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Short Proofs of Explicit Formulas to Boundary Value Problems for Polyharmonic Equations Satisfying Lopatinskii Conditions

Petar Popivanov † and Angela Slavova *,†

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria
* Correspondence: slavova@math.bas.bg
† These authors contributed equally to this work.

Abstract: This paper deals with Lopatinskii type boundary value problem (bvp) for the (poly) harmonic differential operators. In the case of Robin bvp for the Laplace equation in the ball B_1 a Green function is constructed in the cases $c > 0, c \notin -\mathbf{N}$, where c is the coefficient in front of u in the boundary condition $\frac{\partial u}{\partial n} + cu = f$. To do this a definite integral must be computed. The latter is possible in quadratures (elementary functions) in several special cases. The simple proof of the construction of the Green function is based on some solutions of the radial vector field equation $\Lambda u + cu = f$. Elliptic boundary value problems for $\Delta^m u = 0$ in B_1 are considered and solved in Theorem 2. The paper is illustrated by many examples of bvp for $\Delta u = 0$, $\Delta^2 u = 0$ and $\Delta^3 u = 0$ in B_1 as well as some additional results from the theory of spherical functions are proposed.

Keywords: Laplace operator; biharmonic and polyharmonic operators; Dirichlet, Neumann and Robin boundary value problems; Green function for elliptic second order operator; solutions into explicit form of boundary value problems; Lopatinskii (elliptic) boundary conditions

MSC: 31B30; 31A30; 35J05; 35J40; 35J25; 35C05; 35C15

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Citation: Popivanov, P.; Slavova, A. Short Proofs of Explicit Formulas to Boundary Value Problems for Polyharmonic Equations Satisfying Lopatinskii Conditions. *Mathematics* **2022**, *10*, 4413. https://doi.org/10.3390/math10234413

Academic Editor: Snezhana Hristova

Received: 26 October 2022
Accepted: 21 November 2022
Published: 23 November 2022

1. Introduction and Formulation of the Main Results

1. We shall begin with the classical Robin problem for Laplace operator in the unit ball in $\mathbf{R}^n$, $n \geq 2$, namely

$$\left| \begin{array}{l} \Delta u = g \ \text{ in } \ B_1 = \{x \in \mathbf{R}^n, |x| < 1\} \\ \frac{\partial u}{\partial n} + cu = f \ \text{ on } \ S_1 = \partial B_1, c = const \in \mathbf{R}^1. \end{array} \right. \tag{1}$$

Denote by $\varepsilon(x) = \begin{cases} ln\frac{1}{|x-y|}, n = 2, x, y \in \mathbf{R}^2 \\ \frac{1}{n-2}|x-y|^{2-n}, n \geq 3, x, y \in \mathbf{R}^n \end{cases}$ the fundamental solution of the Laplace equation $\Delta u = \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2} = g$ in B_1, i.e., $\Delta\varepsilon = \delta(x)$ in $\mathbf{R}^2$, $\Delta\varepsilon = -\omega_n\delta(x)$, where $\omega_n = \frac{(2\pi)^{n/2}}{\Gamma(\frac{n}{2})}$ is the area of the unit sphere S_1 in $\mathbf{R}^n$ and $\frac{\partial}{\partial n}$ stands for the unit outward normal to S_1, i.e., $\frac{\partial}{\partial n}|_{S_1} = \frac{\partial}{\partial \rho}|_{\rho=1}$. The theory of (1) is well developed in different scales of spaces: Hölder's, $C^{k,\alpha}$, $0 < \alpha < 1$ [1], Sobolev's, H^s [2] and many others. Therefore, it is interesting to find out explicit formulas for the solution of (1) in the cases $c > 0$, $c = 0$, $-n_0 - 1 < c < -n_0$, $n_0 \in \mathbf{N}$ and $c = -n_0$. In the cases $c = 0$, $-c \in \mathbf{N}$ (1) possesses a kernel—one dimensional for $c = 0$ and multidimensional for $-c \in \mathbf{N}$. Certainly, then a solution exists if several orthogonality type conditions are satisfied by g, f. The Green function for (1) (see [1,3]) is the function with representation

$$G_R(x,y) = \frac{1}{\omega_n}(\varepsilon(x-y) + g(x,y)), \tag{2}$$

where $g(x,y)$ is harmonic function in B_1 with respect to y for fixed $x \in B_1$ and conversely, $g \in C^1(\bar{B}_1)$ and satisfies the following boundary value problem:

$$\left| \begin{array}{l} \Delta_y g(x,y) = 0, x, y, \in B_1, x \ \text{fixed} \\ \frac{\partial G_R}{\partial n_y} + cG_R = 0 \ \text{ on } \ y \in S_1, x \ \text{ fixed } \ \text{in} \ B_1 \end{array} \right. \tag{3}$$

i.e., $(\frac{\partial}{\partial n_y} + c)g = -(\frac{\partial}{\partial n_y} + c)\varepsilon|_{y \in S_1}$.

In the Neumann case $c = 0 : \int_{S_1} f dS_1 = \int_{B_1} g dy$. Moreover, for $c = 0$ we shall consider the Green-Neumann function G_N for which the boundary condition $\frac{\partial G_N}{\partial n_y}|_{y \in S_1} = -\frac{1}{\omega_n}$. Both the functions G_R, $c \neq 0$; G_N, $c = 0$ are symmetric with respect to their arguments $(x,y) \in B_1 \times B_1$. Finally if a solution $u \in C^1(\bar{B}_1) \cap C^2(B_1)$ of (1) exists then it has the following form

$$u(x) = -\int_{B_1} G_R(x,y) g(y) dy + \int_{S_1} G(x,y) f(y) dS_y. \tag{4}$$

The Green function in the case $n = 2$ is very well known, while in the case $n = 3$ it was constructed in [4] in 1935. In the general case $n \geq 4$ and following the previous considerations G_N was constructed in [5].

We propose an elementary proof of the above result for (1).

Theorem 1. *Assume that $c > 0$. Then the Green function of (1) is given by the formula*

$$G_R(x,y) = \frac{1}{\omega_n}[\varepsilon(x-y) + \varepsilon(y|x| - \frac{x}{|x|}) + (n - 2(1+c)) \int_0^1 t^{c-1} \varepsilon(ty|x| - \frac{x}{|x|}) dt]. \tag{5}$$

We point out that for $x, y \in \bar{B}_1 \setminus 0 \Rightarrow |ty|x| - \frac{x}{|x|}|^2 = |tx|y| - \frac{y}{|y|}|^2 = |tx - y|^2 + (1 - t^2|x|^2)(1 - |y|^2)$ and $G_R|_{y \in S_1} = \frac{1}{\omega_n}(2\varepsilon(x,y) + (n - 2(1+c)) \int_0^1 t^{c-1} \varepsilon(tx - y) dt)$ for each $x \in B_1$.

