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RESEARCH ARTICLE

On standing waves and gradient-flow for the Landau–De Gennes model of nematic liquid crystals

Daniele Barbera[1] · Vladimir Georgiev[1,2,3]

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Abstract

The article treats the existence of standing waves and solutions to gradient-flow equation for the Landau–De Gennes models of liquid crystals, a state of matter intermediate between the solid state and the liquid one. The variables of the general problem are the velocity field of the particles and the Q-tensor, a symmetric traceless matrix which measures the anisotropy of the material. In particular, we consider the system without the velocity field and with an energy functional unbounded from below. At the beginning we focus on the stationary problem. We outline two variational approaches to get a critical point for the relative energy functional: by the Mountain Pass Theorem and by proving the existence of a least energy solution. Next we describe a relationship between these solutions. Finally we consider the evolution problem and provide some Strichartz-type estimates for the linear problem. By several applications of these results to our problem, we prove via contraction arguments the existence of local solutions and, moreover, global existence for initial data with small L^2-norm.

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✉ Vladimir Georgiev
georgiev@dm.unipi.it

Daniele Barbera
daniele.barbera96@gmail.com

1 Department of Mathematics, University of Pisa, Largo B. Pontecorvo 5, Pisa 56127, Italy

2 Faculty of Science and Engineering, Waseda University, 3-4-1, Okubo, Shinjuku-ku, Tokyo 169-8555, Japan

3 Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, Acad. Georgi Bonchev Str., Block 8, Sofia 1113, Bulgaria

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1 Introduction

There are different models describing nematic liquid crystals (we avoid long list of references, but refer to the recent one [25] that gives a very detailed panorama of the state of the art). We are interested in the Q-tensor model, i.e. De Gennes type model (see [1, 9, 16]) for a molecule in a flux of nematic liquid crystals, which is a state of matter intermediate between the solid state and the liquid one. Concerning the physical modeling of nematic liquid crystals, we refer to [1–4, 23, 25]. The model has been actively studied during the last years both on torus [7, 24], bounded domains [13], in $\mathbb{R}^n$ [17, 18], and in exterior domains [8, 19, 20]. In general the model is described by a system of Navier–Stokes equations and an equation for the Q-tensor:

$$\begin{cases} \partial_t Q + (u\cdot\nabla)Q - (\omega(u)Q - Q\omega(u)) = \Delta_x Q - L[\partial F(Q)], \\ \partial_t u - \mathrm{div}_x(u\otimes u) + \nabla_x p = \Delta_x u + \mathrm{div}_x(-Q\Delta_x Q + \Delta_x Q Q - \nabla_x Q \odot \nabla_x Q), \\ \mathrm{div}_x u = 0, \end{cases}$$

where $Q = (q_{ij})_{i,j=1,2,3}$ and

$$\omega(u) := \frac{1}{2}\left(\nabla_x u - \nabla_x^t u\right), \quad L[A] := A - \frac{1}{3}\,\mathrm{tr}(A)\,\mathrm{Id}, \quad A \in M(3,\mathbb{R}),$$
$$(\nabla_x Q \odot \nabla_x Q)_{ij} := \sum_{\alpha,\beta} \partial_{x_i} q_{\alpha\beta}\, \partial_{x_j} q_{\alpha\beta}, \quad \Delta Q = (\Delta q_{ij})_{i,j}, \quad i,j = 1,2,3.$$

We are looking for the functions

$$u\colon \mathbb{R}^+\times\mathbb{R}^3 \to \mathbb{R}^3, \quad Q\colon \mathbb{R}^+\times\mathbb{R}^3 \to S_0(3,\mathbb{R}), \quad p\colon \mathbb{R}^3 \to \mathbb{R},$$

where $S_0(3,\mathbb{R})$ denotes the space of 3×3 symmetric matrices with zero trace. The function u represents the velocity field of the molecules, p represents the pressure of the fluid and Q is the *De Gennes Q-tensor*.

In the model, F denotes the energy of liquid crystals and it is given by

$$F(Q) = \frac{a}{2}\,|Q|^2 + \frac{b}{3}\,\mathrm{tr}(Q^3) + \frac{c}{4}\,|Q|^4, \quad Q \in M(3,\mathbb{R}),$$

where $a, b, c \in \mathbb{R}$ and $|\cdot|$ is the Frobenius norm. It is easy to prove that

$$\partial F(Q) = aQ + b(Q^2)^{\mathrm{T}} + c|Q|^2 Q \quad \text{for all} \quad Q \in M(3,\mathbb{R}).$$

In the case $u = 0$ the model reduces to the gradient flow equation

$$\partial_t Q = \Delta_x Q - L[\partial F(Q)].$$

Our first goal is to find standing waves, i.e. solutions to the equation

$$\begin{aligned} \Delta Q = L[\partial F(Q)] &= \partial F(Q) - \frac{1}{3}\operatorname{tr}(\partial F(Q))\mathrm{Id} \\ &= aQ + b\Big(Q^2 - \frac{1}{3}|Q|^2\mathrm{Id}\Big) + c|Q|^2 Q \end{aligned} \tag{1}$$

for $Q \in H^1(\mathbb{R}^3; S_0(3,\mathbb{R}))$.

We shall use the following parametrization for the matrices of $S_0(3,\mathbb{R})$:

$$Q(q(x)) = \begin{pmatrix} q_1(x) & q_3(x) & q_4(x) \\ q_3(x) & q_2(x) & q_5(x) \\ q_4(x) & q_5(x) & -q_1(x) - q_2(x) \end{pmatrix}, \quad q = (q_1, q_2, q_3, q_4, q_5).$$

It can be proved that every critical point q of the following functional:

$$\begin{aligned} J(q) &= \int \frac{1}{2}|\nabla Q(q)|^2 + F(Q(q))\,dx \\ &= \int |\nabla q|^2 + \nabla q_1 \cdot \nabla q_2\,dx \\ &\qquad + \int a(|q|^2 + q_1 q_2) + \frac{b}{3}\operatorname{tr}(Q(q)^3) + c(|q|^2 + q_1 q_2)^2\,dx \end{aligned}$$

gives a matrix $Q(q)$ which satisfies (1).

The part of our work, related to the stationary problem, was announced in [12]. However, for completeness we present the proofs or key points in the proofs, so that the article is self-contained.

2 Least energy solution

2.1 Existence of a least energy solution

We study the case $a > 0$ and $c < 0$.

Definition 2.1 Let $d \geqslant 3$, $n \geqslant 2$, $p = \frac{2d}{d-2}$, $G \in C^1(\mathbb{R}^n \setminus \{0\})$ and

$$\mathcal{C} := \Big\{ v \in L^p(\mathbb{R}^d; \mathbb{R}^n) \mid \nabla v \in L^2(\mathbb{R}^d; M(n,d,\mathbb{R})),\ G(v) \in L^1(\mathbb{R}^d) \Big\};$$

let q be a solution of the system

$$\begin{cases} -2\Delta v - \alpha(\Delta v_1 e_2 + \Delta v_2 e_1) = g(v) & \text{in } \mathcal{D}', \\ v \in \mathcal{C}, \end{cases} \tag{2}$$

where $\alpha \in [0, 1]$ and e_1, e_2 are the following vectors of $\mathbb{R}^n$:

$$e_1 = (1, 0, \ldots, 0)^{\mathrm{T}}, \quad e_2 = (0, 1, 0, \ldots, 0)^{\mathrm{T}},$$

and

$$g(v) := \begin{cases} (\partial_{v_i} G(v))_{i=1}^n, & v \neq 0, \\ 0, & v = 0. \end{cases}$$

We say that q is a *least energy solution* of (2) if $q \in \mathcal{C} \setminus \{0\}$ and

$$J(q) = \inf \bigl\{ J(v) \mid v \in \mathcal{C} \setminus \{0\},\ v \text{ is a solution of (2)} \bigr\},$$

where

$$J(v) := \int |\nabla v|^2 + \alpha \nabla v_1 \nabla v_2 - G(v)\, dx, \quad v \in \mathcal{C}.$$

Following [6], we consider a function $G \in C^1(\mathbb{R}^n \setminus \{0\})$ which satisfies the following properties for $p = \frac{2d}{d-2}$:

- $G(0) = 0$;
- $\limsup_{|v| \to +\infty} |v|^{-p} G(v) \leqslant 0$;
- $\limsup_{|v| \to 0} |v|^{-p} G(v) \leqslant 0$;
- there exists $\xi_0 \in \mathbb{R}^n$ such that $G(\xi_0) > 0$;
- for all $\gamma > 0$ there exists C_γ such that

$$|G(v + w) - G(v)| \leqslant \gamma [|G(v)| + |v|^p] + C_\gamma [|G(w)| + |w|^p + 1] \tag{3}$$

  for all $v, w \in \mathbb{R}^n$;
- there exists a constant $C > 0$ such that

$$g(v) \leqslant C + C|v|^{p-1} \quad \text{for all } \ v \in \mathbb{R}^n.$$

In [6] the existence of a least energy solution is established for $\alpha = 0$. We need a small generalization to show it for $\alpha \in [0, 1]$.

Theorem 2.2 *Let $\alpha \in [0, 1]$, $d \geqslant 3$, $n \geqslant 2$ and $G : \mathbb{R}^n \to \mathbb{R}$ satisfy the above hypotheses, then there exists a least energy solution $q : \mathbb{R}^d \to \mathbb{R}^n$ of (2).*

Proof The proof is almost the same as in [6], so we just list the steps:
Step 1: Let us consider the problem

$$T = \inf \left\{ \int |\nabla q|^2 + \alpha \nabla q_1 \nabla q_2\, dx \;\Big|\; q \in \mathcal{C}, \int G(q)\, dx \geqslant 1 \right\}.$$

Let $\{q^j\}_j$ be a minimizing sequence for T, then $\{\nabla q^j\}_j$ is bounded in $L^2(\mathbb{R}^d;\mathbb{R}^n)$ and $\{q^j\}_j$ is bounded in $L^p(\mathbb{R}^d;\mathbb{R}^n)$. It can be proved also that, passing to a subsequence,

$$q^j \xrightarrow{L^p} q, \quad \nabla q^j \xrightarrow{L^2} \nabla q,$$

for $q \in \mathcal{C}\setminus\{0\}$.

Step 2: $\int G(q)\,dx = 1$ and

$$\int |\nabla q|^2 + \alpha \nabla q_1 \nabla q_2\,dx = T.$$

In particular, $\nabla q^j \to \nabla q$ in $L^2(\mathbb{R}^d;\mathbb{R}^n)$ and therefore $q^j \to q$ in $L^p(\mathbb{R}^d;\mathbb{R}^n)$.

