\mathbb{C}^3$ with respect to the standard Hermitian metric of $\mathbb{C}^3$. Note that this map is neither holomorphic, nor anti-holomorphic. The Riemannian manifold $\mathbb{CP}^2$ is self-dual, but not anti-self-dual. Hence the Atiyah-Hitchin-Singer almost-complex structure $\mathcal{J}_1$ is integrable on the negative twistor space $\mathcal{Z}_-$, but not on the positive one. Under the diffeomeorphism $\mathcal{Z}_- \cong F_3$, the structure $\mathcal{J}_1$ coincides with the complex structure of F_3. Details can be found in [25], for example.

Remark 2. Example 24 is one of the reasons that many authors prefer to consider the negative twistor space $\mathcal{Z}_-$. But, as we shall see, in some cases it is more appropriate to deal with the positive twistor space $\mathcal{Z}_+$.

Example 25. Consider the unit four-ball B^4 with its metric of constant negative sectional curvature. Then the positive and the negative twistor spaces are isomorphic to

$$\{[q_1, q_2] \in \mathbb{CP}^3 : |q_1| > |q_2|\},$$

the projection map being given by

$$[q_1, q_2] \to \overline{q}_1 q_2 |q_1|^{-1}; \quad q_1, q_2 \in \mathbb{H} \cong \mathbb{C}^2.$$

Example 26. Let $\mathbb{C}B_2$ be the unit four-ball with the Kähler metric of constant negative holomorphic sectional curvature, the Bergman metric. This metric is self-dual, but not anti-self-dual. Set

$$\mathbb{A}_2 = \mathbb{CP}^2 \setminus \mathbb{C}B_2 = \{[z] \in \mathbb{CP}^2 : -|z_0|^2 + |z_1|^2 + |z_2|^2 > 0\}.$$

Then the positive and negative twistor spaces of $\mathbb{C}B_2$ are isomorphic to

$$\{([z], [w]) \in \mathbb{A}_2 \times \mathbb{A}_2^* : -z_0 w_0 + z_1 w_1 + z_2 w_2 = 0\}.$$

Note that any Kähler metric of constant holomorphic sectional curvature on a complex surface is self-dual; it is anti-self-dual exactly when its scalar curvature vanishes.

5. The energy functional on maps into twistor spaces. Let (M, g) and (N, h) be Riemannian manifolds. If D is a relatively compact open subset of M, the energy functional assigns to every map $\varphi : M \to N$ the integral

$$E_D(\varphi) = \int_D \|\varphi_*\|^2_{g,h} vol$$

where, at a point $p \in M$, $\|\varphi_*\|_{g,h}$ is the Hilbert-Schmidt norm of the linear map $\varphi_{*p} : T_pM \to T_{\varphi(p)}N$ taken with respect to g and h.

Example 27. Let $M = \mathbb{R}^3$ and $N = \mathbb{R}$ be equipped with their standard metrics. The energy of a smooth function $\varphi : \mathbb{R}^3 \to \mathbb{R}$ is the classical Dirichlet integral

$$E_D(\varphi) = \int_D [(\frac{\partial\varphi}{\partial x_1})^2 + (\frac{\partial\varphi}{\partial x_2})^2 + (\frac{\partial\varphi}{\partial x_3})^2] dx_1 dx_2 dx_3.$$

A variation of a map $\varphi : M \to N$ is a 1-parameter family $\{\varphi_s\}$ of smooth maps $\varphi_s : M \to N$ with $\varphi_0 = \varphi$, smoothly depending on s. In other words, this is a smooth map $\Phi : M \times (-\varepsilon, \varepsilon) \to N$ such that $\Phi(p, 0) = \varphi(p)$ for all $p \in M$; in these terms, $\varphi_s(p) = \Phi(p, s)$.

Recall that a smooth map $\varphi : (M, g) \to (N, h)$ is called harmonic if it is a critical point of the energy functional for all relatively compact open subset of M and all variations of the map φ.

Let J be a compatible almost-complex structure on a Riemannian manifold (M, g) with twistor space $(\mathcal{Z}, h_t)$. The almost-complex structure J defines a section $J : M \ni p \to J_p \in \mathcal{Z}$ of the twistor bundle $\pi : \mathcal{Z} \to M$. Conversely, any section of this bundle is a compatible almost-complex structure on (M, g). We shall consider two classes of variations of the map J: (1) All φ_s are sections of the twistor bundle $\mathcal{Z}$ (i.e. compatible almost-complex structures); (2) φ_s are just smooth maps $M \to \mathcal{Z}$, not necessarily sections. Variations of class (1), hence of class (2), always exist, see, for example, [22]. Fix one of the classes (1) or (2). The map J is a critical point of the energy functional if for every variation $\{\varphi_s\}$ of J in the given class, and for every relatively compact open set D in M

$$\frac{d}{ds} E_D(\varphi_s)|_{s=0} = 0.$$

Let ∇ be the Levi-Civita connection of (M, g). If K is a compatible almost-complex structure on (M, g), we have $K_*X = X^h + \nabla_X K$ for every $X \in TM$ where X^h is the horizontal lift of X and $\nabla_X K$ is the vertical part of K_*X. Then $\|K_*\|^2_{g,h_t} = t\|\nabla K\|^2_g + \dim M$. Hence

$$E_D(K) = t \int_D \|\nabla K\|^2_g vol + (\dim M) vol(D).$$

Therefore if J is a critical point of the energy functional for one of metrics h_t, it is a critical point for all of these metrics. Another obvious consequence is that the Kähler structures ($\nabla K = 0$) provide the absolute minimum of the energy functional on the space of sections of $\mathcal{Z}$.

G. Bor, L. Hernández-Lamoneda and M. Salvai [10] have given sufficient conditions for an almost-Hermitian structure to minimize the energy functional on the space of sections of the twistor bundle. We state their main result in a slightly different form.

Theorem 3. ([10]) *Let (M, g) be a compact Riemannian manifold and let J be a compatible almost-complex structure on it. Suppose that*

(1) $\dim M = 4$, *the manifold (M, g) is anti-self-dual and the almost-Hermitian structure (g, J) is Hermitian or almost-Kähler,*

or

(2) $\dim M = 6$ *and the almost-Hermitian structure* (g, J) *is nearly-Kähler, i.e.* $(\nabla_X J)(X) = 0$, $X \in TM$, *where* ∇ *is the Levi-Civita connection of* g,
or

(3) $\dim M \geq 6$, (M, g) *is locally conformally flat and the almost-Hermitian structure* (g, J) *is locally conformally Kähler.*

Then the almost-complex structure J *is an energy minimizer.*

Examples of non-Kähler minimizers of the energy functional on the space of sections of the twistor bundle.

Example 28 ([10])**.** C. LeBrun [50] has constructed anti-self-dual Hermitian structures on the blow-ups $(S^1 \times S^3)\sharp\, n\overline{\mathbb{CP}^2}$ of the Hopf surface $S^1 \times S^3$. Blow-ups do not affect the first Betti number, so any blow up of the Hopf surface has $b_1 = 1$, hence it does not admit a Kähler metric.

Example 29. I. Kim [46] has shown the existence of anti-self-dual non-Kähler almost-Kähler structures on $\mathbb{CP}^2\sharp\, n\overline{\mathbb{CP}^2}$, $n \geq 11$, $(S^2 \times \Sigma)\sharp\, n\overline{\mathbb{CP}^2}$, Σ being a Riemann surface of genus ≥ 2, and $(S^2 \times T^2)\sharp\, n\overline{\mathbb{CP}^2}$, $n \geq 6$.

Example 30. The metric on S^6 is locally conformally flat and the almost-Hermitian structure of S^6 is nearly-Kähler.

Example 31. The Calabi-Eckmann Hermitian structure on $S^1 \times S^{2m+1}$ is locally conformally flat and locally conformally Kähler [32, 66].

II. Harmonic and pseudo-harmonic almost-Hermitian structures

6. "Distinguished" almost-Hermitian structures. It is well known that if a Riemannian manifold (M, g) admits an almost-Hermitian structure, it possesses many such structures. In [22, 24], for example, this is shown by considering an almost-Hermitian structure on (M, g) as a section of the twistor bundle $\pi : \mathcal{Z} \to M$. Thus, it is natural to look for "reasonable" criteria that distinguish some of these structures.

E. Calabi and H. Gluck [15] have proposed to single out as "the best ones" those almost-Hermitian structures J on (M, g) whose image $J(M)$ in the twistor space $\mathcal{Z}$ of (M, g) is of minimal volume with respect to metric h_1 on $\mathcal{Z}$ defined by (6). They have proved that the almost-Hermitian structure on the 6-sphere S^6 considered above can be characterized by that property.

Motivated by the harmonic map theory, C. Wood [71, 72] has suggested to consider as "optimal" those almost-Hermitian structures $J : (M, g) \to (\mathcal{Z}, h_1)$ that are critical points of the energy functional under variations through sections of $\mathcal{Z}$. In general, these critical points are not harmonic maps, but, by analogy, they are referred to as "harmonic almost-complex structures" in [72]; they are also called "harmonic sections" in [71].

Let us also note that these two approaches are not equivalent, but in both of them, the twistor space plays an essential role.