In the Neumann case $c = 0$ the integral in (5) is divergent at $t = 0$ as $\varepsilon(\frac{x}{|x|}) = \frac{1}{n-2}$ for $n \geq 3$ but it is convergent at $t = 0$ for $n = 2$ as $ln|\frac{1}{\frac{x}{|x|}}| = 0$. In order to construct G_N for $n \geq 3$ one considers $\int_0^1 \left[\varepsilon(ty|x| - \frac{x}{|x|}) - \frac{1}{n-2} \right] \frac{dt}{t}$ as $\varepsilon(ty|x| - \frac{x}{|x|}) - \frac{1}{n-2}$ remains harmonic with respect to y for fixed $x \neq 0$. If $n = 2$ and $c = 0$

$$G_N = \frac{1}{2\pi} \left(ln \frac{1}{|x-y|} + ln \frac{1}{|y|x| - \frac{x}{|x|}|} \right), \frac{\partial G_N}{\partial n_y}|_{S_1} = -1.$$

If $c > 0$ then in $\mathbf{R}^2$:

$$G_R(x,y) = \frac{1}{2\pi}(ln \frac{1}{|x-y|} + ln \frac{1}{|y|x| - \frac{x}{|x|}|} - 2c \int_0^1 t^{c-1} ln \frac{1}{|ty|x| - \frac{x}{|x|}|} dt + \frac{1}{c}) =$$

$$\frac{1}{2\pi} \left(\frac{1}{c} + ln \frac{1}{|x-y|} - ln \frac{1}{|y|x| - \frac{x}{|x|}|} + 2 \int_0^1 \frac{(x,y) - t|x|^2|y|^2}{|t|x|y - \frac{x}{|x|}|^2} t^c dt \right).$$

Further on we shall find the Green-Robin function for $-n_0 - 1 < c < -n_0$, $n_0 \in \mathbf{N}$ in the form of a series of a full system of homogeneous harmonic polynomials H_k of degree k having the property of orthonormality on $L_2(S_1)$.

Example 1. *Consider the Robin problem (1) on the unit circle $B_1 = \{x \in \mathbf{R}^2 : |x| < 1\}$ with $g \equiv 0$ on B_1 and look for solution of the standard from*

$$u(\rho,\varphi) = A_0 + \sum_{n=1}^{\infty}(A_n cosn\varphi + B_n sinn\varphi), \tag{6}$$

while $f \in C^2(S_1) : f(\Theta) = a_0 + \sum_{n=1}^{\infty}(a_n cosn\Theta + b_n sinn\Theta)$, $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\Theta)d\Theta$, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\Theta)cosn\Theta d\Theta$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\Theta)sinn\Theta d\Theta$. Substituting (6) in (1) one gets the system: $cA_0 = a_0$, $(n+c)A_n = a_n$, $(n+c)B_n = b_n$, $n \geq 1$. So 3 cases appear:

(1) $\left| \begin{array}{l} -c \notin \mathbf{N} \\ c < 0, \end{array} \right.$ *, (2) $c = 0$,* $\left| \begin{array}{l} c = -n_0 \in -\mathbf{N} \\ a_{n_0} = b_{n_0} = 0 \end{array} \right.$ *, (3) $c > 0$.*

Denote $y = P \in S_1$ in polar coordinates $(1,\Theta) \in S_1$ and $x = P_0 = (\rho,\varphi) \in B_1$, i.e., $\forall t \in (0,1) : tP_0 = (t\rho,\varphi) \in B_1$. Then $|tP_0 - P|^2 = |tx - y|^2 = (t^2|x|^2 - 2t(x,y) + 1) = (t^2\rho^2 - 2t\rho cos(\varphi - \Theta) + 1)$, $y = S^1$.

In the case (3) $c > 0$

$$u(\rho,\varphi) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\Theta)[\frac{1}{c} + 2\Re\sum_{n=1}^{\infty}\frac{z^n}{n+c}]d\Theta,$$

where $z = \rho e^{i(\varphi-\Theta)}$. One can easily see that if $g(z) = \sum_{n=1}^{\infty}\frac{z^n}{n+c}$, then $g(z) = \int_0^1 \frac{t^{c-1}dt}{1-zt} - \frac{1}{c} = \frac{F(1,c,c+1,z)-1}{c}$, $F(1,c,c+1,z)$, $|z| < 1$ being the Gauss hypergeometric function; $F(1,c,c+1,z)$ is analytic in $\mathbf{C}^1 \setminus \{0 \leq z < \infty\}$.

Evidently, $\Re g(z) = -\frac{1}{c} + \int_0^1 \frac{(1-\rho tcos(\varphi-\Theta))t^{c-1}}{|tP_0-P|^2}dt$, $\Im g(z) = \int_0^1 \frac{sin(\varphi-\Theta)t^c}{|tP_0-P|^2}dt$, $\Re z^n = \rho^n cosn(\varphi - \Theta)$, $\Im z^n = \rho^n sinn(\varphi - \Theta)$.

In the case (1) $-n_0 - 1 < c < -n_0$ for some $n_0 \in \mathbf{N}$

$$g(z) = \sum_{n=1}^{n_0+1}\frac{z^n}{n+c} + \sum_{n=n_0+2}^{\infty}\frac{z^n}{n+c} = I + II.$$

One can easily see that $II = z^{n_0+1}\sum_{m=1}^{\infty}\frac{z^m}{m+\tilde{c}}$, where $\tilde{c} = c + n_0 + 1 \in (0,1)$, i.e., $II = \frac{z^{n_0+c}}{\tilde{c}}(F(1,\tilde{c},\tilde{c}+1,z) - 1)$. There are no problems to find $\Re II$.

Case (2) $c = -n_0 \in -\mathbf{N}_0$. Then we have

$$\tilde{g}(z) = \sum_{n=1}^{n_0-1}\frac{z^n}{n-n_0} + \sum_{n=n_0+1}^{\infty}\frac{z^n}{n-n_0} = III + IV,$$

$IV = z^{n_0}\sum_{m=1}^{\infty}\frac{z^m}{m} = -z^{n_0}log_0(1-z)$, $|z| < 1$, $log_0(1-z)$ is analytic in $\mathbf{C}^1 \setminus \{1 \leq z < \infty\}$. One can get that $\Re log_0(1-z) = log_0|P - P_0| = log_0|x-y|$, $\Im log_0(1-z) = -arctg\frac{\rho sin(\varphi-\Theta)}{1-\rho cos(\varphi-\Theta)}$ etc. In this case (1) possesses the kernel $\{\rho^{n_0}cosn_0\varphi, \rho^{n_0}sinn_0\varphi\}$ while the Fourier coefficients of $f(\Theta) : a_{n_0} = b_{n_0} = 0$.