Step 3: Let $\overline{q}(x) = q(\theta x)$ with $\theta = \frac{d}{T(d-2)}$, then in $\mathcal{D}'$ there holds

$$-\Delta\overline{q} - \alpha(\Delta\overline{q}_1 e_2 + \Delta\overline{q}_2 e_1) = g(\overline{q}) = \nabla G(\overline{q}).$$

Step 4: If $q \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^n)\cap\mathcal{C}$ is a solution of (2), then there holds

$$\int |\nabla q|^2 + \alpha\nabla q_1\nabla q_2\,dx = \frac{d}{d-2}\int G(q)\,dx.$$

Step 5: For all $v \in L^\infty_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^n)\cap\mathcal{C}$ non-zero and satisfying (2), we have

$$0 < \int |\nabla\overline{q}|^2 + \alpha\nabla\overline{q}_1\nabla\overline{q}_2 - G(\overline{q})\,dx \leqslant \int |\nabla v|^2 + \alpha\nabla v_1\nabla v_2 - G(v)\,dx.$$

Step 6: We have the following regularity result:

Theorem 2.3 *Let $q \in \mathcal{C}$ be a solution of (2) with $g(q) \in L^1_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^n)$, then it satisfies the following properties:*

- $q \in W^{2,t}_{\mathrm{loc}}(\mathbb{R}^d;\mathbb{R}^n)$ *for any* $t < \infty$;
- $q \in L^\infty(\mathbb{R}^d;\mathbb{R}^n)$;
- q *tends to* 0 *as* $|x| \to +\infty$.

Moreover, if there are $C, \delta > 0$ such that

$$(2v + \alpha v_1 e_2 + \alpha v_2 e_1)\cdot g(v) \leqslant -C|v|^2 \quad \text{for all} \quad v \in B(0,\delta), \tag{4}$$

then q decays exponentially as $|x| \to +\infty$.

This result corresponds to [6, Theorem 2.3] for $\alpha = 0$. The proof for $\alpha \in [0,1]$ is analogous. Thanks to this theorem, we gain that $\overline{q}$ is a least energy solution in $\mathcal{C}$. □

Let us prove the following easy observation.

Lemma 2.4 *If $d \geqslant 3$, $p = \frac{2d}{d-2}$, $G\colon \mathbb{R}^n \to \mathbb{R}$ is continuous in $\mathbb{R}^n$ and*

$$\lim_{|v|\to+\infty} |v|^{-p}|G(v)| = 0, \tag{5}$$

then G satisfies the condition (3).

Proof Let us suppose by contradiction that there is $\gamma > 0$ such that for any $m \in \mathbb{N}$ we can find $u_m, w_m \in \mathbb{R}^n$ which satisfy

$$|G(u_m + w_m) - G(u_m)| > \gamma[|G(u_m)| + |u_m|^p] + m[|G(w_m)| + |w_m|^p + 1].$$

Divide the above inequality by $m[|G(w_m)| + |w_m|^p + 1]$, we have

$$\begin{aligned}\frac{\gamma[|G(u_m)| + |u_m|^p]}{m[|G(w_m)| + |w_m|^p + 1]} + 1 &< \frac{|G(u_m + w_m) - G(u_m)|}{m[|G(w_m)| + |w_m|^p + 1]}\\ &\leqslant \frac{|G(u_m + w_m)| + |G(u_m)|}{m[|G(w_m)| + |w_m|^p + 1]}.\end{aligned}$$

Let us show that $\{u_m + w_m\}_m$ is bounded: if $\{u_m + w_m\}_m$ were unbounded, then passing to a subsequence we can suppose that $|u_m + w_m| \to +\infty$ as $m \to +\infty$. Then, for any $C > 0$, by our hypothesis (5), $|G(u_m + w_m)| \leqslant C|u_m + w_m|^p$, so

$$\begin{aligned}\frac{\gamma[|G(u_m)| + |u_m|^p]}{m[|G(w_m)| + |w_m|^p + 1]} + 1 &< \frac{C|u_m + w_m|^p + |G(u_m)|}{m[|G(w_m)| + |w_m|^p + 1]}\\ &\leqslant \frac{C2^{p-1}(|u_m|^p + |w_m|^p) + |G(u_m)|}{m[|G(w_m)| + |w_m|^p + 1]}.\end{aligned}$$

Let us distinguish two cases:

- Suppose $|u_m| \leqslant R$ for some $R > 0$. This means that $|w_m| \to +\infty$. By continuity of G, we can find $K > 0$ such that $|G(u_m)| \leqslant K$. Therefore

$$1 \leqslant \frac{C2^{p-1}(R^p + |w_m|^p) + K}{m[|G(w_m)| + |w_m|^p + 1]} \lesssim \frac{|w_m|^p + K}{m|w_m|^p} \xrightarrow{m\to+\infty} 0,$$

  which is a contradiction.
- Suppose $\{u_m\}_m$ is unbounded; then passing to a subsequence, we can suppose $|u_m| \to +\infty$ as $m \to +\infty$. Again by our hypothesis, for any $C > 0$, $|G(u_m)| \leqslant C|u_m|^p$, then

$$\begin{aligned}\frac{\gamma|u_m|^p}{m[|G(w_m)| + |w_m|^p + 1]} + 1 &\leqslant \frac{\gamma[|G(u_m)| + |u_m|^p]}{m[|G(w_m)| + |w_m|^p + 1]} + 1\\ &< \frac{C(2^{p-1} + 1)|u_m|^p + C2^{p-1}|w_m|^p}{m[|G(w_m)| + |w_m|^p + 1]}.\end{aligned}$$

If we take $C = \gamma/(2^{p-1}+1)$, we get

$$1 \leqslant \frac{C2^{p-1}|w_m|^p}{m[|G(w_m)| + |w_m|^p + 1]} \lesssim \frac{1}{m} \xrightarrow{m\to+\infty} 0,$$

which is a contradiction.

Thus, $\{u_m + w_m\}_m$ is bounded. Now we are ready to conclude:

- If $\{u_m\}_m$ is unbounded, passing to a subsequence we have $|G(u_m)| \leqslant \gamma|u_m|^p$. So,

$$\begin{aligned}\frac{\gamma[|G(u_m)| + |u_m|^p]}{m[|G(w_m)| + |w_m|^p + 1]} + 1 &< \frac{|G(u_m + w_m)| + |G(u_m)|}{m[|G(w_m)| + |w_m|^p + 1]}\\ &\leqslant \frac{|G(u_m + w_m)| + \gamma|u_m|^p}{m[|G(w_m)| + |w_m|^p + 1]}.\end{aligned}$$

  By continuity of G, we have that $\{G(u_m + w_m)\}_m$ is bounded and therefore

$$1 \leqslant \frac{\gamma|G(u_m)|}{m[|G(w_m)| + |w_m|^p + 1]} + 1 < \frac{|G(u_m + w_m)|}{m[|G(w_m)| + |w_m|^p + 1]} \xrightarrow{m\to+\infty} 0,$$

  which is a contradiction.
- If $\{|u_m|\}_m$ is bounded, there is $K > 0$ such that $|G(u_m + w_m)|, |G(u_m)| \leqslant K$. Therefore

$$1 \leqslant \frac{\gamma[|G(u_m)| + |u_m|^p]}{m[|G(w_m)| + |w_m|^p + 1]} + 1 < \frac{|G(u_m + w_m)| + |G(u_m)|}{m[|G(w_m)| + |w_m|^p + 1]} \leqslant \frac{2K}{m} \to 0,$$

  which is a contradiction. □

In our case $n = 5$, $d = 3$, $p = 6$, $\alpha = 1$ and

$$G(q) = -F(Q(q)) = -a(|q|^2 + q_1q_2) - \frac{b}{3}\operatorname{tr}(Q(q))^3 - c(|q|^2 + q_1q_2)^2. \quad (6)$$

Taking into account Lemma 2.4 it is easy to show that $g(q) = \nabla G(q)$ satisfies all conditions of Theorem 2.2. Moreover, it is easy to see that it also satisfies (4). Thus we have proved:

Corollary 2.5 *Let $a, b, c \in \mathbb{R}$ with $a > 0$ and $c < 0$, then there exists a least energy solution $\overline{q}$ in $H^1(\mathbb{R}^3; \mathbb{R}^5)$ for the system*

$$-2\Delta q - \Delta q_1 e_2 - \Delta q_2 e_1 = -a(|q|^2 + q_1q_2) - \frac{b}{3}\operatorname{tr}(Q(q))^3 - c(|q|^2 + q_1q_2)^2.$$

Moreover, $\overline{q}$ decays exponentially as $|x| \to +\infty$.

2.2 Least energy solution as a saddle point

Let us recall the functional related to our model:

$$J(q) := \int |\nabla q|^2 + \nabla q_1 \cdot \nabla q_2 + a(|q|^2 + q_1 q_2) \\ + \frac{b}{3}\operatorname{tr}(Q(q)^3) + c(|q|^2 + q_1 q_2)^2 \, dx. \tag{7}$$

We assumed $a > 0$ and $c < 0$. This choice of signs for the parameters gives J a particular structure: since $c < 0$, the functional is unbounded from below; however, the sign of a implies that J is positive near the zero. Such information suggests existence of a saddle point for the functional.

Definition 2.6 Let X be a Hilbert space and $J: X \to \mathbb{R}$ be a differentiable function, then we say that $\{q_n\}_n \subseteq X$ is a *Palais–Smale sequence* if $\{J(q_n)\}_n$ is bounded and $J'(q_n) \to 0$ in X^* as $n \to \infty$.

We say that $J: X \to \mathbb{R}$ satisfies the *Palais–Smale condition* (P.S.) when, for each Palais–Smale sequence, there is a convergent subsequence in X.

Theorem 2.7 (Mountain Pass Theorem [22]) *Let X be a Hilbert space and $J \in C^1(X)$ be such that:*

- *J satisfies P.S.;*
- *$J(0) = 0$ and there exist $\rho, \alpha > 0$ and $e \in X$ such that*
  - *if $\|q\| = \rho$ then $J(q) \geqslant \alpha$;*
  - *$\|e\| > \rho$ and $J(e) < 0$;*

then, for a fixed $\Gamma := \{\gamma \in C([0, 1]; X) \mid \gamma(0) = 0, \gamma(1) = e\}$, the quantity

$$M := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t))$$

is a critical value of J in X.

Remark 2.8 Thanks to the geometric hypotheses of the theorem, it is easy to see that any critical point related to M is not 0.

It is not difficult to check that our $J(q)$ belongs to $C^1(H^1(\mathbb{R}^d; \mathbb{R}^n))$ and that it satisfies the geometric hypotheses of the Mountain Pass Theorem. Conversely, it is difficult to prove the P.S. condition is fulfilled. For this reason we use the Mountain Pass Theorem for J restricted to the space $H^1_{\text{rad}}(\mathbb{R}^d; \mathbb{R}^n)$: this set is weakly closed in $H^1(\mathbb{R}^d; \mathbb{R}^n)$, in particular it is a Hilbert space. Moreover, its elements satisfy the following proposition:

Proposition 2.9 *Given $r \in \left(2, \frac{2d}{d-2}\right)$ with $d \geqslant 3$, then $H^1_{\text{rad}}(\mathbb{R}^d; \mathbb{R}^n) \hookrightarrow L^r(\mathbb{R}^d; \mathbb{R}^n)$ is a compact embedding.*

Its proof follows from [21, Lemma 1].

Proposition 2.10 *The function $J: H^1_{\text{rad}}(\mathbb{R}^3; \mathbb{R}^5) \to \mathbb{R}$ from (7) with $a > 0$ and $c < 0$ has a critical point $\overline{q} \neq 0$.*

Proof Let us denote $X := H^1_{\text{rad}}(\mathbb{R}^3;\mathbb{R}^5)$ and

$$\|q\|_{H^1_*} := \int |\nabla q|^2 + \nabla q_1 \nabla q_2 + a(|q|^2 + q_1 q_2)\,dx,$$

which is equivalent to $\|\cdot\|_{H^1}$.