Forgetting the bundle structure of $\mathcal{Z}$, we can consider the almost-Hermitian structures that are critical points of the energy functional under variations through arbitrary maps $M \to \mathcal{Z}$, not just sections. These structures are genuine harmonic maps from (M, g) into $(\mathcal{Z}, h_1)$, and we refer to [33] for basic facts about such maps. This point of view has been taken in [24, 26] where the problem of when an almost-Hermitian structure on a Riemannian four-manifold is a harmonic map from the manifold into its twistor space is discussed. Also, in [23, 22], the problem of when the Atiyah-Hitchin-Singer or Eells-Salamon almost-complex structure is a harmonic map from the twistor space $(\mathcal{Z}, h_t)$ of a Riemannian 4-manifold into the twistor space of $\mathcal{Z}$ is explored.

Two conformal metrics have the same twistor spaces, thus it is natural to consider the twistor spaces in the context of conformal geometry, see, for example, [4, 37]. The harmonic map technique has a useful extension in the conformal geometry due to G. Kokarev [49] who has introduced the notion of a pseudo-harmonic map. The pseudo-harmonic maps share some important properties with the harmonic maps [49], to mention here only the unique continuation one. So, it is natural to ask when an almost-Hermitian structure on a conformal manifolds determines a pseudo-harmonic map from the manifold into its twistor space. This problem is studied in [27].

6.1. Harmonic sections of twistor spaces. The Euler-Lagrange equation for the critical points of the energy functional under variations through sections of the twistor space (harmonic sections) has been found by C. Wood [71, 72]. A compatible almost-complex structure J on a Riemannian manifold (M, g) is a harmonic section if and only if

$$[J, \nabla^* \nabla J] = 0, \tag{7}$$

where the bracket means "commutator", ∇ is the Levi-Civita connection of (M, g) and ∇^* is its formally adjoint operator.

Clearly any Kähler structure ($\nabla J{=}0$) is a harmonic section; in fact, as we have mentioned, the Kähler structures are absolute minima of the energy functional on the space of sections of the twistor bundle.

Examples of non-Kähler harmonic sections ([72])**.**

Example 32. The almost-Hermitian structure on S^6 is a harmonic section. More generally, every nearly-Kähler structure is so.

Example 33. The Abena-Thurston almost-Kähler structure is a harmonic section.

Example 34. The Calabi-Eckman Hermitian structure on $S^{2n+1} \times S^{2m+1}$ is a non-Kähler harmonic section.

Example 35. The Nagano-Sasaki almost-Kähler structure on the tangent bundle TM of a Riemmanian manifold (M, g) is a non-Kähler harmonic section if and only if the manifold is non-flat and has harmonic curvature, i.e. $\delta R = 0$; if $\dim M = 2$ this structure is a harmonic section if and only if M is of constant curvature. Note that the identity $\delta R = 0$ implies that the scalar curvature of M is constant. Every Einstein manifold provides an example of manifold with harmonic curvature.

If $\Omega(X, Y) = g(JX, Y)$ is the fundamental 2-form of the almost-Hermitian structure (J, g), equation (7) is equivalent to

$$(\nabla^* \nabla\, \Omega)(X, Y) = (\nabla^* \nabla\, \Omega)(JX, JY), \quad X, Y \in TM. \tag{8}$$

Note that for the rough Laplacian $\nabla^* \nabla$ we have $\nabla^* \nabla\, \Omega = -\operatorname{Trace} \nabla^2 \Omega$, see, for example, [6].

Let $\widehat{\Omega}$ be the section of $\Lambda^2 TM$ corresponding to the 2-form Ω under the isomorphism $\Lambda^2 TM \cong \Lambda^2 T^* M$ determined by the metric on $\Lambda^2 TM$ defined by (3). Thus, $g(\widehat{\Omega}, X \wedge Y) = \Omega(X, Y)$, and if $E_1, \ldots, E_m, JE_1, \ldots, JE_m$ is an orthonormal frame of TM, $\widehat{\Omega} =$

$2\sum_{k=1}^{m} E_k \wedge JE_k$. Denote by $\mathcal{R}(\Omega)$ the 2-form corresponding to $\mathcal{R}(\widehat{\Omega})$. Then we have $\mathcal{R}(\Omega)(X,Y) = g(\mathcal{R}(\widehat{\Omega}), X \wedge Y)$.

Using the Weitzenböck formula, one can prove the following useful observation.

Lemma 1 ([20]). *A compatible almost-complex structure J on a Riemannian manifold (M,g) is a harmonic section if and only if*

(9) $$\Delta\Omega(X,Y) - \Delta\Omega(JX,JY) = \mathcal{R}(\Omega)(X,Y) - \mathcal{R}(\Omega)(JX,JY), \quad X,Y \in TM,$$

where Δ is the Laplace-de Rham operator of (M,g).

7. Harmonicity of the the Atiyah-Hitchin-Singer and Eells-Salamon almost-complex structures. Lemma 1 has been used in [20] to prove the following statement.

Theorem 4 ([20])**.** *Let (M,g) be an oriented Riemannian 4-manifold and let $(\mathcal{Z}_-,h_t)$ be its negative twistor space. Then:*

(i) The Atiyah-Hitchin-Singer almost-complex structure $\mathcal{J}_1$ on $(\mathcal{Z}_-,h_t)$ is a harmonic section if and only if (M,g) is a self-dual manifold.

(ii) The Eells-Salamon almost-complex structure $\mathcal{J}_2$ on $(\mathcal{Z}_-,h_t)$ is a harmonic section if and only if (M,g) is a self-dual manifold with constant scalar curvature.

This theorem can be considered as a variational interpretation of the self-duality condition.

Example 36. Changing the orientation of M gives the corresponding result for the positive twistor space $(\mathcal{Z}_+,h_t)$. Recall that the Fubini-Study metric on $\mathbb{CP}^2$ is self-dual, but not anti-self-dual. Thus, the structure $\mathcal{J}_1$ on the negative twistor space of $\mathbb{CP}^2$ is a harmonic section, while on the positive twistor space it is not. A similar remark holds for $\mathcal{J}_2$.

Now, consider again a compatible almost-complex structure J on a Riemannian manifold (M,g) as a map from the manifold into its twistor space $(\mathcal{Z},h_t)$. If this map is a critical point of the energy functional under variations of arbitrary maps $M \to \mathcal{Z}$, not just sections, it is a bona fide harmonic map. Let us recall the corresponding Euler-Lagrange equation it this case. Let $J^*T\mathcal{Z}$ be the pull-back of the tangent bundle $T\mathcal{Z}$ under the map $J : M \to \mathcal{Z}$. Then we can consider the differential $J_* : TM \to T\mathcal{Z}$ as a section of the vector bundle $\mathrm{Hom}(TM, J^*T\mathcal{Z}) \to M$. Let D^* be the connection on $J^*T\mathcal{Z}$ induced by the Levi-Civita connection D on $T\mathcal{Z}$. The Levi-Civita connection ∇ on TM and the connection D^* on $J^*T\mathcal{Z}$ induce a connection $\widetilde{\nabla}$ on $\mathrm{Hom}(TM, J^*T\mathcal{Z})$. Recall that the second fundamental form of the map J is, by definition, the bilinear form

$$II_J(X,Y) = (\widetilde{\nabla}_X J_*)(Y), \quad X,Y \in TM.$$

For tangent vectors X,Y of M at a point p, the value of the second fundamental form lies in the fibre of $J^*T\mathcal{Z}$ at p, the latter being isomorphic to $T_{J(p)}\mathcal{Z}$. Note also that the form $II_J(X,Y)$ is symmetric since the connections D and ∇ are torsion-free.

The map $J : (M,g) \to (\mathcal{Z},h_t)$ is harmonic if and only if

$$\mathrm{Trace}_g \widetilde{\nabla} J_* = 0.$$

Recall also that the map $J : (M,g) \to (\mathcal{Z},h_t)$ is totally geodesic (i.e. maps geodesics to geodesics) exactly when $\widetilde{\nabla} J_* = 0$.

By the Vilms theorem (see, for example, [6, Theorem 9.59]), the projection map $\pi : (\mathcal{Z},h_t) \to (M,g)$ is a Riemannian submersion with totally geodesic fibres, a property which can also be proved by an easy direct computation. Taking into account this fact, one can compute the Levi-Civita connection of the metric h_t in terms of the Levi-Civita connection and the curvature of the base manifold (M,g). Then one can derive the following formula for the second fundamental form of J which has a crucial role in studying harmonicity of the map $J:(M,g)\to(\mathcal{Z},h_t)$.

Proposition 1 ([22, 23, 24]). *For every $X,Y\in T_pM$,*

$$\widetilde{\nabla}J_*(X,Y)=\frac{1}{2}\mathcal{V}(\nabla^2_{XY}J+\nabla^2_{YX}J)$$
$$-t[(R((J\circ\nabla_X J)^\wedge)Y)^h_{J(p)}+(R((J\circ\nabla_Y J)^\wedge)X)^h_{J(p)}],$$

where $\mathcal{V}$ means "the vertical component" and $\nabla^2_{XY}J=\nabla_X\nabla_Y J-\nabla_{\nabla_X Y}J$ is the second covariant derivative of J.