If $c = 0$ we obtain Dini's formula

$$u(\rho,\varphi) = A + \int_{-\pi}^{\pi} f(\Theta)ln\frac{1}{|1+\rho^2-2\rho cos(\varphi-\Theta)|}d\Theta.$$

In $\mathbf{R}^n$, $n \geq 3$ the construction of the Green function is reduced to the calculation of the integrals of the form

$$\int_0^1 t^{c-1}|t|x|y - \frac{x}{|x|}|^{2-n}dt, \tag{7}$$

i.e., $\int_0^1 \frac{t^{c-1}dt}{(R(t))^{\frac{n-2}{2}}}$, where the second order polynomial with respect to t $R(t) = t^2|x|^2|y|^2 - 2t(x,y) + 1 \geq 0$ for each $t, x, y \in B_1 \setminus 0$. We put $a = 1$, $b = -2(x,y)$, $c_1 = |x|^2|y|^2 \Rightarrow$

$R(t) = a + bt + c_1t^2$, *the discriminant* $\Delta = 4(|x|^2|y|^2 - (x,y)^2) \geq 0$ *and* $\Delta > 0$ *if* x, y *are not collinear.*

Example 2. *(a)* $c = \frac{n}{2} - 1 > 0$, $n \geq 3$. *Then*

$$G_R = \frac{1}{\omega_n}(\varepsilon(x-y) + \varepsilon(y|x| - \frac{x}{|x|})).$$

(b) $n = 4$, $c = 1/2 \Rightarrow n - 2(c+1) = 1$. *Then*

$$G_R = \frac{1}{\omega_4}\left(\varepsilon(x-y) + \varepsilon(y|x| - \frac{x}{|x|}) + \int_0^1 t^{1/2}\varepsilon(t|x|y - \frac{x}{|x|}dt\right).$$

Put $f(t)\Big|\begin{array}{l} t=1 \\ t=0 \end{array} = f(t) - f(0)$. *After some calculations (see 160.01, 160.11 from [6]) one obtains that*

$$\int_0^1 \frac{t^{1/2}dt}{R(t)} = \frac{1}{\sqrt{b_1}}\{\frac{1}{2\sqrt{c}}ln\frac{\sqrt{c_1} - b_1 + 1}{\sqrt{c_1} + b_1 + 1} + \frac{b_1}{\sqrt{c_1\Delta}}arctg\frac{4\sqrt{c_1\Delta}}{-b - b^2 + 2\sqrt{c_1}}\},$$

where $b_1 = \sqrt{2\sqrt{ac_1} - b} = \sqrt{2}\sqrt{|x||y| + (x,y)} \quad - b - b^2 + 2\sqrt{c_1} = 2(|x||y| + (x,y) - 2(x,y)^2)$. *In fact,* $arctg\, x + arctg\, y = arctg\frac{x+y}{1-xy}$ *if* $xy < 1$ *and similar results hold for* $x > 0$, $xy > 1$; $x < 0$, $xy > 1$.

(c) $n = 5$, $c = 1$. *Then we must compute (see 380.03 [6])*

$$\int_0^1 \frac{dt}{(R(t))^{3/2}} = \frac{4c_1t + 2b}{\Delta\sqrt{R(t)}}\Big|_{t=0}^{t=1} = \frac{1}{|x|^2|y|^2 - (x,y)^2}[\frac{|x|^2|y|^2 - (x,y)}{||x|y - \frac{x}{|x|}|} + (x,y)].$$

The integral (7) can be found in quadratures (elementary functions) if and only if c is integer for n-odd and c is rational for n-even ((7) is a differential binomial).

In calculating the integrals of the type (7) the following recurrent formulas could be very useful: $\int \frac{t^m}{\sqrt{R^{2p+1}}}$, $m \in \mathbf{N}$, $p \in \mathbf{N}_0$, *2.263.1, 2.263.2, p. 82, p. 83 from [7] and 2.263.3, p. 83 from the same book in the case* $m = 0$. *We note that for* $s \in \mathbf{N}: \sqrt{R^s(t)} = \begin{cases} 1, t = 0 \\ |y|x| - \frac{x}{|x|}|^s, t = 1. \end{cases}$

This is our last example for constructing of Green function for (1), $c > 0$ in the multidimensional case.

Example 3. *(a) Assume that* $n > 6$, *n is even,* $n = 2m + 2$, $m > 2$, $c = 1$. *Put* $m = k + 1$, $k > 1$ *and find the corresponding Green function for (1). We have to compute via formula 2.171, p. 79 from [8] the integral*

$$\int_0^1 \frac{dt}{R(t)^m} = \int_0^1 \frac{dt}{R(t)^{k+1}} = \frac{2c_1t + b}{2k+1}\sum_{i=0}^{k-1}\frac{2^i(2k+1)\dots(2k-2i+1)c_1}{k\dots(k-i)\Delta^i(R(t))^{k-i}}\Big|_{t=0}^{t=1} +$$

$$2^k\frac{(2k-1)!!c_1^k}{k!\Delta^k}\int_0^1 \frac{dt}{R(t)},$$

$$\int_0^1 \frac{dt}{R(t)} = \frac{2}{\sqrt{\Delta}}arctg\frac{b + 2c_1t}{\sqrt{\Delta}}\Big|_{t=0}^{t=1} = \frac{1}{\sqrt{|x|^2|y|^2 - (x,y)^2}}arctg\frac{\sqrt{|x|^2|y|^2 - (x,y)^2}}{1 - (x,y)}.$$

Here $1 < k = m - 1 = \frac{n-2}{2} - 1 = \frac{n-4}{2}$.

(b) $c = 1$, n -odd, $n > 5$, i.e., $n = 2m+1$, $m = k+1$, $m > 2$, $k > 1$. According to (5) we shall calculate

$$\int_0^1 \frac{dt}{\sqrt{(R(t))^{2k+1}}} = \frac{2(2c_1 t + b)}{\sqrt{R(t)^{2k-1}}(2k-1)\Delta}\left\{1 + \sum_{i=1}^{k-1} \frac{8^i (k-1)\dots(k-i)}{(2k-3)\dots(2k-2i-1)} \frac{c_1^i}{\Delta^i} R^i(t)\right\}\Big|_{t=0}^{t=1}.$$

(see Formula 2.263, p. 96 from [8]).

Certainly, $2c_1 t + b = \begin{cases} -2(x,y), t = 0 \\ 2|x|^2|y|^2 - 2(x,y), t = 1 \end{cases}$, $1 < k = m-1 < \frac{n-3}{2}$.

2. Our next step is to formulate several properties of the radial vector field

$$\Lambda = \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}$$

which are useful in the study of the properties of the polyharmonic operators in B_1. So consider the equation

$$\Lambda u + cu = f(x) \in C^k(\bar{B}_1), k \geq 2, c = const \in \mathbf{R}^1 \tag{8}$$

looking for smooth solutions $u \in C^k(\bar{B}_1)$. Concerning the kernel $\Lambda + c$, we know that its solutions are functions, positively homogeneous of order $(-c)$. Therefore, dim ker$(\Lambda + c) \cap C^1(B_1) = +\infty$ for $c \leq -1$, dim ker$\Lambda \cap C^1(B_1) = 1$, ker$(\Lambda + c) \cap C^1(B_1) = \{0\}$ for $c > 0$. Assume that $c = -n_0$, $n_0 \in \mathbf{N}$. Then dim ker$(\Lambda + c) \cap C^\infty(B_1) < \infty$ as ker$(\Lambda + c)$ contains the polynomials, homogeneous of order n_0. Many of the results proposed below are valid for $c \in \mathbf{C}^1$.

Proposition 1. *(i) Assume that $\Re\, c > 0$. Then the Equation (8) possesses the solution*

$$u(x) = \int_0^1 f(tx) t^{c-1} dt \in C^k(\bar{B}_1) \tag{9}$$

if $f \in C^k(\bar{B}_1)$; $u(x) \in C^k(\bar{B}_1)$. Shortly, we shall write $u = (\Lambda + c)^{-1} f$.