The idea is to show that J satisfies the hypotheses of the Mountain Pass Theorem. Firstly, we verify the geometric hypotheses:

- Obviously, $J(0) = 0$.
- Let $\rho > 0$ and q be such that $\|q\|_{H^1_*} = \rho$.

$$\begin{aligned}\int |\nabla q|^2 + \nabla q_1 \nabla q_2 + a(|q|^2 + q_1 q_2) + \frac{b}{3}\operatorname{tr}(Q(q)^3) + c(|q|^2 + q_1 q_2)^2\,dx \\ = \|q\|^2_{H^1_*} + \int \frac{b}{3}\operatorname{tr}(Q(q)^3) + c(|q|^2 + q_1 q_2)^2\,dx.\end{aligned}$$

  Then, by Sobolev embeddings, there is $K > 0$ such that

$$J(q) \geqslant \rho^2 - K\|q\|^3_{H^1_*} - K\|q\|^4_{H^1_*} = \rho^2(1 - K\rho - K\rho^2).$$

  Therefore, we can find ρ sufficiently small and $\alpha > 0$ such that $J(q) \geqslant \alpha$ for any $\|q\|_{H^1_*} = \rho$.
- Fix $\overline{q} \in X$ different from 0 and let $q_\lambda := \lambda\overline{q}$ for $\lambda > 0$.

$$\begin{aligned}J(q_\lambda) &\lesssim \|\nabla q_\lambda\|^2_{L^2} + a\|q_\lambda\|^2_{L^2} + |b|\|q_\lambda\|^3_{L^3} + c\|q_\lambda\|^4_{L^4} \\ &= \lambda^2\|\nabla\,\overline{q}\|^2_{L^2} + a\lambda^2\|\overline{q}\|^2_{L^2} + \lambda^3|b|\|\overline{q}\|^3_{L^3} + c\lambda^4\|\overline{q}\|^4_{L^4}.\end{aligned}$$

  Therefore, since $c < 0$, $J(q_\lambda) \to -\infty$ as $\lambda \to +\infty$, so we can choose $e = q_\lambda$ with $\lambda \gg 1$.

Now let us show that J satisfies the P.S. condition: let $\{q^k\}_k \subseteq X$ be a Palais–Smale sequence, that is

$$\begin{cases} -2\Delta q^k - \Delta q^k_1 e_2 - \Delta q^k_2 e_1 + \nabla_q F(q^k) = dJ(q^k) \to 0 \quad \text{in} \quad X^*; \\ \left|\int |\nabla q^k|^2 + \nabla q^k_1 \nabla q^k_2 + F(q^k)\,dx\right| \leqslant K \quad \text{for all} \quad k \in \mathbb{N}. \end{cases}$$

We recall that, for all $v \in X$, $\|dJ(q^k)[v]\| \leqslant \delta_k\|v\| \to 0$, where $\delta_k := \|dJ(q^k)\|_{X^*}$. In particular,

$$\left|\int 2\nabla q^k \cdot \nabla v + \nabla q^k_1 \nabla v_2 + \nabla q^k_2 \nabla v_1 + \nabla_q F(q^k) \cdot v\,dx\right| \to 0. \tag{8}$$

If we take $v = q^k$, we get

$$\begin{cases} \left| \int 2|\nabla q^k|^2 + 2\nabla q_1^k \nabla q_2^k + \nabla_q F(q^k)\cdot q^k \, dx \right| = |dJ(q^k)[q^k]| \leqslant \delta_k \|q^k\|_{H^1_*}; \\ \left| \int |\nabla q^k|^2 + \nabla q_1^k \nabla q_2^k + F(q^k) \, dx \right| \leqslant K. \end{cases}$$

Firstly we consider

$$\begin{cases} -\int 2|\nabla q^k|^2 + 2\nabla q_1^k \nabla q_2^k + \nabla_q F(q^k)\cdot q^k \, dx \leqslant \delta_k \|q^k\|_{H^1_*}; & \text{(U1)} \\ \int |\nabla q^k|^2 + \nabla q_1^k \nabla q_2^k + F(q^k) \, dx \leqslant K. & \text{(U2)} \end{cases}$$

Calculate (U2) + $\frac{1}{2}$(U1):

$$\int F(q^k) - \frac{1}{2} \nabla_q F(q^k)\cdot q^k \, dx \leqslant \frac{\delta_k}{2} \|q^k\|_{H^1_*} + K.$$

One can check that

$$\nabla_q [F(Q(q))] = a q_a + b q_b + 2c(|q|^2 + q_1 q_2) q_a,$$

with

$$q_a := \begin{pmatrix} 2q_1 + q_2 \\ 2q_2 + q_1 \\ 2q_3 \\ 2q_4 \\ 2q_5 \end{pmatrix}, \quad q_b := \begin{pmatrix} q_3^2 - q_5^2 - q_2^2 - 2q_1q_2 \\ q_3^2 - q_4^2 - q_1^2 - 2q_1q_2 \\ 2(q_1q_3 + q_2q_3 + q_4q_5) \\ 2(q_3q_5 - q_2q_4) \\ 2(q_3q_4 - q_1q_5) \end{pmatrix}.$$

Notice that

$$q_a \cdot q = |Q(q)|^2, \quad q_b \cdot q = \operatorname{tr}(Q(q)^3).$$

Thanks to this, we get

$$\begin{aligned} &\int F(q^k) - \frac{1}{2} \nabla_q F(q^k)\cdot q^k \, dx \\ &\qquad = \int -\frac{b}{6} \operatorname{tr}(Q(q^k)^3) - \frac{c}{4} |Q(q^k)|^4 dx \leqslant \frac{\delta_k}{2} \|q^k\|_{H^1_*} + K. \end{aligned} \tag{9}$$

Now, consider

$$\begin{cases} \int 2|\nabla q^k|^2 + 2\nabla q_1^k \nabla q_2^k + \nabla_q F(q^k)\cdot q^k \, dx \leqslant \delta_k \|q^k\|_{H^1_*}; & \text{(L1)} \\ \int |\nabla q^k|^2 + \nabla q_1^k \nabla q_2^k + F(q^k)\, dx \geqslant -K. & \text{(L2)} \end{cases}$$

Calculate (L1) − 2(L2):

$$\begin{aligned} &\int \nabla_q F(q^k)\cdot q^k - 2F(Q(q^k))\, dx \\ &\qquad = \int \frac{b}{3}\operatorname{tr}(Q(q^k)^3) + \frac{c}{2}|Q(q^k)|^4 \, dx \leqslant 2K + \delta_k \|q^k\|_{H^1_*}. \end{aligned} \tag{10}$$

Combining (9) and (10), we get

$$\left| \int \frac{b}{3}\operatorname{tr}(Q(q^k)^3) + \frac{c}{2}|Q(q^k)|^4 \, dx \right| \leqslant \delta_k \|q^k\|_{H^1_*} + 2K.$$

We have

$$\|q^k\|^2_{H^1_*} + \int b\operatorname{tr}(Q(q^k)^3) + c|Q(q^k)|^4 \, dx = dJ(q^k)[q^k] \leqslant \delta_k \|q^k\|_{H^1_*}.$$

Since $c < 0$, we have

$$\begin{aligned} &\int b\operatorname{tr}(Q(q^k)^3) + c|Q(q^k)|^4 \, dx \\ &\qquad \geqslant \int b\operatorname{tr}(Q(q^k)^3) + \frac{3}{2}c|Q(q^k)|^4 \, dx \geqslant -6K - 3\delta_k \|q^k\|_{H^1_*}, \end{aligned}$$

therefore,

$$\|q^k\|^2_{H^1_*} \leqslant 4\delta_k \|q^k\|_{H^1_*} + 6K.$$

So $\{q^k\}_k$ is bounded in X. Passing to a subsequence, we can suppose $q^k \rightharpoonup \overline{q}$ in X.

Now let us show the strong convergence. Take $v = q^k - \overline{q}$ in (8), then

$$\begin{aligned} dJ(q^k)[q^k - \overline{q}] = &\int 2\nabla q^k \cdot \nabla(q^k - \overline{q}) + \nabla q_1^k \nabla(q^k - \overline{q})_2 + \nabla q_2^k \nabla(q^k - \overline{q})_1 \, dx \\ &+ \int a q_a^k (q^k - \overline{q}) + b q_b^k (q^k - \overline{q}) + c|Q(q^k)|^2 q_a^k (q^k - \overline{q})\, dx. \end{aligned}$$

Denote

$$I_k := \int 2\nabla\, \overline{q} \cdot \nabla(q^k - \overline{q}) + \nabla\, \overline{q}_1 \nabla(q^k - \overline{q})_2 + \nabla\, \overline{q}_2 \nabla(q^k - \overline{q})_1 + a\overline{q}_a (q^k - \overline{q})\, dx.$$

It is clear that $q_a^k - \overline{q}_a = (q^k - \overline{q})_a$, so

$$dJ(q^k)[q^k - \overline{q}] - I_k = \|q^k - \overline{q}\|^2_{H^1_*} + \int bq_b^k(q^k - \overline{q}) + c|Q(q^k)|^2 q_a^k(q^k - \overline{q})\,dx.$$

We notice that $I_k = 2\langle \overline{q}, q^k - \overline{q}\rangle_{H^1_*}$ so, by weak convergence, $I_k \to 0$. We are then in the following situation:

$$\|q^k - \overline{q}\|^2_{H^1_*} + \int bq_b^k(q^k - \overline{q}) + c|Q(q^k)|^2 q_a^k(q^k - \overline{q})dx \to 0.$$

We need to show that the last two terms tend to 0 as $k \to +\infty$. Thanks to Proposition 2.9, we can suppose also that $q^k \to \overline{q}$ in $L^4(\mathbb{R}^3; \mathbb{R}^5)$. Then

$$\begin{cases} \left|\int |Q(q^k)|^2 q_a^k(q^k - \overline{q})\,dx\right| \lesssim \int |q^k|^3|q^k - \overline{q}|\,dx \leqslant \|q^k\|^3_{L^4}\|q^k - \overline{q}\|_{L^4}; \\ \left|\int q_b^k(q^k - \overline{q})\,dx\right| \lesssim \int |q^k|^2|q^k - \overline{q}|\,dx \leqslant \|q^k\|^2_{L^{8/3}}\|q^k - \overline{q}\|_{L^4}. \end{cases}$$

Since $8/3 \in [2, 6]$ and $\{q^k\}_k$ is bounded in $H^1(\mathbb{R}^3; \mathbb{R}^5)$, we are done. □

The least energy solution $\overline{q}$ built in the Sect. 2.1 can be seen as a saddle point of the functional $J(q)$, similarly to the one given by the Mountain Pass Theorem:

$$J(\overline{q}) = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J(\gamma(t))$$

where

$$\Gamma := \left\{\gamma \in C^1([0, 1]; H^1(\mathbb{R}^d; \mathbb{R}^n)) \mid \gamma(0) = 0,\ J(\gamma(1)) < 0\right\}.$$