Remark 3. The coefficient of the curvature terms (the horizontal part of $\widetilde{\nabla}J_*(X,Y)$) differs from that in [22, 23] by a factor $\dfrac{2}{n}$, $n=\dim M$, since the inner product on the endomorphisms of TM used therein is $\dfrac{1}{n}\operatorname{Trace}A^tB$, while it is $\dfrac{1}{2}\operatorname{Trace}A^tB$ here (as well as in [24]).

Remark 4. For every $I\in\mathcal{Z}$, we have the orthogonal decomposition

$$A(T_{\pi(I)}M)=\mathcal{V}_I\oplus\{S\in A(T_{\pi(I)}M):\ IS-SI=0\}.$$

This implies that the Euler-Lagrange equation $[J,\nabla^*\nabla J]=0$ is equivalent to the condition that the vertical part of $\nabla^*\nabla J=-\operatorname{Trace}\nabla^2 J$ vanishes. Thus, by Proposition 1, J is a harmonic section if and only if

$$\mathcal{V}\ \operatorname{Trace}\widetilde{\nabla}J_*=0. \tag{10}$$

Proposition 1 implies immediately the following.

Corollary 1. *If (M,g,J) is Kähler, the map $J:(M,g)\to(\mathcal{Z},h_t)$ is a totally geodesic isometric imbedding.*

Proposition 1 has been used in [23] to prove the following.

Theorem 5. ([22, 23]) *Each of the Atiyah-Hitchin-Singer and Eells-Salamon almost-complex structures on the negative twistor space $\mathcal{Z}_-$ of an oriented Riemannian four-manifold (M,g) determines a harmonic map from $(\mathcal{Z}_-,h_t)$ into its twistor space if and only if (M,g) is either self-dual and Einstein, or is locally the product of an open interval in $\mathbb{R}$ and a 3-dimensional Riemannian manifold of constant curvature.*

Remark 5. Every manifold that is locally the product of an open interval in $\mathbb{R}$ and a 3-dimensional Riemannian manifold of constant curvature c is locally conformally flat with constant scalar curvature $6c$. It is not Einstein unless $c=0$, i.e. Ricci flat.

Remark 6. According to Theorems 4 and 5, the conditions under which $\mathcal{J}_1$ or $\mathcal{J}_2$ is a harmonic section or a harmonic map do not depend on the parameter t of the metric h_t. Taking certain special values of t, we can obtain a metric h_t with nice properties (cf., for example, [17, 18, 51]).

8. Almost-Hermitian structures on 4-manifolds that are harmonic maps from the manifold into its twistor space. Recall that the metrics h_t on the twistor space of a manifold M depends on the choice of a connection on M. In this section, we consider the following two cases: (i) M is a Riemannian manifold endowed with its Levi-

Civita connection; (ii) M is a Riemannian manifold endowed with a metric connection with totally skew-symmetric torsion.

Let J be a compatible almost-complex structure on M. Considering M with the orientation determined by J, the map from M into the twistor space determined by J takes its values in the positive twistor space.

In what follows, the positive twistor space $\mathcal{Z}_+$ is simply called "the twistor space" of (M,g) and is denoted by $\mathcal{Z}$.

In the next two paragraphs, (M,g) denotes a four-dimensional Riemannian manifold with a compatible almost-complex structure J. The twistor space is considered as the unit sphere subbundle of Λ^2_+TM. Under this interpretation of the twistor space, the map from M into $\mathcal{Z}$ determined by J is the section $\mathfrak{J}=J^\wedge$ of Λ^2_+TM where $J^\wedge$ is defined via (4). Note also that for the curvature terms in Proposition 1 we have $(J\circ\nabla_X J)^\wedge = J^\wedge\times\nabla_X J^\wedge$, the vector-cross product in the fibres of Λ^2_+TM.

As is well-known, in dimension four, there are three basic classes in the Gray-Hervella classification [39] of almost-Hermitian structures: Hermitian, almost-Kähler (symplectic) and Kähler structures. If (g,J) is Kähler, the map $J:(M,g)\to(\mathcal{Z},h_t)$ is a totally geodesic isometric imbedding. So, we shall discuss the cases when the structure (g,J) is Hermitian, i.e. J is integrable and when (g,J) is almost-Kähler.

8.1. The base manifold is endowed with the Levi-Civita connection. Denote the Ricci tensor of (M,g) by ρ and let ρ^* be the $*$-Ricci tensor of the almost-Hermitian manifold (M,g,J). Recall that the latter is defined by

$$\rho^*(X,Y)=\mathrm{Trace}\{Z\to R(JZ,X)JY\}.$$

Note that $\rho^*(X,Y)=\rho^*(JY,JX)$ by the identity $g(R(JZ,X)JY,Z)=$
$-\,g(R(Z,JY)J(JX),JZ)$. Equivalently, $\rho^*(X,JX)=0$.

The case of integrable J. Suppose that the almost-complex structure J is integrable. Denote by B the vector field on M dual to the Lee form $\theta=-\delta\Omega\circ J$ with respect to the metric g.

Using Proposition 1, one can prove the following.

Theorem 6 ([24]). *Suppose that the almost-complex structure J is integrable. Then the map $\mathfrak{J}:(M,g)\to(\mathcal{Z},h_t)$ is harmonic if and only if $d\theta$ is a $(1,1)$-form w.r.t. J and $\rho(X,B)=\rho^*(X,B)$ for every $X\in TM$.*

The proof of this result and (10) give

Corollary 2. *The map $\mathfrak{J}:(M,g)\to(\mathcal{Z},h_t)$ representing an integrable almost-Hermitian structure J on (M,g) is a harmonic section if and only if the 2-form $d\theta$ is of type $(1,1)$ w.r.t. J.*

Remark 7. The 2-form $d\theta$ of a Hermitian surface (M,g,J) is of type $(1,1)$ if and only if the $\star$-Ricci tensor ρ^* is symmetric ([24]).

Remark 8. In higher dimensions, Theorem 6 holds for locally conformally Kähler manifolds (M,g,J) with $\dim M\geq 6$. This follows from Theorem 10 below.

The case of symplectic *J*. Recall that an almost-Hermitian manifold is called almost-Kähler (or symplectic) if its fundamental 2-form is closed.

The Nijenhuis tensor $N(X,Y)$ of J is skew-symmetric, so it determines a section of the bundle $\mathrm{Hom}(\Lambda^2TM,TM)$ denoted again by N. Denote by Λ^2_0TM the subbundle of Λ^2_+TM orthogonal to $\mathfrak{J}$ (thus $\Lambda^2_0T_pM=\mathcal{V}_{\mathfrak{J}(p)}$).

Under this notation we have the following.

Theorem 7 ([24]). *Let (M, g, J) be an almost-Kähler 4-manifold. Then the map $\mathfrak{J} : (M, g) \to (\mathcal{Z}, h_t)$ is harmonic if and only if the $*$-Ricci tensor ρ^* is symmetric and*

$$\mathrm{Trace}\{\Lambda^2_0 TM \ni a \to R(\tau)(N(a))\} = 0.$$

The proof makes use of Proposition 1 and the Weitzenböck formula.

Example 37 ([24]). **Primary Kodaira surfaces.** Every primary Kodaira surface M can be obtained in the following way [48, p.787]. Let $\varphi_k(z, w)$ be the affine transformations of $\mathbb{C}^2$ given by

$$\varphi_k(z, w) = (z + a_k, w + \overline{a}_k z + b_k),$$

where a_k, b_k, $k = 1, 2, 3, 4$, are complex numbers such that

$$a_1 = a_2 = 0, \quad Im(a_3\overline{a}_4) = mb_1 \neq 0, \quad b_2 \neq 0$$

for some integer $m > 0$. The maps φ_k generate a group G of transformations acting freely and properly discontinuously on $\mathbb{C}^2$, and M is the quotient space $\mathbb{C}^2/G$. The complex two-dimensional manifold $M = \mathbb{C}^2/G$ is compact.

It is well-known that M can also be described as the quotient of $\mathbb{C}^2$ endowed with a group structure by a discrete subgroup Γ. The multiplication on $\mathbb{C}^2$ is defined by

$$(a, b).(z, w) = (z + a, w + \overline{a}z + b), \quad (a, b), (z, w) \in \mathbb{C}^2,$$

and Γ is the subgroup generated by (a_k, b_k), $k = 1, \dots, 4$ (see, for example, [11]). Every Γ-left-invariant object on $\mathbb{C}^2$ descends to a globally defined object on M and both of them will be denoted by the same symbol. Thus, instead on M, we shall deal with Γ-left-invariant objects on $\mathbb{C}^2$.

As in [21], take a frame of left-invariant (under the group $\mathbb{C}^2$) vector fields $A_1, \dots, A_4$ such that

$$[A_1, A_2] = -2A_4, \quad [A_i, A_j] = 0 \ \text{ otherwise.} \tag{11}$$

These identities are satisfied, for example, by the following left-invariant frame

$$A_1 = -\frac{\partial}{\partial x} - x\frac{\partial}{\partial u} + y\frac{\partial}{\partial v}, \quad A_2 = \frac{\partial}{\partial y} + y\frac{\partial}{\partial u} + x\frac{\partial}{\partial v}, \quad A_3 = \frac{\partial}{\partial u}, \quad A_4 = \frac{\partial}{\partial v}, \tag{12}$$

where $x + iy = z$, $u + iv = w$.