(ii) Put $u_l(x) = \frac{(-1)^{l-1}}{(l-1)!} \int_0^1 t^{c-1} (ln\ t)^{l-1} f(tx) dt$, $l \geq 1$, $f \in C^k(\bar{B}_1)$. Then for $\Re\, c > 0$ u_l satisfies the equation

$$(\Lambda + c)^l u_l = f(x), k \geq 2, u \in C^k(\bar{B}_1).$$

Shortly we write $u_l = (\Lambda + c)^{-l} f$.

(iii) Consider the Equation (8) with the additional condition $u|_{\rho=1} = 0$, where $\rho = |x|$, $\tilde{\Theta} = \frac{x}{|x|}$, $x = \rho\tilde{\Theta}$. Then this boundary value problem is satisfied for $|x| > 0$ by

$$u(x) = -\int_1^{1/\rho} f(xt) t^{c-1} dt \in C^k(\bar{B}_1 \setminus \{0\}).$$

Then solution $u \in C^k(\bar{B}_1)$ if and only if for each $\tilde{\Theta} \in S^1$ the integral $\int_0^1 t^{c-1} f(t\tilde{\Theta}) dt = 0$.

In other words, if for some $\tilde{\Theta}_0 \in S_1$ the integral $\int_0^1 t^{c-1} f(t\tilde{\Theta}_0) dt \neq 0$ it follows that $u(\rho\tilde{\Theta}_0)$ develops power type nonlinearity ρ^{-c} for $\rho \to 0$.

Remark 1. *Consider the Equation (8) with $\Re\, c < 0$. Then it possesses for $\rho > 0$ the smooth solution*

$$u(x) = \int_{1/\rho}^1 f(xt) t^{c-1} dt = \rho^{|c|} \int_1^\rho f(t\tilde{\Theta}) t^{c-1} dt,$$

where $x = \rho \frac{x}{|x|} = \rho\tilde{\Theta}$, $\tilde{\Theta} \in S_1$. The latter function is smooth at the origin.

If $c = 0$ then $\Lambda u = f$ in B_1 implies that $f(0) = 0$. Therefore, the general solution of $\Lambda u = f$ is

$$u(x) = \int_0^1 \frac{f(tx)}{t} dt + A, A = const.,$$

as $\frac{f(tx)}{t} \in C^{k-1}(B_1)$.

We shall remind several classical results concerning the Dirichlet, Robin and Neumann boundary value problem [1]:

$$(D) \quad \left| \begin{array}{l} \Delta u = g \ \text{ in } \ B_1, g \in C^{k,\alpha}(\bar{B}_1) \\ u|_{S_1} = f \in C^{k+2,\alpha}(S_1), \end{array} \right.$$

$$(R) \quad \left| \begin{array}{l} \Delta u = g \ \text{ in } \ B_1 \\ \frac{\partial u}{\partial n} + cu = f \in C^{k+1,\alpha}(S_1). \end{array} \right.$$

$$(N) \quad \left| \begin{array}{l} \Delta u = g \ \text{ in } \ B_1 \\ \frac{\partial u}{\partial n} = f \in C^{k+1,\alpha}(S_1). \end{array} \right.$$

It is well known that for (D) problem there exists a unique smooth solution $u \in C^{k+2,\alpha}(\bar{B}_1)$, while for $c \notin \{o, -1, -2, \dots, -n, \dots\}$ the problem (R) possesses a unique solution $u \in C^{k+2,\alpha}(\bar{B}_1)$; if $c = -n_0 \in -\mathbf{N}_0$ there exists a kernel of finite dimension and one can find $u \in C^{k+2,\alpha}(\bar{B}_1)$ if (f, g) satisfies $\int_{B_1} H_{n_0} g dx = \int_{S_1} H_{n_0} f dS_1$ orthogonality conditions. Certainly, in this case the solution of (R) is not unique. Suppose that $g \equiv 0$. Then one can reduce (R) to (D), respectively (N) to (D). To do this we shall use the commutator property $[\Delta, L] = 2\Delta$ and Proposition 1 (i).

Concerning (D), $g \equiv 0$ we have the Poisson formula [6]: $u(x) = \frac{1-\rho^2}{\omega_n} \int_{S_1} \frac{f(y)}{|x-y|^n} dS_y$, $n \geq 2, \rho = |x|$.

3. Now we shall consider Lopatinskii type boundary value problem for the polyharmonic operator $\Delta^m u = 0$ in B_1, $m \geq 2$. Almansi proved in [1] that each polyharmonic function in the ball u can be presented in the form

$$u(x) = \sum_{l=1}^{m} (\rho^2 - 1)^{l-1} u_l, \Delta u_l = 0 \ \text{ in } \ B_1. \tag{10}$$

(see also [7]).

One can easily see that $\Delta L^2 = L^2\Delta + 4L\Delta + 4\Delta$ and by induction verify that $\Delta u = 0 \Rightarrow \Delta L^k u = 0$, $\Delta(\rho^k \frac{du^k}{d\rho^k}) = 0$, $\Delta^p((\rho^2 - 1)^q u) = 0$ if $p > q$ and $[\Delta^k, L] = c_k \Delta^k$, $c_k = const \neq 0$.

We shall study the following boundary value problem:

$$\left| \begin{array}{l} \Delta^m u = 0 \ \text{ in } \ B_1, m \geq 2, n \geq 3 \\ B_0(u) = \varphi_1 \ \text{ on } \ S_1 \\ \dots \\ B_{m-1}(u) = \varphi_m \ \text{ on } \ S_1, \end{array} \right. \tag{11}$$

where $B_0(u) = u$, $B_1(u) = \sum_{l=1}^n a_l D_{x_l} u + b(x) D_\rho u \dots$ $B_j(u) = \sum_{k+|\alpha|=j} a_{k\alpha}(x) D_x^\alpha D_\rho^k u$, $j = 1, 2, \dots, m-1$.

The boundary operators B_j have smooth coefficients. As usual $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}_0^n$, $D_{x_j} = \frac{1}{i}\frac{\partial}{\partial x_j}$, $D_\rho = \frac{1}{i}\frac{\partial}{\partial \rho}$. The vector fields $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial \rho}$ are linearly dependent in $\mathbf{R}^n$.

Theorem 2. *Consider boundary value problem (11) with smooth right-hand sides φ_j, $j = 0, 1, \dots, m-1$ and suppose that for each integer $j = 1, 2, \dots, m-1$ the expression*

$$A_j(x) = \sum_{k+|\alpha|=j} a_{k\alpha}(x) D_x^\alpha D_\rho^k((\rho^2 - 1)^j) \neq 0 \ \text{ at } \ S_1 \tag{12}$$

Then (11) has a unique solution of the form (10) where the functions u_j satisfy the Dirichlet problems for Laplace equation in the ball B, namely

$$\left| \begin{array}{l} \Delta u_1 = 0, B_1 \\ u_1|_{S_1} = \psi_1, \end{array} \right. \left| \begin{array}{l} \Delta u_2 = 0, B_1 \\ u_2|_{S_1} = \psi_2, \end{array} \right. \ldots, \left| \begin{array}{l} \Delta u_m = 0, B_1 \\ u_m|_{S_1} = \psi_m, \end{array} \right. \tag{13}$$

$\psi_1 = \varphi_1$ and ψ_j for $j \geq 2$ is expressed by φ_j as well as by $\varphi_1, \ldots, \varphi_{j-1}$ and their derivatives up to some order.