In fact, with the same techniques as in [14], one can prove

Theorem 2.11 *Let $J : H^1(\mathbb{R}^d; \mathbb{R}^n) \to \mathbb{R}$ with $d \geqslant 3$ and $n \geqslant 2$ be defined as*

$$J(q) = \int |\nabla q|^2 + \alpha \nabla q_1 \nabla q_2 - G(q)\,dx,$$

where $\alpha \in [0, 1]$; let G satisfy the hypotheses of Theorem 2.2 and condition (4). Let us also suppose that there exists $\rho_0 > 0$ such that

$$0 < \|q\|_{H^1_*} \leqslant \rho_0 \implies (d-2)\int |\nabla q|^2 + \alpha \nabla q_1 \nabla q_2\,dx - d\int G(q)\,dx > 0,$$

where

$$\|v\|^2_{H^1_*} = \|v\|^2_{L^2} + \|\nabla v\|^2_* := \|v\|^2_{L^2} + \int |\nabla v|^2 + \alpha \nabla v_1 \nabla v_2\,dx, \quad v \in H^1(\mathbb{R}^d; \mathbb{R}^n);$$

let $\overline{q} \in \mathcal{C}$ *be the least energy solution built in Theorem 2.2, then*

$$J(\overline{q}) = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J(\gamma(t)).$$

Proof We give a sketch of the proof:

Step 1: Since G satisfies condition (4), thanks to Theorem 2.3 we have that $\overline{q} \in H^1(\mathbb{R}^d; \mathbb{R}^n)$. Let

$$\gamma(t) := \begin{cases} \overline{q}\left(\frac{Lx}{t}\right), & t > 0 \\ 0, & t = 0, \end{cases}$$

with $L > 0$. It can be seen that, for L sufficiently small, $J(\gamma(1)) < 0$, so $\gamma \in \Gamma$. On the other hand, thanks to Step 4 in the proof of Theorem 2.2, we have

$$\begin{aligned} \frac{d}{dt} J(\gamma(t)) &= \frac{d-2}{L^{d-2}} t^{d-3} \|\nabla \overline{q}\|_*^2 - \frac{d}{L^d} t^{d-1} \int G(\overline{q})\, dx \\ &= \frac{d-2}{L^{d-2}} t^{d-3} \|\nabla \overline{q}\|_*^2 \left(1 - \frac{t^2}{L^2}\right). \end{aligned}$$

So $J(\overline{q}) = J(\gamma(L)) = \max_{t \in [0,1]} J(\gamma(t))$. In particular, $J(\overline{q}) \geqslant M$, where

$$M := \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J(\gamma(t)).$$

Step 2: Let

$$\begin{aligned} \mathcal{P} &:= \left\{ q \in H^1(\mathbb{R}^d; \mathbb{R}^n) \setminus \{0\} \;\middle|\; \|\nabla q\|_*^2 = \frac{d}{d-2} \int G(q)\, dx \right\}; \\ \mathcal{S} &:= \left\{ q \in H^1(\mathbb{R}^d; \mathbb{R}^n) \setminus \{0\} \;\middle|\; \int G(q)\, dx = 1 \right\}. \end{aligned}$$

One can check that

$$\phi(q) := q\left(\frac{x}{L_q}\right), \quad \text{with } L_q := \left(\frac{d-2}{d} \int |\nabla q|^2 + \alpha \nabla q_1 \nabla q_2\, dx\right)^{1/2}$$

is invertible as $\phi \colon \mathcal{S} \to \mathcal{P}$.

Step 3: It can be seen from the proof of Theorem 2.2 that $\overline{q} = \phi(q)$ with q such that

$$\|\nabla q\|_*^2 = \inf \left\{ \|\nabla v\|_*^2 \;\middle|\; v \in \mathcal{C},\ \int G(v)\, dx \geqslant 1 \right\}.$$

On the other hand, $\overline{q} \in H^1(\mathbb{R}^d;\mathbb{R}^n)$, so $q \in H^1(\mathbb{R}^d;\mathbb{R}^n)$ and we also proved that $\int G(q)\,dx = 1$. Then

$$\|\nabla q\|_*^2 = \inf\left\{\|\nabla v\|_*^2 \,\middle|\, v \in H^1(\mathbb{R}^d;\mathbb{R}^n),\ \int G(v)\,dx = 1\right\} = \inf_{v\in\mathcal{S}} \|\nabla v\|_*^2.$$

Thanks to the previous step,

$$\begin{aligned}\inf_{v\in\mathcal{P}} J(v) = \inf_{v\in\mathcal{S}} J(\phi(v)) &= \frac{2}{d}\left(\frac{d-2}{d}\right)^{d/2-1} \inf_{v\in\mathcal{S}} \|\nabla v\|_*^d \\ &= \frac{2}{d}\left(\frac{d-2}{d}\right)^{d/2-1} \|\nabla q\|_*^d = J(\phi(q)) = J(\overline{q}).\end{aligned}$$

Step 4: For any $\gamma \in \Gamma$, we have that $\gamma([0,1]) \cap \mathcal{P} \neq \varnothing$. Let

$$S(q) := (d-2)\int |\nabla q|^2 + \alpha \nabla q_1 \nabla q_2\,dx - d\int G(q)\,dx, \quad q \in H^1(\mathbb{R}^d;\mathbb{R}^n).$$

We notice that $S(q) = dJ(q) - 2\|\nabla q\|_*^2$. For any $\gamma \in \Gamma$ we then have

$$S(\gamma(0)) = 0, \quad S(\gamma(1)) \leqslant dJ(\gamma(1)) < 0.$$

It is clear that $S \in C(H^1(\mathbb{R}^d;\mathbb{R}^n))$, so we can find $t_0 \in [0,1]$ such that $S(\gamma(t_0)) = 0$. We also know by hypothesis that

$$0 < \|q\|_{H^1_*} \leqslant \rho_0 \implies S(q) > 0.$$

Therefore, $\|\gamma(t_0)\| > \rho_0$. In particular, $\gamma(t_0) \neq 0$, so $\gamma(t_0) \in \mathcal{P}$.

Thus

$$J(\overline{q}) = \inf_{v\in\mathcal{P}} J(v) \leqslant \max_{t\in[0,1]} J(\gamma(t)) \text{ for all } \gamma \in \Gamma \quad \Longrightarrow \quad J(\overline{q}) \leqslant M. \qquad \square$$

Just to conclude, we note that

Proposition 2.12 *If $G(q)$ is continuous and satisfies*

$$\limsup_{|q|\to 0^+} \frac{G(q)}{|q|^2} \leqslant -\nu < 0, \quad \limsup_{|q|\to +\infty} \frac{G(q)}{|q|^p} \leqslant 0,$$

then there exists $\rho_0 > 0$ such that

$$0 < \|q\|_{H^1_*} \leqslant \rho_0 \implies S(q) > 0.$$

So our functional (7) satisfies also the additional hypothesis of Theorem 2.11.

3 Evolution equation

Consider the system

$$\begin{cases} \partial_t Q - \Delta Q + aQ = G(Q), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ Q(0,x) = Q_0(x), \end{cases} \tag{11}$$

where $G(Q) = bQ^2 + c|Q|^{p-1}Q$ with $a > 0$, $b, c \in \mathbb{R}$ and $p \in (1,5)$.

Let us recall that by Duhamel's formula any solution Q of (11) satisfies the following equality:

$$Q(t,x) = e^{(\Delta-a)t}Q_0(x) + \int_0^t e^{(\Delta-a)(t-\tau)}G(Q(\tau,x))\,d\tau,$$

where $e^{(\Delta-a)t}f(x) = e^{-at}K_t * f(x)$ with $K_t(x) := \frac{e^{-|x|^2/4t}}{(4\pi t)^{3/2}}$.

Proposition 3.1 *Let $t > 0$ and $1 \leqslant r \leqslant q \leqslant \infty$, then*

$$\|e^{(\Delta-a)t}f\|_{L^q_x} \lesssim t^{-\frac{n}{2}\left(\frac{1}{r}-\frac{1}{q}\right)}e^{-at}\|f\|_{L^r}.$$

The proof follows using the well-known estimates on K_t and the Young inequality.

3.1 Strichartz estimates

Consider the problem

$$\begin{cases} (\partial_t - \Delta + a)Q(t,x) = F(t,x), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ Q(0,x) = Q_0(x) \end{cases} \tag{12}$$

with Q_0 and F taken in a suitable space to be chosen later.

Our aim is to obtain certain estimates in the space $L^q(\mathbb{R}^+; L^r(\mathbb{R}^3; S_0(3,\mathbb{R})))$ for $q, r \in (1,+\infty)$. We will use the norm

$$\|f\|_{L^q_t(I;L^r_x(\mathbb{R}^3))} := \left[\int_I \|f(t,\cdot)\|^q_{L^r}\,dt\right]^{1/q}, \quad I \subseteq \mathbb{R}^+.$$

Theorem 3.2 *Let $I \subseteq \mathbb{R}^+$, then for $q \in [1,+\infty)$ and $r \in (1,\infty)$ the dual of $L^q_t(I; L^r_x)$ is isometric to $L^{q'}_t(I; L^{r'}_x)$ with the duality pair*

$$\langle f, g\rangle_{L^q_t L^r_x} := \int_I \langle f(t,\cdot), g(t,\cdot)\rangle_{L^r_x}\,dt \quad \textit{for all} \quad f \in L^q(I; L^r_x), \quad g \in L^{q'}_t(I; L^{r'}_x).$$

In particular, when $q > 1$, these spaces are reflexive.

The proof follows from [10, Theorems 8.20.3 and 8.20.5, pp. 602–607].

Let us consider the following spacial class of exponents:

Definition 3.3 Let $\sigma > 0$, then (q, r) is said to be a σ-*admissible couple* if $q, r \geqslant 2$, $(q, r, \sigma) \neq (2, \infty, 1)$ and

$$\frac{1}{q} + \frac{\sigma}{r} \leqslant \frac{\sigma}{2}.$$

If the equality holds, the couple is called *strictly* σ-*admissible*. Moreover, when $\sigma > 1$, the couple $\left(2, \frac{2\sigma}{\sigma-1}\right)$ is called the *endpoint*.

Thanks to [15, Theorem 1.2] and the embedding theorems for H^s with $s > 0$ (which can be found in [5, pp. 153–154]) we get

Proposition 3.4 *Let* $a \geqslant 0$, *then*

$$\left\| e^{(\Delta-a)t} f \right\|_{L_t^q L_x^r} \lesssim \|f\|_{H^s}$$

where $q, r \geqslant 2$, $s \in \left[0, \frac{n}{2} - \frac{2}{q}\right)$ *and*

$$\frac{1}{q} + \frac{n}{2r} = \frac{n}{4} - \frac{s}{2}.$$

Let us obtain an estimate for the term $\int_0^t e^{(\Delta-a)(t-\tau)} F(\tau, x)\, d\tau$.

Proposition 3.5 *Let* $\lambda > 0$ *and* $I, J \subseteq \mathbb{R}^+$ *be such that* $l(I) = l(J) = \lambda$ *and* $d(I, J) \approx \lambda$, *then, for all* $r, \widetilde{r} \in [2, +\infty]$ *and* $q, \widetilde{q} \in [1, +\infty]$,

$$\left\| \int_0^t e^{\Delta(t-\tau)} F(\tau, x)\, d\tau \right\|_{L_t^q(J; L_x^r)} \lesssim \lambda^{\beta(q,\tilde{q},r,\tilde{r})} \|F\|_{L_t^{\tilde{q}'}(I; L_x^{\tilde{r}'})}, \quad F \in L_t^{\tilde{q}'}(I; L_x^{\tilde{r}'}), \quad (13)$$

where

$$\beta(q, \widetilde{q}, r, \widetilde{r}) = -\frac{n}{2}\left(\frac{1}{\widetilde{r}'} - \frac{1}{r}\right) + \frac{1}{q} + \frac{1}{\widetilde{q}}.$$

Proof Firstly $F(\cdot, x)$ is defined only on I but, preserving the notation we can suppose that it is defined on $\mathbb{R}^+$ with support on I. If $\sup J < \inf I$, then the left-hand side of (13) is equal to zero, since $\tau \notin I$. In this case the inequality is obvious.