Let g be the Riemannian metric for which the frame $A_1, \dots, A_4$ is orthonormal.

Complex structures on primary Kodaira surfaces. It is a result by K. Hasegawa [41] that every complex structure on M is induced by a left-invariant complex structure on $\mathbb{C}^2$. It is not hard to see ([52, 21]) that a left-invariant almost-complex structure J on $\mathbb{C}^2$ compatible with the metric g is integrable if and only if it is given by

$$JA_1 = \varepsilon_1 A_2, \quad JA_3 = \varepsilon_2 A_4, \quad \varepsilon_1, \varepsilon_2 = \pm 1. \tag{13}$$

Denote by θ the Lee form of the Hermitian structure (g, J), where $J = J_{\varepsilon_1,\varepsilon_2}$ is defined by means of the latter identities.

Using (11), one can easily find that the non-zero covariant derivatives $\nabla_{A_i} A_j$ are

$$\nabla_{A_1} A_2 = -\nabla_{A_2} A_1 = -A_4, \quad \nabla_{A_1} A_4 = \nabla_{A_4} A_1 = A_2, \quad \nabla_{A_2} A_4 = \nabla_{A_4} A_2 = -A_1.$$

This implies that the Lie form is $\theta(X) = -2\varepsilon_1 g(X, A_3)$. Therefore

$$B = -2\varepsilon_1 A_3, \quad \nabla\theta = 0. \tag{14}$$

Thus, $d\theta = 0$. Set for short $R_{ijk} = R(A_1, A_j)A_k$. Then the non-zero R_{ijk} are

$$R_{121} = -3A_2, \quad R_{122} = 3A_1, \quad R_{141} = A_4,$$
$$R_{144} = -A_1, \quad R_{242} = A_4, \quad R_{244} = -A_2.$$

Set also $\rho_{ij} = \rho(A_i, A_j)$, $\rho^*_{ij} = \rho^*(A_i, A_j)$. Then

$$(15) \qquad \begin{gathered} \rho_{ij} = 0 \quad \textit{and} \quad \rho^*_{ij} = 0 \quad \textit{except} \\ \rho_{11} = \rho_{22} = -2, \quad \rho_{44} = 2, \quad \rho^*_{11} = \rho^*_{22} = -3. \end{gathered}$$

It follows from (14), (15) and Theorem 6 that the map $\mathfrak{J} : (M, g) \to (\mathcal{Z}, h_t)$ corresponding to J is harmonic.

It is easy to give an explicit description of the twistor space $(\mathcal{Z}, h_t)$ ([21]), since $\Lambda^2_+ M$ admits a global orthonormal frame defined by

$$s_1^+ = \varepsilon_1 A_1 \wedge A_2 + \varepsilon_2 A_3 \wedge A_4, \quad s_2^+ = A_1 \wedge A_3 + \varepsilon_1\varepsilon_2 A_4 \wedge A_2,$$
$$s_3^+ = \varepsilon_2 A_1 \wedge A_4 + \varepsilon_1 A_2 \wedge A_3.$$

Then we have a natural diffeomorphism $f : \mathcal{Z} \cong M \times S^2$ defined by $\sum\limits_{k=1}^{3} x_k s_k^+(p)) \to (p, x_1, x_2, x_3)$ under which J becomes the section $p \to (p, 1, 0, 0)$. Denote the pushforward of the metric h_t under f again by h_t. For $x = (x_1, x_2, x_3) \in S^2$, set

$$u_1(x) = \varepsilon_1\varepsilon_2(-x_3, 0, x_1), \quad u_2(x) = \varepsilon_2(x_2, -x_1, 0),$$
$$u_3(x) = 0, \quad u_4(x) = \varepsilon_1(0, x_3, -x_2).$$

The differential f_* sends the horizontal lifts A_i^h $i = 1, \ldots, 4$, at a point $\sigma = \sum\limits_{k=1}^{3} x_k s_k^+(p) \in \mathcal{Z}$ to the vectors $A_i + u_i$ of $TM \oplus TS^2$. Then, if $X, Y \in T_pM$ and $P, Q \in T_xS^2$,

$$h_t(X + P, Y + Q) = g(X, Y) + t \left\langle P - \sum_{i=1}^{4} g(X, A_i)u_i(x), Q - \sum_{j=1}^{4} g(Y, A_j)u_j(x) \right\rangle$$

where $\langle \cdot, \cdot \rangle$ is the standard metric of $\mathbb{R}^3$.

Almost-Kähler structures on primary Kodaira surfaces. Now suppose again that J is a left-invariant almost-complex structure on $\mathbb{C}^2$ compatible with the metric g. Then the almost-Hermitian structure (g, J) is almost-Kähler (symplectic) if and only if J is given by ([52, 21])

$$JA_1 = -\varepsilon_1 \sin\varphi A_3 + \varepsilon_1\varepsilon_2 \cos\varphi A_4, \quad JA_2 = -\cos\varphi A_3 - \varepsilon_2 \sin\varphi A_4,$$
$$JA_3 = \varepsilon_1 \sin\varphi A_1 + \cos\varphi A_2, \quad JA_4 = -\varepsilon_1\varepsilon_2 \cos\varphi A_1 + \varepsilon_2 \sin\varphi A_2,$$
$$\varepsilon_1, \varepsilon_2 = \pm 1, \ \varphi \in [0, 2\pi).$$

The Nijenhuis tensor of such a structure is not zero, so it is not Kählerian. Suppose that $J = J_{\varepsilon_1,\varepsilon_2}$ is determined by these identities and set

$$(16) \qquad \begin{gathered} E_1 = A_1, \quad E_2 = -\varepsilon_1 \sin\varphi A_3 + \varepsilon_1\varepsilon_2 \cos\varphi A_4, \\ E_3 = \cos\varphi A_3 + \varepsilon_2 \sin\varphi A_4, \quad E_4 = A_2. \end{gathered}$$

Then $E_1, \ldots, E_4$ is an orthonormal frame of TM for which $JE_1 = E_2$ and $JE_3 = E_4$.

Computing the curvature components $R(E_i,E_j)E_k$, then $\rho^*(E_i,E_j)$, one can see that the $*$-Ricci tensor is symmetric. Define an orthonormal frame s_l^+, $l=1,2,3$, of Λ^2_+TM by means of $E_1,\dots,E_4$ via (5). Computing the values $N(E_i,E_j)$ of the Nijenhuis tensor, one gets

$$\mathrm{Trace}\{\Lambda^2_0TM \ni a \to R(\tau)(N(a))\} = R(s_2)(N(s_2)) + R(s_3)(N(s_3)) = 0.$$

Thus, by Theorem 7, J defines a harmonic map $(M,g)\to(\mathcal{Z},h_t)$.

As in the preceding case, it is easy to find an explicit description of the twistor space $\mathcal{Z}$ of M and the metric h_t ([21]). The frame $\{s_1^+,s_2^+,s_3^+\}$ gives rise to an obvious diffeomorphism $f:\mathcal{Z}\cong M\times S^2$ under which J becomes the map $p\to(p,1,0,0)$. The differential f_* of this diffeomorphism sends the horizontal lifts E_i^h, $i=1,\dots,4$, to $E_i+u_i\in TM\oplus TS^2$, where

$$u_1(x) = (x_3\varepsilon_1\varepsilon_2\cos\varphi, x_3\varepsilon_2\sin\varphi, -x_1\varepsilon_1\varepsilon_2\cos\varphi - x_2\varepsilon_2\sin\varphi),$$

$$u_2(x) = (x_2\varepsilon_1\varepsilon_2\cos\varphi, -x_1\varepsilon_1\varepsilon_2\cos\varphi, 0), \quad u_3(x) = (x_2\varepsilon_2\sin\varphi, -x_1\varepsilon_2\sin\varphi, 0)$$

$$u_4(x) = (-x_3\varepsilon_2\sin\varphi, x_3\varepsilon_1\varepsilon_2\cos\varphi, x_1\varepsilon_2\sin\varphi - x_2\varepsilon_1\varepsilon_2\cos\varphi).$$

for $x=(x_1,x_2,x_3)\in S^2$. Then, if $X,Y\in T_pM$ and $P,Q\in T_xS^2$,

$$(17)\quad h_t(X+P,Y+Q) = g(X,Y) + t\left\langle P - \sum_{i=1}^4 g(X,E_i)u_i(x), Q - \sum_{j=1}^4 g(Y,E_j)u_j(x)\right\rangle.$$

Example 38 ([24]). **Four-dimensional Lie groups with a non-integrable left-invariant almost-Kähler structure.** By a result of A. Fino [36], for every left-invariant (non-integrable) almost-Kähler structure (g,J) with J-invariant Ricci tensor on a 4-dimensional Lie group M, there exists an orthonormal frame of left-invariant vector fields $E_1,\dots,E_4$ such that

$$JE_1 = E_2, \quad JE_3 = E_4$$

and

$$[E_1,E_2]=0, \quad [E_1,E_3] = sE_1 + \frac{s^2}{t}E_2, \quad [E_1,E_4] = \frac{s^2-t^2}{2t}E_1 - sE_2$$

$$[E_2,E_3] = -tE_1 - sE_2, \quad [E_2,E_4] = -sE_1 - \frac{s^2-t^2}{2t}E_2, \quad [E_3,E_4] = -\frac{s^2+t^2}{t}E_3$$

where s and $t\neq 0$ are real numbers. Using this table one can compute the $*$-Ricci and Nijenhius tensors. The computation shows that J defines a harmonic map by virtue of Theorem 7.