Example 4. $\left| \begin{array}{l} \Delta^2 u = 0 \text{ in } B_1, n \geq 3 \\ u|_{S_1} = \varphi_1 \\ B_1(u) = \varphi_2 = \sum_{l=1}^n a_l(x) D_{x_l} u + b(x) D_\rho u|_{S_1}. \end{array} \right.$

Then condition (12) takes the form

$$A_1(x) = \sum_{j=1}^n x_j a_j(x) + b(x) \neq 0 \text{ on} S_1,$$

$$u = (\rho^2 - 1)u_2 + u_1; \left| \begin{array}{l} \Delta u_1 = 0 \\ u_1|_{S_1} = \varphi_1 \end{array} \right. \left| \begin{array}{l} \Delta u_2 = 0 \\ u_2|_{S_1} = i \frac{\varphi_2 - B_1(u_1)}{2A_1}|_{S_1} \end{array} \right.$$

There are many paper on the subject but we quote only [9,10].

The paper is organized as follows. Section 1 deals with introduction and formulation of the main results. In Section 2 some additional results from the theory of spherical functions are given. The short proofs are contained in Section 3.

2. Additional Results from the Theory of Spherical Functions

We shall follow here [11]. The spherical change of the variables is defined by

$$\begin{array}{l} x_1 = \rho cos\Theta_1, \rho = |x|, n \geq 3 \\ x_2 = \rho sin\Theta_1 cos\Theta_2 \\ \ldots \\ x_{n-1} = \rho sin\Theta_1 sin\Theta_2 \ldots sin\Theta_{n-2} cos\Theta_{n-1} \\ x_n = \rho sin\Theta_1 sin\Theta_2 \ldots sin\Theta_{n-2} sin\Theta_{n-1}, \end{array}$$

where the polar angles are: $0 \leq \Theta_j \leq \pi$, $1 \leq j \leq n-2$, $-\pi \leq \Theta_{n-1} \leq \pi$, $\frac{x}{|x|} = (\tilde{\Theta}_1, \ldots, \tilde{\Theta}_n) \in S_1$ and $x \to (\rho, \tilde{\Theta})$, $|x| = \rho$.

In this coordinates $\Delta = \frac{\partial^2}{\partial \rho^2} + \frac{n-1}{\rho} \frac{\partial}{\partial \rho} + \frac{1}{\rho^2} \delta_{\tilde{\Theta}}$, $\delta_{\tilde{\Theta}}$ being the Laplace-Beltrami operator on S_1. $\delta_{\tilde{\Theta}}$ admits a sequence of eigenvalues $\{-\lambda_k\}$ with multiplicity μ_k equal to the number of linearly independent homogeneous harmonic polynomials of degree k, i.e., $\delta_{\tilde{\Theta}} v = -\lambda_k v$, $v \in C^\infty(S_1)$. In other words if $H_k(\xi)$ is harmonic polynomial in $B_1 \subset \mathbf{R}^n$, $H_k(\lambda \xi) = \lambda^k H_k(\xi)$, $|\xi| = \rho$, then $\rho^{-k} H_k(\xi) = H_k(\frac{\xi}{|\xi|}) = H_k(\tilde{\Theta})$, $\tilde{\Theta} \in S_1$. Thus, the spherical harmonic $H_k(\tilde{\Theta}) = Y_k(\Theta)$ of degree k is continuous on S_1. Moreover, $\lambda_k = k(n+k-2)$, $H_0 = 1$. From [11] it is known that there exists a full system of orthogonal spherical harmonics of degree $k \geq 0$ on $L_2(S_1)$: $\{H_k^{(i)}$ and such that $\frac{1}{\omega_n} \int_{S_1} H_k^{(i)}(x) H_m^{(j)}(x) dS = \delta_{ij} \delta_{km}$. The quantity of these polynomials $H_k^{(i)}$ is $h_k = (2k+n-2) \frac{(k+n-3)!}{k!(n-2)!} = O(k^{n-2})$, $k \to \infty$, $i = 1, 2, \ldots, h_k$. Each harmonic function u in B_1 can be written in the form of following series of spherical polynomials:

$$u(\rho, \tilde{\Theta}) = \sum_{k=0}^\infty \rho^k \sum_{i=1}^{h_k} a_k^{(i)} H_k^{(i)}(\frac{x}{|x|}), \rho = |x|, \frac{x}{|x|} = \tilde{\Theta}.$$

For the Dirichlet problem $f(\tilde{\Theta}) = u|_{\rho=1}$ we have that $a_k^{(i)} = \frac{1}{\omega_n}\int_{S_1} f(\tilde{\Theta})H_k^{(i)}(\tilde{\Theta})dS_1$, $dS_1 = (sin\Theta_1)^{n-2}\dots sin\Theta_{n-2}d\Theta_1\dots d\Theta_{n-1}$, $\delta H_k^{(i)}(\tilde{\Theta}) = -k(k+n-2)H_k^{(i)}(\tilde{\Theta}), i = 1, 2, \dots, h_k$ and one can prove that

$$\sum_{i=1}^{k}(H_k^{(i)}(\xi))^2 = h_k, \forall \xi \in S_1.$$

After the spherical change the vector field $\Lambda \to \rho\frac{\partial}{\partial\rho}, \rho \geq 0$. So $\Lambda|_{S_1} = \frac{d}{d\rho}|_{S_1} = \frac{d}{dn}|_{S_1}$ and $Ker(\rho\frac{d}{d\rho} + c) \cap C^1[0,1) = \begin{cases} 0, c > 0 \\ A = const, c = 0. \\ A\rho^{-c}, c \leq -1 \end{cases}$ Certainly, $\rho^{-c} \in C^\infty$ iff $c \in -\mathbf{N}_0$.

A direct proof of Theorem 1 can be given by using the following Proposition 2 (Lemma 2.1 from [5]).

Proposition 2. *Let* $\varepsilon = \frac{|x-y|^{2-n}}{n-2}$ *be the fundamental solution of the Laplace equation for* $n \geq 3$. *Then*

$$\varepsilon(x-y) = \sum_{k=0}^{\infty}\frac{1}{2k+n-2}\frac{|x|^k}{|y|^{k+n-2}}\sum_{i=1}^{h_k}H_k^{(i)}(\frac{x}{|x|})H_k^{(i)}(\frac{y}{|y|}) \tag{14}$$

for $|x| < |y| \leq 1$ *and*

$$\varepsilon(x-y) = \sum_{k=0}^{\infty}\frac{1}{2k+n-2}\frac{|y|^k}{|x|^{k+n-2}}\sum_{i=1}^{h_k}H_k^{(i)}(\frac{x}{|x|})H_k^{(i)}(\frac{y}{|y|}) \tag{15}$$

for $1 \geq |x| > |y| > 0$.