Let us suppose now that $\sup I < \inf J$. Then

$$\begin{aligned} &\left\| \int_0^t e^{\Delta(t-\tau)} F(\tau)\, d\tau \right\|_{L_x^r} \\ &\leqslant \int_0^t \left\| e^{\Delta(t-\tau)} F(\tau) \right\|_{L_x^r} d\tau \lesssim \int_0^t \frac{1}{|t-\tau|^{\frac{n}{2}\left(\frac{1}{\tilde{r}'} - \frac{1}{r}\right)}} \|F(\tau)\|_{L_x^{\tilde{r}'}}\, d\tau \end{aligned}$$

which follows from Proposition 3.1.

$$\left\| \int_0^t e^{\Delta(t-\tau)} F(\tau)\, d\tau \right\|_{L_t^q(J;L_x^r)} \lesssim \left\| \int_0^t \frac{\|F(\tau)\|_{L_x^{\tilde{r}'}}}{|t-\tau|^{\frac{n}{2}\left(\frac{1}{\tilde{r}'}-\frac{1}{r}\right)}}\, d\tau \right\|_{L_t^q(J)} = \left\| \int_I \frac{\|F(\tau)\|_{L_x^{\tilde{r}'}}}{|t-\tau|^{\frac{n}{2}\left(\frac{1}{\tilde{r}'}-\frac{1}{r}\right)}}\, d\tau \right\|_{L_t^q(J)}$$

since $I \subseteq [0, t]$ for $t \in J$. If $\widetilde{q} < +\infty$ by the Hölder inequality

$$\int_I \left\| \frac{1}{|t-\tau|^{\frac{n}{2}\left(\frac{1}{\tilde{r}'}-\frac{1}{r}\right)}} \right\|_{L_t^q(J)} \|F(\tau)\|_{L_x^{\tilde{r}'}} d\tau \leqslant \left[\int_I \left\| \frac{1}{|t-\tau|^{\frac{n}{2}\left(\frac{1}{\tilde{r}'}-\frac{1}{r}\right)}} \right\|_{L_t^q(J)}^{\tilde{q}} d\tau \right]^{1/\tilde{q}} \|F\|_{L_t^{\tilde{q}'}(I;L_x^{\tilde{r}'})}.$$

By hypothesis, $d(I, J) = \lambda$, $t \in J$ and $\tau \in I$, then $|t-\tau| \geqslant \lambda$ so

$$\left[\int_I \left\| \frac{1}{|t-\tau|^{\frac{n}{2}\left(\frac{1}{\tilde{r}'}-\frac{1}{r}\right)}} \right\|_{L_t^q(J)}^{\tilde{q}} d\tau \right]^{1/\tilde{q}} \leqslant \lambda^{-\frac{n}{2}\left(\frac{1}{\tilde{r}'}-\frac{1}{r}\right)} \left[\int_I \|1\|_{L_t^q}^{\tilde{q}} d\tau \right]^{1/\tilde{q}} = \lambda^{-\frac{n}{2}\left(\frac{1}{\tilde{r}'}-\frac{1}{r}\right)+\frac{1}{q}+\frac{1}{\tilde{q}}},$$

where we have used that $|I| = |J| = \lambda$. For $\widetilde{q} = \infty$ it is easier:

$$\int_I \left\| \frac{1}{|t-\tau|^{\frac{n}{2}\left(\frac{1}{\tilde{r}'}-\frac{1}{r}\right)}} \right\|_{L_t^q(J)} \|F(\tau)\|_{L_x^{\tilde{r}'}} d\tau \leqslant \lambda^{-\frac{n}{2}\left(\frac{1}{\tilde{r}'}-\frac{1}{r}\right)+\frac{1}{q}} \|F\|_{L^1(I;L_x^{\tilde{r}'})}.$$

□

Theorem 3.6 *For any $r, \widetilde{r} \in [2, +\infty)$ and $q, \widetilde{q} \in (1, +\infty)$ such that*

$$\frac{1}{q} + \frac{1}{\widetilde{q}} = \frac{n}{2}\left(\frac{1}{\widetilde{r}'} - \frac{1}{r}\right), \quad \frac{1}{q} + \frac{1}{\widetilde{q}} < 1$$

we have the following inequality:

$$\left\| \int_0^t e^{\Delta(t-\tau)} F(\tau)\, d\tau \right\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}.$$

Proof If we denote

$$A[F](t,x) := \int_0^t e^{\Delta(t-\tau)} F(\tau, x)\, d\tau, \quad F \in L_t^{\tilde{q}'}\left(\mathbb{R}^+; L_x^{\tilde{r}'}\right)$$

by duality we get

$$\|A[F]\|_{L_t^q L_x^r} = \sup\left\{|\langle A[F], G\rangle| \;\middle|\; \|G\|_{L_t^{q'} L_x^{r'}} \leqslant 1\right\}.$$

Let us define the bilinear form

$$B(F,G) := \langle A[F], G\rangle = \int_0^\infty \langle A[F](t), G(t)\rangle_{L_x^r}\, dt.$$

To show that

$$|B(F,G)| \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \|G\|_{L_t^{q'} L_x^{r'}},$$

it is useful to rewrite the operator B as

$$B(F,G) = \int_{\mathbb{R}^+} \int_0^t \left\langle e^{\Delta(t-\tau)} F, G\right\rangle_{L_x^r} dt d\tau = \iint_{\{\tau<t\}} \left\langle e^{\Delta(t-\tau)} F, G\right\rangle_{L_x^2} d\tau dt.$$

Then we conclude following the proof of [11, Theorem 1.4]. □

Before passing to the main theorem of this section, let us remark that

$$\frac{1}{q} + \frac{1}{\widetilde{q}} < 1 \iff \frac{1}{\widetilde{r}'} - \frac{1}{r} < \frac{2}{n} \iff \frac{1}{r} + \frac{1}{\widetilde{r}} > \frac{n-2}{n}.$$

Let us also assume the condition of Proposition 3.4 (in the case $s = 0$) holds:

$$\frac{1}{q} + \frac{n}{2r} = \frac{n}{4} \iff \frac{1}{q} = \frac{n}{2}\left(\frac{1}{2} - \frac{1}{r}\right).$$

Then the condition on the indices in Theorem 3.6 becomes

$$\frac{1}{\widetilde{q}} = \frac{n}{2}\left(\frac{1}{\widetilde{r}'} - \frac{1}{r}\right) - \frac{1}{q} = \frac{n}{2}\left(\frac{1}{\widetilde{r}'} - \frac{1}{2}\right) = \frac{n}{2}\left(\frac{1}{2} - \frac{1}{\widetilde{r}}\right).$$

This means that (q, r) and $(\widetilde{q}, \widetilde{r})$ are strictly $\frac{n}{2}$-admissible.

Theorem 3.7 *Let $a \geqslant 0$, $n \geqslant 3$, $r, \widetilde{r}, q, \widetilde{q} \in [2, +\infty)$ be such that (q, r) and $(\widetilde{q}, \widetilde{r})$ are strictly $\frac{n}{2}$-admissible and such that $\frac{1}{r} + \frac{1}{\widetilde{r}} > \frac{n-2}{n}$. If $F \in L_t^{\tilde{q}'}(\mathbb{R}^+; L_x^{\tilde{r}'}(\mathbb{R}^n; S_0(3,\mathbb{R})))$ and $Q_0 \in L^2(\mathbb{R}^n; S_0(3,\mathbb{R}))$, then there is a solution Q of (12) such that*

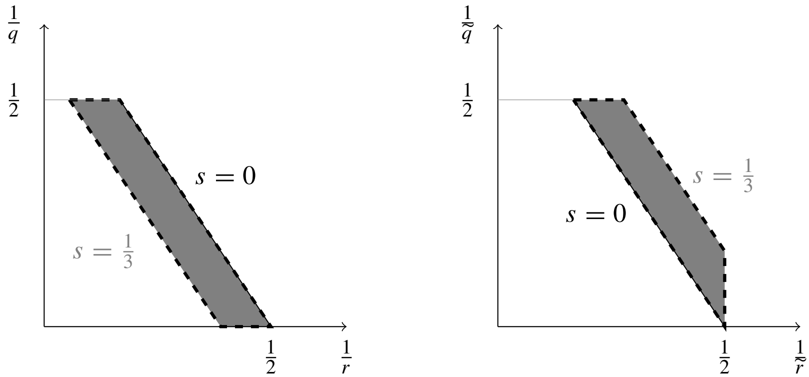
$$\|Q\|_{L_t^q(\mathbb{R}^+; L_x^r)} \lesssim \|Q_0\|_{L^2} + \|F\|_{L_t^{\tilde{q}'}(\mathbb{R}^+; L_x^{\tilde{r}'})}.$$

The proof follows from Proposition 3.4, Theorem 3.6 and the above remarks.

For simplicity, in what follows we shall take $n = 3$. If $Q_0 \in H^s(\mathbb{R}^3; S_0(3, \mathbb{R}))$, then we have more freedom in the choice of couples:

$$\frac{1}{q} + \frac{3}{2r} = \frac{3}{4} - \frac{s}{2} \iff \frac{1}{q} = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{r}\right) - \frac{s}{2};$$
$$\frac{1}{\widetilde{q}} + \frac{1}{q} = \frac{3}{2}\left(\frac{1}{\widetilde{r}'} - \frac{1}{r}\right) \iff \frac{1}{\widetilde{q}} = \frac{3}{2}\left(\frac{1}{\widetilde{r}'} - \frac{1}{2}\right) + \frac{s}{2} = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{\widetilde{r}}\right) + \frac{s}{2}.$$

Certainly, if $Q_0 \in H^s(\mathbb{R}^3; S_0(3, \mathbb{R}))$, then $Q_0 \in H^l(\mathbb{R}^3; S_0(3, \mathbb{R}))$ for every $l \in [0, s]$. If we ignore the condition $\frac{1}{r} + \frac{1}{\widetilde{r}} > \frac{1}{3}$, the two areas representing the admissible couples are the following:



excluding the points on the axes. Precisely, these two sets are defined as

$$\mathcal{D}_s = \left\{(q, r) \in [2, \infty) \times [2, \infty) \;\middle|\; \frac{3}{2}\left(\frac{1}{2} - \frac{1}{r}\right) - \frac{s}{2} \leqslant \frac{1}{q} \leqslant \frac{3}{2}\left(\frac{1}{2} - \frac{1}{r}\right)\right\},$$
$$\widetilde{\mathcal{D}}_s = \left\{(\widetilde{q}, \widetilde{r}) \in [2, \infty) \times [2, \infty) \;\middle|\; \frac{3}{2}\left(\frac{1}{2} - \frac{1}{\widetilde{r}}\right) \leqslant \frac{1}{\widetilde{q}} \leqslant \frac{3}{2}\left(\frac{1}{2} - \frac{1}{\widetilde{r}}\right) + \frac{s}{2}\right\}.$$

Corollary 3.8 *Let $a \geqslant 0$, $s \geqslant 0$, $(q, r) \in \mathcal{D}_s$ and $(\widetilde{q}, \widetilde{r}) \in \widetilde{\mathcal{D}_s}$ be such that $\frac{1}{\widetilde{r}} + \frac{1}{r} > \frac{1}{3}$. If $F \in L_t^{\widetilde{q}'}(\mathbb{R}^+; L_x^{\widetilde{r}'}(\mathbb{R}^3; S_0(3, \mathbb{R})))$ and $Q_0 \in H^s(\mathbb{R}^3; S_0(3, \mathbb{R}))$, then there is a solution Q of (12) such that*

$$\|Q\|_{L_t^q(\mathbb{R}^+; L_x^r)} \lesssim \|Q_0\|_{H^s} + \|F\|_{L_t^{\tilde{q}'}(\mathbb{R}^+; L_x^{\tilde{r}'}(\mathbb{R}^3))}.$$

Remark 3.9 The theorem holds also when one of the couples is an endpoint.