Example 39 ([24]). **Inoue surfaces of type S^0.** Let us recall the construction of these surfaces ([45]). Take a matrix $A\in SL(3,\mathbb{Z})$ with a real eigenvalue $\alpha>1$ and two complex eigenvalues β and $\overline{\beta}$, $\beta\neq\overline{\beta}$. Choose eigenvectors $(a_1,a_2,a_3)\in\mathbb{R}^3$ and $(b_1,b_2,b_3)\in\mathbb{C}^3$ of A corresponding to α and β, respectively. Then the vectors (a_1,a_2,a_3), (b_1,b_2,b_3), $(\overline{b_1},\overline{b_2},\overline{b_3})$ are $\mathbb{C}$-linearly independent. Denote the upper-half plane in $\mathbb{C}$ by $\mathbf{H}$ and let Γ be the group of holomorphic automorphisms of $\mathbf{H}\times\mathbb{C}$ generated by

$$g_o:(w,z)\to(\alpha w,\beta z), \quad g_i:(w,z)\to(w+a_i,z+b_i),\ i=1,2,3.$$

The group Γ acts on $\mathbf{H}\times\mathbb{C}$ freely and properly discontinuously. Then $M=(\mathbf{H}\times\mathbb{C})/\Gamma$ is a compact complex surface known as the Inoue surface of type S^0. It has been shown by F. Tricerri [64] that every such a surface admits a locally conformally Kähler metric g obtained from the Γ-invariant Hermitian metric

$$g = \frac{1}{v^2}(du \otimes du + dv \otimes dv) + v(dx \otimes dx + dy \otimes dy), \quad u + iv \in \mathbf{H}, \quad x + iy \in \mathbb{C}. \tag{18}$$

Instead on M, we work with Γ-invariant objects on $\mathbf{H} \times \mathbb{C}$. Let Ω be the fundamental 2-form of the Hermitian structure (g, J) on $\mathbf{H} \times \mathbb{C}$, J being the standard complex structure. Then

$$d\Omega = \frac{1}{v} dv \wedge \Omega.$$

Hence the Lee form is $\theta = d \ln v$. In particular, $d\theta = 0$, i.e. (g, J) is a locally conformally Kähler structure. Set

$$E_1 = v\frac{\partial}{\partial u}, \quad E_2 = v\frac{\partial}{\partial v}, \quad E_3 = \frac{1}{\sqrt{v}}\frac{\partial}{\partial x}, \quad E_4 = \frac{1}{\sqrt{v}}\frac{\partial}{\partial y}. \tag{19}$$

These are Γ-invariant vector fields constituting an orthonormal frame such that $JE_1 = E_2$, $JE_3 = E_4$, and $E_2 = B$, the dual vector field of θ. In fact, what is important for our considerations is not the specific definition of the vector fields (19), but the property that $E_1, \dots, E_4$ constitute an orthonormal frame of Γ-invariant vector fields such that

$$\begin{gathered} JE_1 = E_2, \quad JE_3 = E_4, \\ [E_1, E_2] = -E_1, \quad [E_2, E_3] = -\frac{1}{2}E_3, \quad [E_2, E_4] = -\frac{1}{2}E_4, \\ \text{all other } [E_i, E_j] = 0. \end{gathered} \tag{20}$$

Then we have the following table for the Levi-Civita connection ∇ of g:

$$\begin{gathered} \nabla_{E_1} E_1 = E_2, \quad \nabla_{E_1} E_2 = -E_1, \\ \nabla_{E_3} E_2 = \frac{1}{2}E_3, \quad \nabla_{E_3} E_3 = -\frac{1}{2}E_2, \quad \nabla_{E_4} E_2 = \frac{1}{2}E_4, \quad \nabla_{E_4} E_4 = -\frac{1}{2}E_2, \end{gathered}$$

and all other $\nabla_{E_i} E_j = 0$. It follows from these identities that $B = E_2$ and

$$\rho(E_k, B) = \rho^*(E_k, B) = 0 \ \text{ for k} = 1, 3, 4, \quad \rho(\mathrm{E}_2, \mathrm{B}) = \frac{1}{2}, \quad \rho^*(\mathrm{E}_2, \mathrm{B}) = -1.$$

By Corollary 2, $\mathfrak{J} : (M, g) \to (\mathcal{Z}, h_t)$ is a harmonic section. However $\mathfrak{J}$ is not a harmonic map by Theorem 6.

8.2. The base manifold is endowed with a metric connection with totally skew-symmetric torsion. Let (N, h), (N', h') be Riemannian manifolds and $f : N \to N'$ a smooth map. If f^*TN' is the pull-back bundle of the bundle TN' under the map f, we can consider the differential $f_* : TN \to TN'$ as a section of the bundle $\mathrm{Hom}(TN, f^*TN')$. Suppose we are given connections D and D' on TN and TN', respectively. Denote by D'^* the connection on f^*TN' induced by the connection D' of TN'. The connections D and D'^* give rise to a connection $\widehat{D}$ on the bundle $\mathrm{Hom}(TN, f^*TN')$. Define a bilinear form on N setting

$$II_f(D, D')(X, Y) = (\widehat{D}_X f_*)(Y), \quad X, Y \in TN.$$

If D and D' are torsion-free, this form is symmetric. Recall that the map f is harmonic if for the Levi-Civita connections ∇ and ∇' of TN and TN',

$$\mathrm{Trace}_h\, II_f(\nabla, \nabla') = 0.$$

Now, suppose that D and D' are metric connections with totally skew-symmetric torsions T and T', i.e. the trilinear form

$$\mathcal{T}(X,Y,Z) = h(T(X,Y),Z), \quad X,Y,Z \in TN,$$

is skew-symmetric, and similarly for $\mathcal{T}'(X,Y,Z) = h'(T'(X,Y),Z)$. Recall that, on a Riemannian manifold, there is a unique metric connection with a given torsion; for an explicit formula see, for example, [40, Sec. 3.5, formula (14)]. Since the torsion 3-forms $\mathcal{T}$ and $\mathcal{T}'$ are skew-symmetric,

$$D_XY = \nabla_XY + \frac{1}{2}T(X,Y), \quad D'_XY = \nabla'_XY + \frac{1}{2}T'(X,Y). \tag{21}$$

A simple computation gives the following.

Proposition 2 ([26])**.** *Let $f : (N,h) \to (N',h')$ be a smooth map of Riemannian manifolds endowed with metric connections D and D' with totally skew-symmetric torsions T and T'. Denote the Levi-Civita connections of (N,h) and (N,h') by ∇ and ∇'. Then*

$$\mathrm{Trace}_h\, II_f(D,D') = \mathrm{Trace}_h\, II_f(\nabla,\nabla').$$

Corollary 3. *A map $f : N \to N'$ is harmonic if and only if*

$$\mathrm{Trace}_h\, II_f(D,D') = 0.$$

This observation suggests that the metric connections with skew-symmetric torsion can be used for studying harmonic maps between Riemannian manifolds endowed with additional structures that are preserved by such connections.

Now, let again (M,g) be an oriented (connected) Riemannian manifold of dimension four, J a compatible almost-complex structure on it and $\mathfrak{J} : M \to \mathcal{Z}$ the map of M into the positive twistor space determined by J.

Let D be a metric connection on M. As we have noted, any metric connection gives rise to a splitting $T\mathcal{Z} = \mathcal{H} \oplus \mathcal{V}$ into horizontal and vertical subbundles which allows one to define a 1-parameter family of Riemannian metrics $h_t = h_t^D$ on the twistor space as in (6).

Denote the Levi-Civita connection of h_t by $\widetilde{D}$. One can compute the second fundamental form

$$II_{\mathfrak{J}}(D,\widetilde{D})$$

along the same lines as in the case of Levi-Civita connections.

Suppose that the metric connection D has totally skew-symmetric torsion T, so $\mathcal{T}(X,Y,Z) = g(T(X,Y),Z)$ is a skew-symmetric 3-form. Let $* : \Lambda^kT^*M \to \Lambda^{4-k}T^*M$, $k = 0,\dots,4$, be the Hodge star operator. Set $\tau = *\mathcal{T}$. Clearly, the 1-form τ uniquely determines the 3-form $\mathcal{T}$, hence the connection D.

Let R^D be the curvature tensor of the connection D. The Ricci tensor and $*$-Ricci tensor of D are defined by

$$\rho_D(X,Y) = \mathrm{Trace}\{Z \to g(R^D(X,Z)Y,Z)\},$$

$$\rho_D^*(X,Y) = \mathrm{Trace}\{Z \to g(R^D(JZ,X)JY,Z)\}.$$

As above, we denote the Lee form of the almost-Hermitian manifold (M,g,J) by θ, so $\theta = -\delta\Omega \circ J$.

Further on, we shall frequently identify TM and T^*M by means of the metric g.