Remark 2. *We remind of the reader that* $\Lambda + c \to \rho\frac{d}{d\rho} + c$ *in* $\mathbf{R}^1$, $\tilde{\Theta}$ *being a parameter. After the change* $\rho = e^t$, $t \in \mathbf{R}^1_-$ ($\rho = e^{-t}$, $t \in \mathbf{R}^1_+$) *the operator* $\rho\frac{d}{d\rho}$ *for* $\rho \in (0,1]$ *becomes* $\frac{d}{dt}$ *for* $t \in (-\infty, 0]$ ($\rho\frac{d}{d\rho}$ *becomes* $\frac{d}{dt}$ *for* $t \in [0,\infty)$). *We can apply Laplace transform for* $t = 0$ ($\iff \rho = 1$), *i.e.,* $u(0) = 0$ *obtaining for* $\rho\frac{du}{d\rho} + cu = f$, $u|_{\rho=1} = 0 \iff \frac{du}{dt} + cu = f$, $u(0) = 0$ *the algebraic equation* $(s+c)\mathcal{L}(u)(s) = \mathcal{L}(f)(s) \Rightarrow \mathcal{L}(u) = \mathcal{L}(f)\mathcal{L}(e^{-ct})$. *The convolution formula enables us to conclude that* $u(t) = \int_0^t f(\tau)e^{-c(\tau-t)}d\tau$. *Going back to the x - coordinates we come to Proposition 1 (iii).*

3. Proofs of the Main Results

1. We shall begin with the proof of Proposition 1 (i). Then

$$\Lambda u = \int_0^1 t^c\sum_{j=1}^{n}x_j\frac{\partial f}{\partial x_j}(tx)dt = \int_0^1 t^c\frac{d}{dt}f(tx)dt = t^c f(tx)|_{t=0}^1 - cu(x) = f(x) - cu(x).$$

Inductively one proves (iii). In fact,

$$\Lambda u_{l+1} = \frac{(-1)^l}{l!}\int_0^1 t^c(ln\ t)^l\frac{d}{dt}f(tx)dt = \frac{(-1)^l}{l!}[-c\int_0^1 t^{c-1}(ln\ t)^l f(tx)dt -$$

$$l\int_0^1 t^{c-1}(ln\ t)^{l-1}f(tx)dt] = -cu_{l+1} + \frac{(-1)^{l+1}}{(l-1)!}\int_0^1 t^{c-1}(ln\ t)^{l-1}f(tx)dt = -cu_{l+1} + u_l.$$

According to the inductive assumption $(\Lambda + c)^{l+1}u_{l+1} = (\Lambda + c)^l u_l = f(x)$.

Direct calculation shows that u from (iii) is a solution of $(\Lambda + c)u = f$ in $B_1 \setminus \{0\}$, $u|_{S_1} = 0$. Put $v = \int_0^1 t^{c-1}f(tx)dt$ for a solution of $(\Lambda + c)v = f$ in B_1. Then

$$u(x) - v(x) = \int_{1/\rho}^{0}\lambda^{c-1}f(\lambda\tilde{\Theta})d\lambda$$

after the change $t\rho = \lambda$ and the notation $x = \rho\frac{x}{|x|} = \rho\tilde{\Theta}$, $\tilde{\Theta} \in S_1$. In other words, $u(x) = u(\rho\tilde{\Theta})$ does not develop singularity at the origin iff $\int_0^1 \lambda^{c-1} f(\lambda\tilde{\Theta})d\lambda = 0 = v|_{\rho=1} = v(\tilde{\Theta})$, $\forall\tilde{\Theta} \in S_1 \iff u \equiv v$ in B_1, $u|_{\rho=1} = 0$.

Assume that $c \leq -1$, $f \in C^k(\bar{B}_1)$ and $f(x) = O(|x|^{[|c|]+2})$, $x \to 0$. Then the general solution of $(\Lambda + c)u = f(x)$ is given by (9) plus arbitrary linear combination of finitely many smooth functions, which are homogeneous of degree $|c|$; $[|c|]$ stands for the integer part of $|c|$. Evidently. $u(x) = O(|x|^{[|c|]+2})$, $x \to 0$.

2. Concerning the proof of Theorem 1 we shall use the fact that $\Lambda_y u = \frac{\partial u}{\partial n_y}$ for $y \in S_1$. Therefore, $\Lambda_y \varepsilon(y-x) = \frac{-|x|^2|y|^2+(x,y)}{|y|x|-\frac{x}{|x|}|^n}$ and if $\varepsilon_1(x,y)$ stands for the integral term of (5) then according to Proposition 1 (i)

$$\Lambda_y \varepsilon_1(x,y) = \frac{(n-2(1+c))}{n-2}|y|x| - \frac{x}{|x|}|^{2-n}, (x,y) \in B_1 \times B_1.$$

The identity $|y|x| - \frac{x}{|x|}| = |x-y|$ for $y \in S_1$ gives that for $y \in S_1 : \frac{\partial G_R}{\partial n_y} = \frac{-1+2c/(n-2)+1-2c(n-2)}{|x-y|^{n-2}} = 0$. The proof is completed. The proof in the case $n = 2$, $c > 0$ is similar and it is omitted.

Following [4,12] one can give another proof of Theorem 1. To do this we look for g from (2) of the form

$$g(x,y) = \sum_{k=0}^{\infty} b_k |x|^k |y|^k \sum_{i=1}^{h_k} H_k^{(i)}(\frac{x}{|x|}) H_k^{(i)}(\frac{y}{|y|}), \tag{16}$$

where the coefficients are unknown. Fixing $x \in B_1$ we obtain harmonic function with respect to $y \in B_1$ and vice versa; $g(x,y) = g(y,x)$ $\forall(x,y) \in B_1 \times B_1$. We shall consider only the case (14). We write $r = |x|$, $\rho = |y|$ and for $r < \rho \leq 1$ we differentiate (14) and put the expression (14) for $\frac{\partial\varepsilon}{\partial\rho}(x-y)$ and the corresponding expression for $\frac{\partial}{\partial\rho}g(x,y)$ from (16) into the boundary condition from (3): $\rho = 1$. Equalizing to 0 the coefficients in front of hte spherical harmonics we get that for $k \geq 0$

$$(c+k)b_k = \frac{c+2-(k+n)}{2k+n-2}, \tag{17}$$

i.e., for $c \neq 0$, $c \neq -k$, $k \in \mathbf{N}$

$$b_k = \frac{k+n-2-c}{(2k+n-2)(k+c)} = \frac{1}{2k+n-2} + \frac{A}{(k+c)(2k+n-2)}, \tag{18}$$

$A = n - 2(1+c) = const.$

If some $c = -k_0$, $k_0 \in \mathbf{N}$, the coefficients b_k, $k \neq k_0$ are given by (18) but b_{k_0} does not exist.

Conclusion. For each $c \notin -\mathbf{N}$ the Green function for (1) exists and is given by the formula

$$g(x,y) = \sum_{k=0}^{\infty} \frac{|x|^k|y|^k}{2k+n-2} \sum_{i=1}^{h_k} H_k^{(i)}\left(\frac{x}{|x|}\right) H_k^{(i)}\left(\frac{y}{|y|}\right) + \tag{19}$$

$$A\sum_{k=0}^{\infty} \frac{|x|^k|y|^k}{(k+c)(2k+n-2)} \sum_{i=1}^{h_k} H_k^{(i)}(\frac{x}{|x|}) H_k^{(i)}(\frac{y}{|y|}) = I + II.$$

From (14) one easily gets that

$$\varepsilon(x|y| - \frac{y}{|y|}) = \varepsilon(y|x| - \frac{x}{|x|}) = I.$$

Assume $c > 0$. The identity $\int_0^1 t^{c-1+k}dt = \frac{1}{k+c}$ gives us that $A\int_0^1 \varepsilon(ty|x| - \frac{x}{|x|})t^{c-1}dt = II$. This way we have another proof of Theorem 1.