Remark 3.10 The theorem holds also when we work with $J = [0, T]$ instead of $\mathbb{R}^+$: we only need to switch $\widetilde{Q} := Q\mathbb{1}_J(t)$ with u and $\widetilde{F} := F\mathbb{1}_J(t)$ with F, getting

$$\|Q\|_{L_t^q(J; L_x^r)} \leqslant C\Big[\|Q_0\|_{H^s} + \|F\|_{L_t^{\tilde{q}'}(J; L_x^{\tilde{r}'}(\mathbb{R}^3))}\Big].$$

The constant C does not depend on T.

Another important result is the following smoothing inequality.

Proposition 3.11 *Let $s \geqslant 0$ and $a, T > 0$ be constants, let $Q_0 \in H^s(\mathbb{R}^3; S_0(3,\mathbb{R}))$ and $F \in L^2([0,T]; H^{s-1}(\mathbb{R}^3; S_0(3,\mathbb{R})))$, let Q be a solution of (12), then Q satisfies*

$$\|Q\|^2_{L^\infty_t H^s_x} + \|Q\|^2_{L^2_t H^{s+1}_x} \lesssim \|Q_0\|^2_{H^s} + \|F\|^2_{L^2_t H^{s-1}_x}.$$

Proof Firstly we prove the case $s = 0$: let us multiply (12) by $Q(t,x)$:

$$\frac{1}{2}\frac{d}{dt}\|Q(t)\|^2_{L^2_x} + \|\nabla Q(t)\|^2_{L^2_x} + a\|Q(t)\|^2_{L^2_x} \leqslant \left|\langle F(t), Q(t)\rangle_{L^2_x}\right|.$$

Since $a > 0$, it is clear that $\|Q(t)\|_{H^1_x} \simeq \|(a\mathrm{Id} - \Delta)^{1/2} Q(t)\|_{L^2_x}$ for any $t \geqslant 0$. Then we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|Q(t)\|^2_{L^2_x} + \|Q(t)\|^2_{H^1_x} &\lesssim \frac{1}{2}\frac{d}{dt}\|Q(t)\|^2_{L^2_x} + \left\|(a-\Delta)^{1/2}u(t)\right\|^2_{L^2_x}\\ &\lesssim \left|\langle F(t), Q(t)\rangle_{L^2_x}\right|.\end{aligned}$$

By the Plancherel identity, we have

$$\begin{aligned}\left|\langle F(t), Q(t)\rangle_{L^2_x}\right| &= \left|\langle \widehat{F}(t), \widehat{Q}(t)\rangle_{L^2_\xi}\right|\\ &= \left|\left\langle (a+|\xi|^2)^{-1/2}\widehat{F}(t), (a+|\xi|^2)^{1/2}\widehat{Q}(t)\right\rangle_{L^2_\xi}\right|\\ &\leqslant C_\varepsilon \left\|(a+|\xi|^2)^{-1/2}\widehat{F}(t)\right\|^2_{H^{-1}_\xi} + \varepsilon\left\|(a+|\xi|^2)^{1/2}\widehat{Q}(t)\right\|^2_{L^2_\xi}\\ &\leqslant \widetilde{C_\varepsilon}\|F(t)\|^2_{H^{-1}_x} + C_2\varepsilon\|Q(t)\|^2_{H^1_x}.\end{aligned}$$

If we take $\varepsilon < 1/C_2$, then

$$\frac{1}{2}\frac{d}{dt}\|Q(t)\|^2_{L^2_x} + \|Q(t)\|^2_{H^1_x} \lesssim \|F(t)\|^2_{H^{-1}_x}$$

and taking the integral from 0 to T we achieve the result.

For the case $s > 0$ we only need to apply this result to $(\mathrm{Id} - \Delta)^{s/2} Q$. ☐

3.2 Local existence of the solutions

We consider the following spaces:

$$\begin{aligned}S_{T,s} &:= \left\{V(t,x) \in S_0(3,\mathbb{R}) \mid \|V\|_{S_{T,s}} < +\infty\right\},\\ S'_{T,s} &:= \left\{V(t,x) \in S_0(3,\mathbb{R}) \mid \|V\|_{S'_{T,s}} < +\infty\right\},\end{aligned}$$

where

$$\|V\|_{S_{T,s}} := \sup_{(q,r)\in\mathcal{S}} \|V\|_{L_t^q([0,T];L_x^r)}, \quad \|V\|_{S'_{T,s}} := \inf_{(\tilde{q},\tilde{r})\in\tilde{\mathcal{S}}} \|V\|_{L_t^{\tilde{q}'}([0,T];L_x^{\tilde{r}'})},$$

$$\mathcal{S}_s := \{(q,r) \in \mathcal{D}_s \,|\, r \in (2,6)\}, \quad \widetilde{\mathcal{S}}_s := \{(\widetilde{q},\widetilde{r}) \in \widetilde{\mathcal{D}}_s \,|\, \widetilde{r} \in (2,6)\}.$$

In the following, for simplicity, we omit the index s. With respect to the previous theorems, we assume $r, \widetilde{r} < 6$ so that the condition $1/r + 1/\widetilde{r} > 1/3$ is automatically satisfied. In this case, thanks to Proposition 3.4 and Theorem 3.6 we obtain

$$\left\|e^{(\Delta-a)t} Q_0\right\|_{S_T} \lesssim \|Q_0\|_{H^s}, \quad \left\|\int_0^t e^{(\Delta-a)(t-\tau)} F(\tau)\, d\tau\right\|_{S_T} \lesssim \|F\|_{S'_T}.$$

Moreover, Remark 3.10 tells us that this inequality holds for any $T > 0$.

Lemma 3.12 *Let $s \in \left[0, \frac{3}{2}\right)$ and $p \in \left(\frac{5}{3}, \min\left\{5, \frac{7}{3-2s}\right\}\right)$, then there are $\varepsilon_0, \delta_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, $\delta \in (0, \delta_0)$ there exist*

$$\gamma_1, \gamma_2 > 0, \quad (q_1, r_1), (q_2, r_2), (k_1, h_1), (k_2, h_2) \in \mathcal{D}_s,$$
$$(\widetilde{q}, \widetilde{r}) \in \widetilde{\mathcal{D}}_0, \quad l_1, \widetilde{l}_1 \in \left(0, \frac{5}{3}\right), \quad l_2, \widetilde{l}_2 \in (0, p)$$

such that $l_1 + \widetilde{l}_1, l_2 + \widetilde{l}_2 \in \left[\frac{5}{3}, 5\right)$ and

$$\|Q^p\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \leqslant T^{\gamma_1} \|Q\|_{L_t^{q_1} L_x^{r_1}}^{l_1} \|Q\|_{L_t^{k_1} L_x^{h_1}}^{\tilde{l}_1} + T^{\gamma_2} \|Q\|_{L_t^{q_2} L_x^{r_2}}^{l_2} \|Q\|_{L_t^{k_2} L_x^{h_2}}^{\tilde{l}_2}.$$

All the couples, the parameters γ_1, γ_2 and $l_1, \widetilde{l}_1, l_2, \widetilde{l}_2$ depend on the choice of ε and δ.

The proof is technical and follows from several uses of interpolation and Hölder inequalities.

Remark 3.13 If we want to capture all the powers $p \in \left[\frac{5}{3}, 5\right)$, it is necessary to assume that $s \geqslant \frac{4}{5}$.

We are finally ready to prove the existence and uniqueness of the solution in S_T.

Theorem 3.14 *Let $s \geqslant 0$, $p \in \left[\frac{5}{3}, 5\right)$ be such that $p < \frac{7}{3-2s}$, let $a \geqslant 0$ and $b, c \in \mathbb{R}$. If $Q_0 \in H^s(\mathbb{R}^3; S_0(3,\mathbb{R}))$ with $\|Q_0\|_{H^s} \leqslant \frac{R}{2}$ for some $R > 0$, then there is $T = T(R) > 0$ such that there exists a unique $Q \in B(0,R) \subseteq S_T$ which satisfies*

$$\begin{cases} (\partial_t - \Delta + a) Q(t,x) = G(Q) := bQ^2 + c|Q|^{p-1}Q, & (t,x) \in [0,T]\times\mathbb{R}^3, \\ Q(0,x) = Q_0(x). \end{cases}$$

Proof The idea of the proof is to apply the Banach Fixed Point Theorem to the operator

$$K(Q) := \frac{1}{(2\pi)^{\frac{3}{2}}} \left[e^{(\Delta-a)t} Q_0(x) + \int_0^t e^{(\Delta-a)(t-\tau)} G(Q(\tau,x))\, d\tau \right]$$

on the ball $B(0, R)$ of the Banach space

$$\mathcal{B}_T := \bigcap_{i=1}^{4} L_t^{q_i}\big([0,T]; L_x^{r_i}\big) \cap L_t^{k_i}\big([0,T]; L_x^{h_i}\big)$$

for T and R sufficiently small (the couples (q_i, k_i) are the ones of Lemma 3.12 for both the powers in $G(Q)$). In particular, for any $Q_1, Q_2 \in \mathcal{B}_T$, we have the inequality

$$\begin{aligned}
&\|K(Q_1) - K(Q_2)\|_{\mathcal{B}} \leqslant \|K(Q_1) - K(Q_2)\|_{S_T} \\
&\quad = \left\| \int_0^t e^{(\Delta - a)(t-\tau)}(G(Q_1(\tau)) - G(Q_2(\tau)))\, d\tau \right\}_{S_T} \\
&\quad \lesssim \|G(Q_1) - G(Q_2)\|_{S'_T} \leqslant \|G(Q_1) - G(Q_2)\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \\
&\quad \lesssim \Big[T^\alpha(\|Q_1\|_{\mathcal{B}} + \|Q_2\|_{\mathcal{B}}) + T^\beta\big(\|Q_1\|_{\mathcal{B}}^{p-1} + \|Q_2\|_{\mathcal{B}}^{p-1}\big)\Big]\|Q_1 - Q_2\|_{\mathcal{B}}.
\end{aligned}$$

Therefore, if T and R are sufficiently small, then there exists a unique $Q \in B(0, R) \subseteq \mathcal{B}$ solution of the Cauchy problem.