The case of integrable J**.** Using Corollary 3 and the formula obtained for $II_{\mathfrak{J}}(D,\widetilde{D})$ in [26], one can prove the following.

Theorem 8. ([26]) *An integrable almost-Hermitian structure* J *determines a harmonic map* $\mathfrak{J}:(M,g)\to(\mathcal{Z},h_t^D)$ *if and only if* 2*-form*

$$d\theta - d\tau - \imath_B\mathcal{T}$$

is of type $(1,1)$ *with respect to* J *and for every* $X\in TM$

$$\rho_D(X,B)-\rho_D^*(X,B)-\rho_D(X,\tau)+\rho_D^*(X,\tau)=0.$$

Remark 9. Note that for the Levi-Civita connection, i.e. $\tau=0$, this theorem coincides with Theorem 6.

Remark 10. If D is the metric connection with totally skew-symmetric torsion determined by the 1-form $\tau=\theta$, the conditions of this theorem are trivially satisfied. In fact, in this case, D is Bismut-Strominger connection [7, 62], the unique metric connection with totally skew-symmetric torsion preserving the complex structure ($DJ=0$). Actually, the fact that the map $\mathfrak{J}$ is harmonic when we endow M with the Bismut-Strominger connection follows directly from the formula obtained for the second fundamental form of $\mathfrak{J}$. It is natural to ask if there are other metric connections with totally skew-symmetric torsion on M for which $\mathfrak{J}$ is harmonic.

Example 40 ([26])**. Primary Kodaira surfaces.** In the notation above, let J be the complex structure defined by (13). Denote the dual frame of $A_1,\dots,A_4$ by $\alpha_1,\dots,\alpha_4$. Thus $\theta=-2\varepsilon_1\alpha_3$. Any left-invariant 1-form τ is of the form $\tau=a_1\alpha_1+\dots+a_4\alpha_4$, where $a_1,\dots,a_4$ are real constants. The form

$$d\theta-d\tau-\imath_B\mathcal{T}=2(\varepsilon_1-1)a_4\alpha_1\wedge\alpha_2-2\varepsilon_1(a_2\alpha_1\wedge\alpha_4-a_1\alpha_2\wedge\alpha_4)$$

is of type $(1,1)$ w.r.t. J if and only if $a_1=a_2=0$. Let D be the metric connection with totally skew-symmetric torsion T determined by the form τ. It is not hard to obtain the values of $D_{A_i}A_j=\nabla_{A_i}A_j+\frac{1}{2}T(A_i,A_j)$, $i,j=1,\dots,4$, then to compute the components $R^D(A_i,A_j)A_k$ of the curvature tensor of D. Of course, the next step is to compute $\rho_D(A_i,A_j)$ and $\rho_D^*(A_i,A_j)$. As a result, one sees that the identity

$$\rho_D(Z,B)-\rho_D^*(Z,B)-\rho_D(Z,\tau)+\rho_D^*(Z,\tau)=0$$

is satisfied for every $Z\in TM$ if and only if

$$(1-\varepsilon_1)a_4=0,\quad (1-\varepsilon_1)(2-a_3)a_4=0.$$

Clearly, if $\varepsilon_1=1$, the system is satisfied for any a_3 and a_4, and if $\varepsilon_1\neq-1$, the solution of the system is $a_4=0$, a_3-arbitrary. Thus, on a Kodaira surface, there are many metric connections with totally skew-symmetric torsion for which the complex structures J defines a harmonic map from (M,g) into its twistor space $(\mathcal{Z},h_t^D)$.

Example 41 ([26])**. Inoue surfaces of type S^0.** Recall that the map $\mathfrak{J}:(M,g)\to(\mathcal{Z},h_t)$ is not harmonic, if we use the Levi-Civita connection of (M,g) in order to define the metric h_t, but it is harmonic if we use the Bismut-Strominger connection of (M,g,J). In the notation of Sec. 8.1, denote the dual frame of of the frame of vector fields $E_1,\dots,E_4$ by $\eta_1,\dots,\eta_4$. The 1-forms $\eta_1,\dots,\eta_4$ are Γ-invariant. Take a Γ-invariant 1-form $\tau=a_1\eta_1+\dots+a_4\eta_4$ where $a_1,\dots,a_4$ are real constants. Denote by D the connection which is metric w.r.t. g and has totally skew-symmetric torsion determined by τ. Proceeding in a way similar to that in the preceding example, one can see that the map $\mathfrak{J}:(M,g)\to$

$(\mathcal{Z}, h_t^D)$ is harmonic exactly when τ coincides with the Lee form θ, i.e. D is the Bismut-Strominger connection.

The case of symplectic J. As above, let Λ_0^2TM be the subbundle of Λ_+^2TM orthogonal to $\mathbb{R}\mathfrak{J}$. Set

$$\kappa_D(X) = \mathrm{Trace}\{\Lambda_0^2TM \ni a \to g(\mathcal{R}^D(N(a)\wedge X), a)\}, \quad X \in TM.$$

Note that if D is the Levi-Civita connection,

$$g(\mathcal{R}^D(N(a)\wedge X), a) = g(R^D(a)N(a), X),$$

hence

$$\kappa_D(X) = g(\mathrm{Trace}\{\Lambda_0^2TM \ni a \to R^D(a)(N(a))\}, X).$$

Let ρ^* be the $*$-Ricci tensor of (M,g,J) with respect to the Levi-Civita connection.

Theorem 9. ([60]) *Let (M,g,J) be an almost-Kähler 4-manifold. Then the map $\mathfrak{J} : (M,g) \to (\mathcal{Z}, h_t^D)$ is harmonic if and only if for every $X, Y \in TM$ the tensor*

$$2\rho^*(X,Y) + d\tau(X,Y) - \frac{1}{2}g(N(X,Y),\tau)$$

is of type $(1,1)$ with respect to J and

$$\kappa_D(X) + \rho_D(X,\tau) - \rho_D^*(X,\tau) = 0$$

Remark 11. If D is the Levi-Civita connection, then $\tau = 0$ and this theorem coincides with Theorem 7.

Remark 12. By [38, Proposition 4], an almost-Kähler manifold admits a metric connection D with totally skew-symmetric torsion for which the almost-complex structure J is parallel ($DJ = 0$) if and only if the structure is Kählerian. Thus, in this case the Bismut-Strominger connection is the Levi-Civita one.

Example 42 ([60]). **Primary Kodaira surfaces.** As above, we view each primary Kodaira surface M as the quotient of $\mathbb{C}^2$ considered as a multiplicative group by a subgroup Γ. In general, we do not make any difference between Γ-invariant objects on $\mathbb{C}^2$ and objects on $M = \mathbb{C}^2/\Gamma$. Let $A_1, \dots, A_4$ be a frame of left-invariant vector fields satisfying the commutation relations (11), say the frame (12). Also, let g be the Riemannian metric for which this frame is orthonormal. Let $E_1, \dots, E_4$ be the g-orthonormal frame defined by (16). As we have remarked, if $J = J_{\varepsilon_1,\varepsilon_2}$ is the almost-complex structure for which $JE_1 = E_2$, $JE_3 = E_4$, then (g,J) is an almost-Kähler structure (and any such a structure on M is of this type). Denote the dual frame of $E_1, \dots, E_4$ by $\gamma_1, \dots, \gamma_4$. Any left-invariant 1-form on $\mathbb{C}^2$ is of the form $\tau = a_1\gamma_1 + \dots + a_4\gamma_4$ where $a_1, \dots, a_4$ are real constants. Let D be the metric connection with totally skew-symmetric torsion determined by the form τ. After a long computation, one can see that J determines a harmonic map from (M,g) to $(\mathcal{Z}, h_t)$ if and only if the torsion of the connection D is determined by the form $\tau = a_2\gamma_2 + a_3\gamma_3$, where the constants a_2 and a_3 satisfies the identity $\varepsilon_1 a_2 \cos\varphi + a_3 \sin\varphi = 0$.

Example 43 ([60]). **Four-dimensional Lie groups with a non-integrable left-invariant almost-Kähler structure.** The description of four-dimensional Lie groups M admitting a non-integrable left-invariant almost-Kähler structure (g,J) with J-invariant Ricci tensor found by A. Fino [36] has been given in the preceding paragraph. Let D be the metric connection with totally skew-symmetric torsion determined by a left-invariant form τ. A tedious computation shows that J yields a harmonic map $(M,g) \to (\mathcal{Z}, h_t^D)$ if and only if $\tau = 0$, i.e. D is the Levi-Civita connection of the metric g.

9. Almost-Hermitian structures on Weyl manifold that are pseudo-harmonic maps from the manifold into its twistor space. If g and $g^1 = e^f g$ are conformal metrics, then clearly every g-orthogonal endomorphism of any tangent space T_pM is g^1-orthogonal and vice versa. Hence the Riemannian manifolds (M,g) and (M,g^1) have the same twistor space, i.e. the twistor space depends on the conformal class of g rather than the metric g itself. Recall that the definitions of the Atiyah-Hitchin-Singer and Eells-Salamon almost-complex structures depend on the decomposition $T\mathcal{Z} = \mathcal{H} \oplus \mathcal{V}$ into horizontal and vertical parts, this decomposition being yielded by the Levi-Civita connection of the metric on the base manifold M. As is well known [4], the Atiyah-Hitchin-Singer almost-complex structures on $\mathcal{Z}$ defined by means of two conformal metrics g and $g^1 = e^f g$ coincide as well as their integrability condition (anti/self-duality). Thus, the Atiyah-Hitchin-Singer almost-complex structure is invariant under conformal changes of the metric of M. On the other hand, the Eells-Salamon almost-complex structures is not conformally invariant [34]. In fact, it is not hard to show [28] that among all almost complex structures on the twistor space defined by means of a morphism of the twistor bundle ([29]), the Atiyah-Hitchin-Singer almost-complex structures is the only one which is invariant under conformal changes of the base manifold metric.