Suppose now that $-k_0 - 1 < c < -k_0$ for some $k_0 \in \mathbf{N}$, i.e., $|k_0| < |c| < |k_0| + 1$, $-c \notin \mathbf{N}_0$. Certainly, $[|c|] = k_0$. In this case the Green function for (1) can be written as:

$$G_R(x,y) = \frac{1}{\omega_n}[\varepsilon(x-y) + (n-2(c+1))\sum_{k=0}^{k_0} \frac{|x|^k|y|^k}{(2k+n-2)(k+c)} \sum_{i=1}^{h_k} H_k^{(i)}(\frac{x}{|x|})H_k^{(i)}(\frac{y}{|y|}) + \tag{20}$$

$$(n-2(c+1))\int_0^1 t^{c-1}\left[\varepsilon(ty|x| - \frac{x}{|x|}) - \sum_{k=0}^{k_0}\frac{t^k|x|^k|y|^k}{2k+n-2}\sum_{i=1}^{h_k} H_k^{(i)}(\frac{x}{|x|})H_k^{(i)}(\frac{y}{|y|})\right]dt\,.$$

We can reduce to (D) and solve (R) and (N) problems for Laplace operator $\Delta u = 0$ in the standard way. Put $v = \Lambda u + cu$. Then $\Delta v = \Delta\Lambda u + c\Delta u = (\Lambda + 2)\Delta u + c\Delta u = 0$ in B_1 and $v|_{S_1} = \frac{\partial u}{\partial n} + cu|_{S_1} = f$. Applying Poisson formula for the above (D) problem we obtain $v(x) \in C^{k+2,\alpha}(\bar{B}_1)$. If $c > 0$ we can apply Proposition 1 (i) to find $u = \int_0^1 v(tx)dt$, while $v(x) = \frac{1-\rho^2}{\omega_n}\int_{S_1}\frac{f(y)}{|x-y|^n}dS_y$ etc.

In the (N) case $\int_{S_1} f dS_1 = 0$, $lim_{\rho\to 0}v = lim_{\rho\to 0}\Lambda u = 0$, i.e., $v(0) = 0 \Rightarrow u = \int_0^1 \frac{v(tx)}{t}dt$ etc.

3. We shall prove now Theorem 2. We are looking for the solution of (11) of the form (10) Evidently, $\left| \begin{array}{l} \Delta u_1 = 0 \text{ in } B_1, \\ u_1|_{S_1} = \varphi_1. \end{array}\right.$

For $j \geq 1$

$$B_j(u) = \sum_{l=1}^{m}\sum_{k+|\alpha|=j} a_{k\alpha}(x)D_x^\alpha D_\rho^k((\rho^2-1)^{l-1}u_l).$$

Having in mind that $(\rho^2-1)^{l-1} = (\rho-1)^{l-1}(\rho+1)^{l-1}$ vanishes of sharp order $l-1$ at $S_1 = \{\rho = 1\}$ we shall consider three different cases:

(a) $l - 1 = j \iff l = j+1$

(b) $j + 1 < l \iff l - 1 > j$

(c) $j + 1 > l \iff l - 1 < j$.

In the case (a) $B_j u|_{S_1}$ contains the term

$$\sum_{k+|\alpha|=j} a_{k\alpha}(x)D_x^\alpha D_\rho^k((\rho-1)^j(\rho+1)^j u_{j+1})|_{S_1} = A_j(x)u_{j+1},$$

as $D_x^\alpha D_\rho^k(\rho-1)^j = 0$ if $|\alpha| + k < j$, A_j is given by (12).

(b) implies that $l - 1 > j \Rightarrow D_x^\alpha D_\rho^k((\rho^2-1)^{l-1}u_l)|_{S_1} = 0$ if $|\alpha| + k = j$, i.e., $B_j(u)$ does not contain u_l or its derivatives for $l > j+1$. Assume now that $l - 1 < j$. Then $D_x^\alpha D_\rho^k((\rho^2-1)^{l-1}u_l)$ can contain u_l and its derivatives for $l < j+1$ according to the Leibnitz rule. Certainly, $|\alpha| + k = j$.

One can easily see that for $j \geq 1$

$$\varphi_{j+1} = B_j(u)|_{S_1} = A_j u_{j+1} + \tag{21}$$

a linear combination of $u_1, \ldots, u_j$ and their derivatives of some order.

The boundary value problem (11) decomposes to the solvability of m -Dirichlet boundary value problems of the type (13). As u_1 can be found directly via Poisson formula the other solutions can be constructed inductively via (21), respectively the solution u of (11) is written in the form (10).

We shall complete this paper with several examples for solutions into explicit form of the boundary value problem for Δ^2 and Δ^3.

Example 5. *(a)*

$$\left|\begin{array}{l}\Delta^2 u = 0 \ \text{ in } \ B_1 \\ u|_{S_1} = \varphi_1, Au + B\frac{\partial u}{\partial n} + \Delta u|_{S_1} = \varphi_2\end{array}\right.$$

The solution can be found in the form $u = u_1 + (\rho^2 - 1)u_2$, $\Delta u_1 = 0$, $\Delta u_2 = 0$ *in* B_1. *Evidently,*

$$\left|\begin{array}{l}\Delta u_1 = 0, B_1 \\ u_1|_{S_1} = \varphi_1,\end{array}\right. \tag{22}$$

while $\Delta u = 2nu_2 + 4\Lambda u_2$, $\frac{\partial u}{\partial n}|_{S_1} = \frac{\partial u_1}{\partial \rho}|_{S_1} + 2u_2|_{S_1}$. *Therefore,*

$$\left|\begin{array}{l}\Delta u_2 = 0 \\ 4\frac{\partial u_2}{\partial \rho} + 2(B+n)u_2|_{S_1} = \varphi_2 - Au_1 - B\frac{\partial u_1}{\partial \rho}|_{S_1}\end{array}\right. \tag{23}$$

Assume that $B + n \geq 0$. *At first we find the solution* u_1 *of (22) and then the solution of the Robin problem for (23) via Robin (Neumann) function (5) and [4]. On the other hand, define in* B_1 $v = Au_1 + B\Lambda u_1 + 2(B+n)u_2 + 4\Lambda u_2$. *Evidently,*

$$\left|\begin{array}{l}\Delta v = 0 \ \text{ in } \ B_1 \\ lim_{\rho\to 1} v = Au_1 + B\frac{\partial u_1}{\partial \rho} + 2(B+n)u_2 + 4\frac{\partial u_2}{\partial \rho} = \varphi_2\end{array}\right. \tag{24}$$

Under the assumption $c = \frac{1}{2}(B+n) > 0$ *we find* $v(x) = \int_{S_1} \frac{\varphi_2(y)dS_y}{|x-y|^n}$ *and then solve the equation* $(\Lambda + c)u_2 = \frac{1}{4}(v - Au_1 - B\Lambda u_1)$ *via (9);* $u_1 = \frac{1}{\omega_n}\int_{S_1} \frac{\varphi_1(y)}{|x-y|^n} dS_y$.