Let us see that $Q \in S_T$: let $(q, r) \in \mathcal{S}$, define $\widetilde{\mathcal{B}}_{\tilde{T}} := \mathcal{B}_{\tilde{T}} \cap L_t^q([0, \widetilde{T}]; L_x^r)$ for $\widetilde{T}$ to be defined. Thanks to what we have already proved we get

$$\begin{aligned}
&\|G(Q_1) - G(Q_2)\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}} \\
&\quad \lesssim \Big[\widetilde{T}^\alpha\big(\|Q_1\|_{\mathcal{B}_{\tilde{T}}} + \|Q_2\|_{\mathcal{B}_{\tilde{T}}}\big) + \widetilde{T}^\beta\big(\|Q_1\|_{\mathcal{B}_{\tilde{T}}}^{p-1} + \|Q_2\|_{\mathcal{B}_{\tilde{T}}}^{p-1}\big)\Big]\|Q_1 - Q_2\|_{\mathcal{B}_{\tilde{T}}}, \\
&\quad \leqslant \Big[\widetilde{T}^\alpha\big(\|Q_1\|_{\tilde{\mathcal{B}}_{\tilde{T}}} + \|Q_2\|_{\tilde{\mathcal{B}}_{\tilde{T}}}\big) + \widetilde{T}^\beta\big(\|Q_1\|_{\tilde{\mathcal{B}}_{\tilde{T}}}^{p-1} + \|Q_2\|_{\tilde{\mathcal{B}}_{\tilde{T}}}^{p-1}\big)\Big]\|Q_1 - Q_2\|_{\tilde{\mathcal{B}}_{\tilde{T}}}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\|K(Q_1) - K(Q_2)\|_{\tilde{\mathcal{B}}_{\tilde{T}}} &\leqslant \|K(Q_1) - K(Q_2)\|_{S_{\tilde{T}}} \\
&\lesssim \|G(Q_1) - G(Q_2)\|_{S'_{\tilde{T}}} \leqslant \|G(Q_1) - G(Q_2)\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}.
\end{aligned}$$

So, exactly as before, it can be proved that there exists a unique $\widetilde{Q} \in B(0, R) \subseteq \widetilde{\mathcal{B}}_{\tilde{T}}$ fixed point for K. Moreover the choice of $\widetilde{T}$ is the same as before: it depends only on some global constants (thanks to Remark 3.10) and on R, which can be taken as before. In conclusion, we can choose $\widetilde{T} = T$. On the other hand, $\widetilde{\mathcal{B}}_T \subseteq \mathcal{B}_T$, in particular the R-balls of the two spaces share the same inclusion. So, by uniqueness of the fixed point in the greater ball, $Q = \widetilde{Q}$ and therefore $Q \in L_t^q([0, T]; L_x^r)$. Certainly, this argument can be repeated for all $(q, r) \in \mathcal{S}$, so $Q \in B(0, R) \subseteq S_T$. □

Before finishing this section, let us present some regularity results.

Proposition 3.15 *If $Q \in S_T$ is a solution of*

$$\begin{cases} (\partial_t - \Delta + a)Q(t, x) = G(Q), \\ Q(0, x) = Q_0(x) \in H^1(\mathbb{R}^3; S_0(3, \mathbb{R})) \end{cases} \tag{14}$$

for $p\in\left[\frac{5}{3},5\right)$, *then* $Q\in L^\infty([0,T];H^1(\mathbb{R}^3;S_0(3,\mathbb{R})))\cap L^2([0,T];H^2(\mathbb{R}^3;S_0(3,\mathbb{R})))$.

Proof Thanks to Proposition 3.11, we have

$$\|Q\|^2_{L^\infty_t H^1_x}+\|Q\|^2_{L^2_t H^2_x}\lesssim\|Q_0\|^2_{H^1}+\|G(Q)\|^2_{L^2_t L^2_x}$$
$$\leqslant\|Q_0\|^2_{H^1}+\|Q\|^2_{L^4_t L^4_x}+\|Q\|^2_{L^{2p}_t L^{2p}_x}.$$

Therefore, for a fixed $p\in\left[\frac{5}{3},5\right)$, we only need to see that $(2p,2p)\in\mathcal{D}_1$:

$$\frac{1}{2p}\leqslant\frac{3}{2}\left(\frac{1}{2}-\frac{1}{2p}\right)\iff p\geqslant\frac{5}{3},\quad\frac{1}{2p}\geqslant\frac{3}{2}\left(\frac{1}{2}-\frac{1}{2p}\right)-\frac{1}{2}\iff p\leqslant 5.\qquad\square$$

Proposition 3.16 *If* $Q\in S_T$ *is a solution of* (14) *for* $p\in\left[\frac{5}{3},5\right)$, *then* $Q\in C([0,T];H^1(\mathbb{R}^3;S_0(3,\mathbb{R})))$.

Proof Let us take $s,t\geqslant 0$.

$$Q(t,x)-Q(s,x)$$
$$=\int_s^t e^{(\Delta-a)(t-\tau)}G(Q(\tau,x))\,d\tau$$
$$+\left(e^{(\Delta-a)(t-s)}-\mathrm{Id}\right)\left[\int_0^s e^{(\Delta-a)(s-\tau)}G(Q(\tau,x))\,d\tau\right].$$

Let us see what happens to the first term when $s\to t$: thanks to Proposition 3.1, it is easy to see that for any $p\in\left[\frac{5}{3},5\right)$ and $r\leqslant\frac{6}{p}$,

$$\int_s^t\left\|e^{(\Delta-a)(t-\tau)}G(Q(\tau))\right\|_{L^2_x}d\tau\leqslant\|Q\|^p_{L^\infty_t H^1_x}\int_s^t\frac{1}{(t-\tau)^{\frac{3}{2}\left(\frac{1}{r}-\frac{1}{2}\right)}}\,d\tau\xrightarrow{s\to t}0.$$

Let us evaluate the part related with the L^2-norm of the gradient:

$$\int_s^t\left\|e^{(\Delta-a)(t-\tau)}|Q(\tau)|^{p-1}\nabla Q(\tau)\right\|_{L^2_x}d\tau$$
$$\leqslant\int_s^t\frac{1}{(t-\tau)^{\frac{3}{2}\left(\frac{1}{r}-\frac{1}{2}\right)}}\left\||Q(\tau)|^{p-1}\nabla Q(\tau)\right\|_{L^r_x}d\tau.$$

Thanks to Proposition 3.15, we know that $Q\in L^\infty([0,T];L^q(\mathbb{R}^3;S_0(3,\mathbb{R})))$ and that $\nabla Q\in L^2([0,T];L^q(\mathbb{R}^3;S_0(3,\mathbb{R})))$ for any $q\in[2,6]$. So, if we take $r=\frac{6}{p}$, $\alpha=\frac{6}{r(p-1)}$ and $\beta=\frac{6}{6-r(p-1)}$, we get

$$
\begin{aligned}
&\int_s^t \frac{1}{(t-\tau)^{\frac{3}{2}\left(\frac{1}{r}-\frac{1}{2}\right)}} \left\| |Q(\tau)|^{p-1} \nabla Q(\tau) \right\|_{L_x^r} d\tau \\
&\qquad \leqslant \int_s^t \frac{1}{(t-\tau)^{\frac{3}{2}\left(\frac{1}{r}-\frac{1}{2}\right)}} \|Q(\tau)\|^{p-1}_{L_x^{(p-1)r\alpha}} \|\nabla Q(\tau)\|_{L_x^{\beta r}} d\tau \\
&\qquad \leqslant \|Q\|^{p-1}_{L_t^\infty H_x^1} \int_s^t \frac{\|\nabla Q(\tau)\|_{L_x^{\beta r}}}{(t-\tau)^{\frac{3}{2}\left(\frac{1}{r}-\frac{1}{2}\right)}} d\tau \\
&\qquad \leqslant \|Q\|^{p-1}_{L_t^\infty H_x^1} \|Q\|_{L_t^2 H_x^2} \left[\int_s^t \frac{1}{(t-\tau)^{3\left(\frac{1}{r}-\frac{1}{2}\right)}} d\tau \right]^{1/2} \xrightarrow{s \to t} 0.
\end{aligned}
$$

As for the second term, by similar calculations we have

$$
\begin{aligned}
&\left\| \int_0^s e^{(\Delta-a)(s-\tau)} G(Q(\tau))\, d\tau \right\|_{H_x^1} \\
&\qquad \leqslant C_1 \int_0^s \frac{1}{(s-\tau)^{\frac{3}{2}\left(\frac{1}{r}-\frac{1}{2}\right)}} d\tau + C_2 \left[\int_0^s \frac{1}{(s-\tau)^{3\left(\frac{1}{r}-\frac{1}{2}\right)}} d\tau \right]^{1/2}
\end{aligned}
$$

with $r = \frac{6}{p}$. In particular, this term is bounded for $s \to t$. On the other hand, it is easy to see that $e^{(\Delta-a)\varepsilon} \to \mathrm{Id}$ as $\varepsilon \to 0+$ as a map from $H^1(\mathbb{R}^3; S_0(3, \mathbb{R}))$ to itself. □

3.3 Global existence and decay for $t \to +\infty$

Let us show that any solution of (11) does not blow up in a finite time when the data are "small".

Proposition 3.17 *If $Q \in C([0, T); H^1(\mathbb{R}^3; S_0(3, \mathbb{R})))$ is a solution of*

$$
\begin{cases} (\partial_t - \Delta + a) Q = b Q^2 + c |Q|^{p-1} Q, \\ Q(0, x) = Q_0(x) \in H^1(\mathbb{R}^3; S_0(3, \mathbb{R})), \end{cases} \tag{15}
$$

where $1 < p < 5$ and $a, b, c \in \mathbb{R}$, then

$$
\begin{aligned}
&\frac{d}{dt} \left[\int |Q(t,x)|^2 dx \right] \\
&\quad = 2 \int -|\nabla Q(t,x)|^2 - a |Q(t,x)|^2 + b \operatorname{tr}(Q(t,x)^3) + c |Q(t,x)|^{p+1} dx.
\end{aligned} \tag{16}
$$

It can be gained multiplying the equation by Q.

It can be seen that, for $\|Q_0\|_{H^1} \leqslant \varepsilon$ sufficiently small, the local solution exists globally. However we can do better: we only need to require that $\|Q_0\|_{L^2}$ is small.