Given a smooth manifold M endowed with a class $\mathfrak{c}$ of conformal Riemannian metrics, it is natural to consider connections preserving the conformal class in the sense that for every Riemannian metric g in $\mathfrak{c}$, there exists a 1-form φ_g such that $D_X g = \varphi_g(X)g$ for every $X \in TM$; obviously, such a form is unique. In the particular case $\mathfrak{c} = \{g\}$ and $\varphi_g = 0$, this condition reduces to the requirement that D is a metric connection. Of special interest are those connections preserving $\mathfrak{c}$ that have vanishing torsion. These are the Weyl connections studied in different aspects by many authors.

Let J be an almost-complex structure on M compatible with one, hence with all metrics in the conformal class $\mathfrak{c}$. As above, give M the orientation induced by J and consider J as map from M into the positive twistor space $\mathcal{Z}$ of the Riemannian metrics in $\mathfrak{c}$. In this context, as we have already mentioned, it is natural to ask when the map $J : M \to \mathcal{Z}$ is pseudo-harmonic in the sense of G. Kokarev [49]. Here is the relevant definition.

Let M and M' be smooth manifolds endowed with torsion-free connections D and D', respectively. Let g be a Riemannian metric on M. A smooth map $\varphi : M \to M'$ is called pseudo-harmonic in [49] if the trace of its second fundamental form defined by means of D and D' vanishes, $\mathrm{Trace}_g\, II_\varphi(D,D') = 0$. Clearly, this condition is invariant under conformal changes of the metric g.

Let $(M,\mathfrak{c})$ be a conformal manifold and D a Weyl connection on it. For $g \in \mathfrak{c}$, denote the Levi-Civita connection of g by ∇^g. Then

$$D_X Y = \nabla^g_X Y - \frac{1}{2}[\varphi_g(X)Y + \varphi_g(Y)X - g(X,Y)\varphi_g^\sharp], \tag{22}$$

where the 1-form φ_g is determined by $Dg = \varphi_g \otimes g$ and $\varphi_g^\sharp$ is the dual vector field of the form φ_g with respect to the metric g: $g(\varphi_g^\sharp, Z) = \varphi_g(Z)$. If $g^1 = e^f g$ is another

Riemannian metric in the conformal class, f being a smooth function, then

$$\nabla^{g^1}_X Y = \nabla^g_X Y + \frac{1}{2}[X(f)Y + Y(f)X - g(X,Y)\nabla^g f], \tag{23}$$
$$\varphi_{g^1} = df + \varphi_g, \quad \varphi^\sharp_{g^1} = e^{-f}(\varphi^\sharp_g + \nabla^g f),$$

where $\nabla^g f$ is the gradient of f with respect to g, $g(\nabla^g f, X) = X(f)$ for $X \in TM$.

Note that if g is a Riemannian metric and φ is a 1-form on a manifold, there is a unique Weyl connection for the conformal class $\mathfrak{c}$ of g such that $\varphi_g = \varphi$. Indeed, for any conformal metric $g^1 = e^f g$, set $\varphi_{g^1} = df + \varphi_g$. Then, by (23), the right-hand side of (22) does not depend on the choice of a metric in $\mathfrak{c}$, so identity (22) defines a Weyl connection.

Consider the bundle $\mathcal{Z}$ as a subbundle of the bundle $A(TM)$ of the endomorphisms of TM that are skew-symmetric with respect to one Riemannian metric in $\mathfrak{c}$, hence with respect to all such metrics. Denote the connection on the vector bundle $\mathrm{Hom}(TM,TM)$ induced by D also by D. Endow the bundle $\mathrm{Hom}(TM,TM)$ with the metric $G(a,b) = \frac{1}{2}\mathrm{Trace}_g\{TM \ni X \to g(aX,bX)\}$, where g is a Riemannian metric in $\mathfrak{c}$. Clearly the metric G does not depend on the particular choice of g. It is easy to see that the induced connection D preserves the metric G ($DG = 0$) as well as the bundle $A(TM)$. The horizontal space of the vector bundle $A(TM)$ with respect to D at a point $I \in \mathcal{Z}$ is tangent to $\mathcal{Z}$, and we have the decomposition $T\mathcal{Z} = \mathcal{H} \oplus \mathcal{V}$. Fix a metric $g \in \mathfrak{c}$. As in the preceding sections, using the decomposition $T\mathcal{Z} = \mathcal{H} \oplus \mathcal{V}$, we can define a 1-parameter family of Riemannian metrics on $\mathcal{Z}$ setting $\tilde{g}_t(X^h + V, Y^h + W) = g(X,Y) + tG(V,W)$, $t > 0$. Then the projection map $\pi : (\mathcal{Z}, \widetilde{g}_t) \to (M,g)$ is a Riemannian submersion with totally geodesic fibres. Denote the Levi-Civita connection of $\widetilde{g}_t$ by $\widetilde{D}$, and define a new torsion-free connection $D' = D'_{g,t}$ on $\mathcal{Z}$ as follows. Let $I \in \mathcal{Z}$. Let X, Y be vector fields on M in a neighbourhood of $p = \pi(I)$, and let V, W be vertical vector fields on $\mathcal{Z}$ in a neighbourhood of I. Set

$$(D'_{X^h}Y^h)_I = (D_X Y)^h_I + \frac{1}{2}R^D(X,Y)I, \tag{24}$$
$$D'_V X^h = \widetilde{D}_V X^h, \quad D'_{X^h}V = \widetilde{D}_{X^h}V, \quad D'_V W = \widetilde{D}_V W.$$

Note that the summand $R^D(X,Y)I$ in the first formula is a vertical vector at I. Also, the fibres of the bundle $\pi : \mathcal{Z} \to M$ are still totally geodesic submanifolds when $\mathcal{Z}$ is considered with the connection D'.

Suppose that the almost-complex structure J is integrable. Let

$$\theta = -\frac{2}{n-2}\delta\Omega \circ J$$

be the Lee form of the Hermitian structure (g,J). If $n = 4$, this form satisfies the identity

$$d\Omega = \theta \wedge \Omega. \tag{25}$$

For $n \geq 6$, this identity is satisfied if and only if (g,J) is a locally conformally Kähler structure. If identity (25) is satisfied by a Hermitian structure (g,J), it is also satisfied by any Hermitian structure (g^1, J) with $g^1 = e^f g$, f being a smooth function, since $\Omega^1 = e^f\Omega$ and $\theta^1 = \theta + df$ by the first identity of (23). As is well-known, if $n \geq 6$, identity (25) implies $d\theta = 0$. Note also that, in any dimension, an Hermitian structure (g,J) is locally conformally Kähler if and only if $d\Omega = \theta \wedge \Omega$ and $d\theta = 0$, see, for example, [32, 66]. Clearly, if (g,J) is such a structure, then for any metric g^1 conformal to g, the structure (g^1,J) is also locally conformally Kähler.

The Ricci and the $*$ Ricci tensors are defined by

$$\rho_D(X,Z) = \mathrm{Trace}_g\{Y \to g(R^D(X,Y)Z,Y)\},$$
$$\rho_D^*(X,Z) = \mathrm{Trace}_g\{Y \to g(R^D(JY,X)JZ,Y)\},$$

where g is a metric in the conformal class $\mathfrak{c}$. Clearly, ρ_D and ρ_D^* do not depend on the particular choice of the metric g.

Computing the curvature R^D by means of (22), one can obtain the following formulas for the tensors ρ_D and ρ_D^*.

Proposition 3. ([27]) *Let ρ_g and ρ_g^* be the Ricci tensor and the $\star$-Ricci tensor with respect to a Riemannian metric $g \in \mathfrak{c}$. Then*

$$\rho_D(X,Z) = \rho_g(X,Z) + \frac{n-1}{2}(\nabla^g_X \varphi_g)(Z) - \frac{1}{2}(\nabla^g_Z\varphi_g)(X)$$
$$-\frac{n-2}{4}\big[\|\varphi_g\|^2 g(X,Z) - \varphi_g(X)\varphi_g(Z)\big] - \frac{1}{2}(\delta^g\varphi_g)g(X,Z).$$

$$\rho_D^*(X,Z) = \rho_g^*(X,Z) + (\nabla^g_X\varphi_g)(Z) - \frac{1}{2}\big[(\nabla^g_Z\varphi_g)(X) - (\nabla^g_{JX}\varphi_g)(JZ)\big]$$
$$+\frac{1}{4}\big[\varphi_g(X)\varphi_g(Z) + \varphi_g(JX)\varphi_g(JZ) - \|\varphi_g\|^2 g(X,Z)\big]$$
$$+\frac{1}{2}\big[\delta^g(J^*\varphi_g) - \varphi_g(\delta^g J)\big]g(X,JZ),$$

where the norms and the codifferential are taken with respect to the metric g.