(b). $\left|\begin{array}{l}\Delta^3 u = 0 \ \text{ in } \ B_1, n \geq 3 \\ u|_{S_1} = \varphi_1 \\ Au + \frac{\partial u}{\partial n}|_{S_1} = \varphi_2, A = const. \\ B\frac{\partial^2 u}{\partial n^2} + \Delta^2 u|_{S_1} = \varphi_3, B = const.\end{array}\right.$

The solution u (if it exists) has the form

$$u = u_1 + (\rho^2 - 1)u_2 + (\rho^2 - 1)^2 u_3, \Delta u_j = 0, j = 1, 2, 3.$$

Thus, $\left|\begin{array}{l}\Delta u_1 = 0, B_1 \\ u_1|_{S_1} = \varphi_1,\end{array}\right.$ $\left|\begin{array}{l}\Delta u_2 = 0, B_1 \\ u_2|_{S_1} = \frac{\varphi_2 - Au_1}{2}|_{S_1}.\end{array}\right.$

One can compute that $\Delta^2 u = \Delta^2(\rho^2 - 1)^2 u_3 = 8(n(n+2) + 4(n+1)\Lambda + 4\Lambda^2)u_3$ *and* $\frac{\partial^2 u}{\partial \rho^2}|_{S_1} = \frac{\partial^2 u_1}{\partial \rho^2} + 2u_2 + 4\frac{\partial u_2}{\partial \rho} + 8u_3|_{S_1}$, *i.e.,*

$$\left|\begin{array}{l}\Delta u_3 = 0, B_1 \\ 8\left[n(n+2) + B + 4(n+1)\Lambda + 4\Lambda^2\right]u_3|_{S_1} = \varphi_3 - B(2u_2 + 4\frac{\partial u_2}{\partial \rho} + \frac{\partial^2 u_1}{\partial \rho^2})|_{S_1}.\end{array}\right.$$

One can easily check that $\Lambda|_{S_1} = \frac{\partial}{\partial \rho}|_{S_1}$, $(\Lambda^2 - \Lambda)|_{S_1} = \frac{\partial^2}{\partial \rho^2}|_{S_1}$. *Having in mind the identities* $\Delta\Lambda = (\Lambda + 2)\Delta$ *and* $\Delta L^2 = (\Lambda + 2)^2\Delta$ *we have that* $v = 8(n(n+2) + B + 4(n+1)\Lambda + 4\Lambda^2)u_3 + B(2u_2 + 4\Lambda u_2 + (\Lambda^2 - \Lambda)u_1)$ *is harmonic in* B_1 *and its trace on* $S_1 : v|_{S_1} = \varphi_3$. *Consequently, v is given by the Poisson formula,* u_1, u_2 *are well known and we must solve with respect to* u_3 *the following equation in* B_1

$$\left[n(n+2) + B + 4(n+1)\Lambda + 4\Lambda^2\right]u_3 = \frac{1}{8}[v - B(2u_2 + 4\Lambda u_2 + (\Lambda^2 - \Lambda)u_1] = w(x), x \in B_1. \tag{25}$$

As the roots of the equation $4\lambda^2 + 4(n+1)\lambda + (n+2)n = 0$ *are* $\lambda_1 = -\frac{n}{2}$, $\lambda_2 = \frac{-n-2}{2}$ *we can see that for* $0 \leq B_1 \leq 1$ *the roots of* $P(\lambda) = 4\lambda^2 + 4(n+1)\lambda + (B + n(n+2))$ *remain negative, while for* $B > 1$ *they have negative real parts:* μ_1, μ_2, $\Re\mu_1 < 0$, $\Re\mu_2 < 0$.

To solve (25) we use methods from the operational calculus (Laplace transformation). $\frac{1}{P(\lambda)} = \frac{A_1}{\lambda-\lambda_1} + \frac{A_2}{\lambda-\lambda_2}$, *if* $B \neq 1$, *i.e.,* $\lambda_1 \neq \lambda_2$ *are roots of* $P(\lambda) = 0$, $\Re\lambda_1 < 0$, $\Re\lambda_2 < 0$, *where* $A_j = \frac{1}{P'(\lambda_j)}$, $j = 1, 2$. *If* $B = 1$, *i.e.,* $\lambda_1 = \lambda_2 = -\frac{n+1}{2}$

$$\frac{1}{P(\lambda)} = \frac{1}{4}\frac{1}{(\lambda-\lambda_1)^2}.$$

Having in mind Proposition 1 (i), (ii) we can conclude that $u_3 = A_1(\Lambda-\lambda_1)^{-1}w + A_2(\Lambda-\lambda_2)^{-1}w$, *respectively* $u_3 = \frac{1}{4}(\Lambda-\lambda_1)^{-2}w$ *for* $B = 1$. *Of course,* $-\Re\lambda_j > 0$ *for* $j = 1, 2$.

4. Discussion

The bvp (D),(R),(N) are classical and participate in each manual and handbook on PDE as they describe important physical stationary processes corresponding to the propagation of electromagnetic and sound waves and the heat propagation. The problem was to construct explicitly the Green function for (R) problem in the simple domain B_1. The (D) and (N) problems for $n = 2$ have numerous applications in the complex analysis. B.Wirth in 2019 found and interesting link between the Green function for (N) and the Green function for the forward problem of the electroencephalography (EEG). It is well known that the Robin problem for the electrostatic equilibrium in B_1 is closely linked with (N). Concerning the biharmonic operator, it has applications to the theory of elasticity (clamped plate, the buckled plate problem and others) and solid mechanics. In the field of bvp for polyharmonic operators are actively working K. Dang, F. Gazzola, A. Gomez-Polanko, H. Grunau, V. Karachik, G. Sweers and many others.

Below we propose several possible generalizations of the results of this paper. They are in two different directions. The first one is elliptic (Lopatinskii) type boundary value problem with possible applications in mechanics and the second one is non-elliptic boundary value problem. In the first case a-priori estimates and Fredholm type theorems for existence of classical (Hölder) and generalized (Sobolev) solutions are proved. In the second case the results are a few. For the biharmonic operator overdetermination appears, i.e., the boundary data are not independent. One can construct examples of boundary value problems with infinite dimensional kernel or cokernel too.

Because of the above reasons we shall formulate the following open problems.

1. To construct Green function to the Dirichlet problem for the polyharmonic operator in a domain with $C^{1,1}$ (C^2) boundary. For the classical Laplace operator it was done by Z. Zhao in J. Math. Anal. Appl., 116, 309–334, 1986. It is interesting to estimate from below and above the Green function.

2. To construct the Green function for the non homogeneous polyharmonic operator satisfying elliptic (Lopatinskii) type boundary conditions with variable coefficients in the unit ball B_1 and eventually in some bounded domains with smooth boundary.

Author Contributions: Conceptualization, P.P. and A.S.; methodology, P.P. and A.S.; investigation, P.P. and A.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

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