Proposition 3.18 *If $Q(t,x) \in C([0,T); H^1(\mathbb{R}^3; S_0(3,\mathbb{R})))$ is a solution of (15), where $\frac{5}{3} < p < \frac{7}{3}$, $a > 0$, $b, c \in \mathbb{R}$ and $\|Q_0\|_{L^2} \leqslant \varepsilon$, then there is ε sufficiently small such that $Q(t,\cdot) \in L^2(\mathbb{R}^3)$ and $\|Q(t,\cdot)\|_{L^2_x} \lesssim e^{-at}$ for $t \geqslant 0$.*

Proof We start from equality (16), which can be rewritten as

$$\begin{aligned}
&\frac{d}{dt}\left[e^{2at}\int |Q(t,x)|^2\,dx\right] \\
&\quad = 2e^{2at}\int -|\nabla Q(t,x)|^2 + b\,\mathrm{tr}(Q(t,x)^3) + c|Q(t,x)|^{p+1}\,dx \\
&\quad \lesssim -2e^{2at}\|\nabla Q(t)\|^2_{L^2_x} + e^{2at}\|Q(t)\|^{\frac{3}{2}}_{L^2_x}\|\nabla Q(t)\|^{\frac{3}{2}}_{L^2_x} + e^{2at}\|Q(t)\|^{\frac{5-p}{2}}_{L^2_x}\|\nabla Q(t)\|^{\frac{3(p-1)}{2}}_{L^2_x}.
\end{aligned}$$

If we denote $f(t) := e^{2at}\int |Q(t,x)|^2\,dx$ and $g(t) := 2e^{2at}\int |\nabla Q(t,x)|^2\,dx$, then we have

$$\begin{aligned}
f'(t) + g(t) &\leqslant C_1 e^{-at} f(t)^{\frac{3}{4}} g(t)^{\frac{3}{4}} + C_2 e^{-a(p-1)t} f(t)^{\frac{5-p}{4}} g(t)^{\frac{3(p-1)}{4}} \\
&\leqslant \frac{\widetilde{C_1} e^{-4at}}{\delta^4} f(t)^3 + \frac{\widetilde{C_2} e^{-\frac{4a(p-1)}{7-3p}t}}{\delta^{\frac{4}{7-3p}}} f(t)^{\frac{5-p}{7-3p}} + (\delta^4 + \delta^{\frac{4}{7-3p}})\,g(t)
\end{aligned}$$

for all $\delta > 0$, where we have used that $\frac{3(p-1)}{4} < 1$ for $p < \frac{7}{3}$. Therefore, if we take $\delta^4 + \delta^{\frac{4}{7-3p}} = 1$, then

$$f'(t) \leqslant \overline{C_1} e^{-4at} f(t)^3 + \overline{C_2} e^{-\frac{4a(p-1)}{7-3p}t} f(t)^{\frac{5-p}{7-3p}}.$$

By comparison, we have that $f(t) \leqslant y(t)$ for $y(t)$ which satisfies

$$\begin{cases} y'(t) = \overline{C_1} e^{-4at} y(t)^3 + \overline{C_2} e^{-\frac{4a(p-1)}{7-3p}t} y(t)^{\frac{5-p}{7-3p}}, \\ y(0) = f(0) = \|Q_0\|^2. \end{cases}$$

Let us show that there exists ε sufficiently small such that $y(t)$ is uniformly bounded on $t \in \mathbb{R}^+$. Let us start with the case of just one non-linearity:

$$\begin{cases} y'(t) = \overline{C} e^{-Bt} y(t)^A, \\ y(0) = \|Q_0\|^2. \end{cases}$$

With $A > 1$ and $B, \overline{C} > 0$, it is easy to prove that

$$y(t) = \frac{1}{(A-1)\left(\frac{\overline{C}}{B}e^{-Bt} - \gamma\right)}$$

where

$$\gamma = \frac{\overline{C}}{B} - \frac{1}{(A-1)\|Q_0\|^{2(A-1)}_{L^2}}.$$

If we take $\|Q_0\|_{L^2} \leqslant \varepsilon$ sufficiently small such that $\gamma < 0$, then $y(t) \leqslant -\frac{1}{(A-1)\gamma}$ for all $t \geqslant 0$.

Let us return to the case of two non-linearities. If, by contradiction, $y(t)$ were unbounded, then, for any $\varepsilon > 0$, there would exist $T_\varepsilon > 0$ such that

$$y(t) \leqslant 1 \quad \text{for all} \quad t < T_\varepsilon, \quad y(T_\varepsilon) = 1.$$

If we call $\overline{C} = \max\{\overline{C}_1, \overline{C}_2\}$, $B = \min\left\{4a, \frac{4a(p-1)}{7-3p}\right\}$ and $A = \min\left\{3, \frac{5-p}{7-3p}\right\}$, then $y(t) \leqslant w(t)$ with $w(t)$ solution of

$$\begin{cases} w'(t) = \overline{C}e^{-Bt}w(t)^A, \\ w(0) = \|Q_0\|^2. \end{cases}$$

We already know that $w(t) \leqslant -\frac{1}{(A-1)\gamma}$ and we can take ε sufficiently small so that $\gamma < -\frac{1}{A-1}$. In this case $w(t) < 1$ for any $t \in \mathbb{R}^+$. In particular, $y(T_\varepsilon) \leqslant w(T_\varepsilon) < 1$ which gives us a contradiction. □

Corollary 3.19 *If $Q(t,x) \in C([0,T); H^1(\mathbb{R}^3; S_0(3,\mathbb{R})))$ is a solution of (15), where $\frac{5}{3} < p < \frac{7}{3}$, $a > 0$, $b, c \in \mathbb{R}$ and $\|Q_0\|_{L^2} \leqslant \varepsilon$, then there is ε sufficiently small such that $\|Q(t)\|_{H^1} \lesssim e^{-at}$ for any $t \geqslant 0$.*

Proof Let us take $V \in H^1(\mathbb{R}^3; S_0(3,\mathbb{R}))$ such that $Q(t,x) = e^{-at}V(t,x)$. The function V satisfies

$$\partial_t V - \Delta V = be^{-at}V^2 + ce^{-(p-1)at}|V|^{p-1}V.$$

If we multiply this equation by $\partial_t V$, we get

$$\begin{aligned} &\frac{d}{dt}\left(\int |\nabla V(t,x)|^2\,dx\right) \\ &\quad \leqslant be^{-at}\frac{d}{dt}\left[\int |V(t,x)|^3 dx\right] + ce^{(p-1)at}\frac{d}{dt}\left[\int |V(t,x)|^{p+1}\,dx\right]. \end{aligned} \tag{17}$$

By Proposition 3.17 we have that $\|V\|_{L^2_x} \lesssim 1$ so, integrating (17), we get

$$\begin{aligned} &\int_0^t |\nabla V(t,x)|^2 dx \\ &\quad \leqslant \|Q_0\|^2_{H^1} + \int_0^t be^{-at}\frac{d}{dt}\left[\int |V(t,x)|^3\,dx\right] + ce^{(p-1)at}\frac{d}{dt}\left[\int |V(t,x)|^{p+1}\,dx\right] \\ &\quad \lesssim C(\|Q_0\|_{H^1}) + \int_0^t e^{-a\tau}\|\nabla V(\tau)\|^{\beta_1} + e^{-a(p-1)\tau}\|\nabla V(\tau)\|^{\beta_2}_{L^2}\,d\tau. \end{aligned}$$

For $\beta_1, \beta_2 < 2$, since $x^\beta \leqslant 1 + x^2$ for any $\beta \leqslant 2$ and for any $x \geqslant 0$, we get

$$\|\nabla V(\tau)\|^2_{L^2_x} \lesssim \widetilde{C}\big(\|Q_0\|_{H^1}\big) + \int_0^t \big(e^{-a\tau} + e^{-a(p-1)\tau}\big)\|\nabla V(\tau)\|^2_{L^2_x}\,d\tau.$$

Finally, we apply the Grönwall inequality and obtain

$$\|\nabla V(\tau)\| \leqslant \widetilde{C} e^{\int_0^t e^{-a\tau}+e^{-a(p-1)\tau} d\tau} \lesssim 1 \quad \text{for all} \quad t \geqslant 0.$$

We are done, since $\|\nabla Q(t)\|_{L^2_x} = e^{-at}\|\nabla V(t)\|_{L^2_x}$. □

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References

1. Andrienko, D.: Introduction to liquid crystals. Journal of Molecular Liquids **267**, 520–541 (2018)
2. Ball, J.: The Mathematics of Liquid Crystals. Cambridge CCA Course (2012). https://people.maths.ox.ac.uk/ball/Teaching/cambridge.pdf
3. Ball, J.M., Majumdar, A.: Nematic liquid crystals: From Maier-Saupe to a continuum theory. Molecular Crystals and Liquid Crystals **525**(1), 1–11 (2010)
4. Ball, J.M., Zarnescu, A.D.: Orientability and energy minimization in liquid crystal model. Arch. Ration. Mech. Anal. **202**(2), 493–535 (2011)
5. Bergh, J., Löfström, J.: Interpolation Spaces. Grundlehren der Mathematischen Wissenschaften, vol. 223. Springer, Berlin (1976)
6. Brezis, H., Lieb, E.H.: Minimum action solutions of some vector field equations. Comm. Math. Phys. **97**(1), 97–113 (1984)
7. Cavaterra, C., Rocca, E., Wu, H., Xu, X.: Global strong solutions of the full Navier-Stokes and Q-tensor system for nematic liquid crystal flows in two dimensions. SIAM J. Math. Anal. **48**(2), 1368–1399 (2016)
8. Dai, M., Feireisl, E., Rocca, E., Schimperna, G., Schonbek, M.E.: On asymptotic isotropy for a hydrodynamic model of liquid crystals. Asymptot. Anal. **97**(3–4), 189–210 (2016)
9. De Gennes, P.G., Prost, J.: The Physics of Liquid Crystals. Oxford University Press, New York (1993)
10. Edwards, R.E.: Functional Analysis. Holt, Rinehart and Winston, New York (1965)
11. Foschi, D.: Inhomogeneous Strichartz estimates. Hyperbolic. Differ. Equ. **2**(1), 1–24 (2005)
12. Georgiev, V., Barbera, D.: Least energy solutions for a model of liquid crystals. In: Seventh International Conference on New Trends in the Applications of Differential Equations in Sciences (NTADES 2020). AIP Conference Proceedings 2321, Art. No. 030001 (2021)
13. Iyer, G., Xu, X., Zarnescu, A.D.: Dynamic cubic instability in a 2d Q-tensor model for liquid crystals. Math. Models Methods Appl. Sci. **25**(8), 1477–1517 (2015)
14. Jeanjean, L., Tanaka, K.: A remark on least energy solution in $\mathbb{R}^N$. Amer. Math. Soc. **131**(8), 2399–2408 (2002)
15. Keel, M., Tao, T.: Endpoint Strichartz estimates. Amer. J. Math. **120**(5), 955–980 (1998)
16. Mottram, N.J., Newton, C.J.P.: Introduction to Q-tensor theory (2014). arXiv:1409.3542
17. Paicu, M., Zarnescu, A.D.: Global existence and regularity for the full coupled Navier-Stokes and Q-tensor system. SIAM J. Math. Anal. **43**(5), 2009–2049 (2011)
18. Paicu, M., Zarnescu, A.: Energy dissipation and regularity for a coupled Navier-Stokes and Q-tensor system. Arch. Ration. Mech. Anal. **203**(1), 45–67 (2012)

19. Schonbek, M., Shibata, Y.: Global well-posedness and decay for a Q-tensor model of incompressible nematic liquid crystals in $\mathbb{R}^N$. J. Differential Equations **266**(6), 3034–3065 (2019)
20. Schonbek, M., Shibata, Y.: On the global well-posedness of strong dynamics of incompressible nematic liquid crystals in $\mathbb{R}^N$. J. Evol. Equ. **17**(1), 537–550 (2017)
21. Strauss, W.A.: Existence of solitary waves in higher dimensions. Comm. Math. Phys. **55**(2), 149–162 (1977)
22. Struwe, M.: Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems. Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 34. 2nd edn. Springer, Berlin (1996)
23. Virga, E.G.: Variational Theories for Liquid Crystals. Applied Mathematics and Mathematical Computation, vol. 8. Chapman & Hall, London (1994)
24. Wu, H., Xu, X., Zarnescu, A.: Dynamics and flow effects in the Beris-Edwards system modeling nematic liquid crystals. Arch. Ration. Mech. Anal. **231**(2), 1217–1267 (2019)
25. Zarnescu, A.: Mathematical problems of nematic liquid crystals: between dynamical and stationary problems. Philos. Trans. Roy. Soc. A **379**(2201), Art. No. 20200432 (2021)