As is well-known, $\rho_g(X,Z) = \rho_g(Z,X)$ and $\rho_g^*(X,Z) = \rho_g^*(JZ,JX)$. Proposition 3 and these identities imply the following.

Corollary 4. $\rho_D(X,Z) - \rho_D(Z,X) = \dfrac{n}{2}d\varphi_g(X,Z)$.

$\rho_D^*(X,Z) - \rho_D^*(JZ,JX) = d\varphi_g(X,Z) + d\varphi_g(JX,JZ)$.

Corollary 5. (1) *The Ricci tensor ρ_D is symmetric if and only if $d\varphi_g = 0$.*

(2) *The $*$-Ricci tensor ρ_D^* satisfies the identity $\rho_D^*(X,Z) = \rho_D^*(JZ,JX)$ for all $X,Z \in TM$ if and only if the $(1,1)$-component of $d\varphi_g$ w.r.t. J vanishes.*

Claim (1) is well-known. Note also that, by (23), the 2-form $d\varphi_g$ does not depend on the choice of $g \in \mathfrak{c}$.

Under the notation above, we have the following.

Theorem 10 ([27]). *Let $(M,\mathfrak{c})$ be a conformal n-dimensional manifold with a Weyl connection D, let $g \in \mathfrak{c}$ and set $\varphi = \varphi_g$. Suppose that J is a Hermitian structure on $(M,\mathfrak{c})$ such that Lee form θ of (g,J) satisfies identity (25). Let D' be the torsion-free connection on $\mathcal{Z}$ defined by (24). Then J determines a (D,D')-pseudo-harmonic map from M into the positive twistor space $\mathcal{Z}$ of $(M,\mathfrak{c})$ if and only if the following two conditions are satisfied.*

(i) *The 2-form*

$$d(\theta-\varphi) + \frac{n(n-4)}{2(n-2)}\theta\wedge\varphi$$

is of type $(1,1)$ with respect to J.

(*ii*) *For every tangent vector* $Z \in TM$,

$$\Big(\frac{n}{2}-1\Big)d\varphi((\theta-\varphi)^\sharp, Z) - d\varphi(J(\theta-\varphi)^\sharp, JZ) - (\theta-\varphi)(JZ)d\varphi(J^\wedge)$$
$$-\rho_D((\theta-\varphi)^\sharp, Z) + \rho_D^*(J(\theta-\varphi)^\sharp, JZ) = 0$$

where $\sharp : T^*M \to TM$ *is the isomorphism defined by means the metric* g.

Corollary 6. *In the notation of Theorem* 10, *if* $\dim M = 4$, J *determines a* (D, D')*-pseudo-harmonic map from* M *into the positive twistor space* $\mathcal{Z}$ *of* $(M, \mathfrak{c})$ *if and only if the* 2*-form* $d(\theta-\varphi)$ *is of type* $(1,1)$ *with respect to* J *and*

$$(\theta-\varphi)(JZ)d\varphi(J^\wedge) + \rho_D((\theta-\varphi)^\sharp, Z) - \rho_D^*(J(\theta-\varphi)^\sharp, JZ) = 0 \quad \text{for} \quad Z \in TM.$$

Remark 13. If $\mathfrak{c}$ consists of a single metric g and we set $\varphi_g = 0$, then D is the Levi-Civita connection of g and D' is the Levi-Civita connection of $\widetilde{g}_t$. In this case, Corollary 6 coincides with [24, Theorem 1].

If D is the Weyl connection determined by g and the 1-form $\varphi = \theta$, then conditions (i) and (ii) of Theorem 10 are trivially satisfied, hence $J : M \to \mathcal{Z}$ is a pseudo-harmonic map. This follows directly from the formula for the second fundamental form $II_J(D, D')$ obtained in [27] since $DJ = 0$ by (22) and the following well-known formula

$$2(\nabla^g_X J)(Y) = g(JX, Y)\theta^\sharp - g(\theta^\sharp, Y)JX + g(X, Y)J\theta^\sharp - g(J\theta^\sharp, Y)X.$$

In fact, D is the unique Weyl connection preserving the conformal class of g and such that $DJ = 0$ ([66]).

Example 44 ([27])**. Primary Kodaira surfaces.** Recall that we consider each primary Kodaira surface M as the quotient of $\mathbb{C}^2$ considered as a multiplicative group by a certain subgroup Γ. Let $J = J_{\varepsilon_1,\varepsilon_2}$ be the left-invariant complex structure on $\mathbb{C}^2$ defined by (13). Denote the dual basis of $A_1, \dots, A_4$ by $\alpha_1, \dots, \alpha_4$. Thus $\theta = -2\varepsilon_1\alpha_3$. Also, let again g be the metric for which the frame $A_1, \dots, A_4$ is orthonormal. Any left-invariant 1-form φ is of the form $\varphi = a_1\alpha_1 + \dots + a_4\alpha_4$, where $a_1, \dots, a_4$ are real constants. The Levi-Civita connection ∇ of the metric g is left-invariant. Thus, the connection D on $\mathbb{C}^2$ defined via (22) by means the metric g and the form φ is left-invariant, hence it yields a Weyl connection on M denoted again by D. Applying Theorem 10, one can show that J determines a (D, D')-pseudo-harmonic map $M \to \mathcal{Z}$ if and only if one of the following holds. (1) $\varepsilon_2 = 1$ and $a_1 = a_2 = 0$, a_3, a_4-arbitrary; (2) $\varepsilon_2 = -1$ and a_1, a_2-arbitrary, $a_3 = -2\varepsilon_1$, $a_4 = 0$; (3) $\varepsilon_2 = -1$ and $a_1 = a_2 = 0$, $a_3 \neq -2\varepsilon_1$, $a_4 = 0$. Thus, on a Kodaira surface, there are many Weyl connections for which the complex structures $J_{\varepsilon_1,\varepsilon_2}$ are pseudo-harmonic maps.

Example 45 ([27])**. Inoue surfaces of type S^0.** Let $M = (\mathbf{H} \times \mathbb{C})/\Gamma$ be an Inoue surface of type S^0 endowed with the metric g defined by (18). Let $E_1, \dots, E_4$ be Γ-invariant g-orthonormal vector fields on $\mathbf{H} \times \mathbb{C}$ satisfying conditions (20). Denote the dual frame of $E_1, \dots, E_4$ by $\eta_1, \dots, \eta_4$. Take a Γ-invariant 1-form $\varphi = a_1\eta_1 + \dots + a_4\eta_4$ where $a_1, \dots, a_4$ are real constants. The Levi-Civita connection ∇ of the metric g is Γ-invariant, hence the connection D on $\mathbf{H} \times \mathbb{C}$ defined via (22) by means the metric g and the form φ yields a Weyl connection on M. One can prove that the complex structure J of M is a pseudo-harmonic map from M into its twistor space only when $\varphi = \theta$, i.e. D is the Weyl connection determined by g and the Lee form θ of (g, J).

The next statement is a weaker version of [58, Proposition 1.3]. It is an easy consequence from Theorem 10 (as well from Theorem 8 in the four-dimensional case) and the uniqueness theorem for pseudo-harmonic maps [49] (respectively, for harmonic maps).

Corollary 7. *Let J_1 and J_2 be two Hermitian structures on a (connected) Riemannian manifold (M, g). Suppose that J_1 and J_2 have the same Lee form θ satisfying identity* (25)*. If J_1 and J_2 coincide on an open subset or, more-generally, if they coincide to infinite order at a point, they coincide on the whole manifold M.*

Remark 14. In fact, M. Pontecorvo [58] has proved that the conclusion of the above corollary holds without the assumption on the Lee forms. The idea of his proof is inspired by the theory of pseudo-holomorphic curves. It makes use of the observation in [34] that an almost-Hermitian structure J on a Riemannian manifold M is integrable if and only if the corresponding map $J : M \to \mathcal{Z}$ is pseudo-holomorphic with respect to the almost-complex structure J on M and the Atiyah-Hitchin-Singer almost-complex structure $\mathcal{J}_1$ on $\mathcal{Z}$.

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ПОЧТИ ЕРМИТОВИ СТРУКТУРИ, ТУИСТОРНИ ПРОСТРАНСТВА И ХАРМОНИЧНИ ИЗОБРАЖЕНИЯ

Йохан Давидов

В първата част на статията се припомня понятието за почти комплексна структура и се разглеждат някои топологични препятствия за съществуването на такава структура. Освен това са представени основни факти за туисторните пространства като параметризационни пространства на почти Ермитовите структури, т.е. ортогоналните почти комплексни структури, върху Римановo (или конформно) многообразие. Във втората част се разглежда проблемът кога една почти Ермитова структура върху Риманово или конформно многообразие определя хармонично или псевдо-хармонично изображение от многообразието в неговото туисторно пространство. Обсъждат се неотдавнашни резултати на автора (и съавтори) по този проблем главно в случая на четиримерно базово многообразие.

Тъй като този обзор е предназначен за широка аудитория, включително студенти, и в двете части по-голямо внимание е отделено на изясняващи примери, отколкото на техническите подробности на доказателствата.