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DECAY IN ENERGY SPACE FOR THE SOLUTIONS OF GENERALIZED SCHRÖDINGER–HARTREE EQUATION PERTURBED BY A POTENTIAL

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Abstract

We prove, in any space dimension $d \geq 3$, the decay in the energy space for the defocusing Schrödinger–Hartree (SCH) equations with mass-energy intercritical non-local nonlinearities and perturbed by a potential. We will show also new Morawetz inequalities and estimates, generalizing the previous results appearing in [1].

Key words: nonlinear Schrödinger equations, Schrödinger operators, scattering theory

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1. Introduction. Consider the Cauchy problem associated with the nonlinear defocusing Schrödinger equation with generalized Hartree-type nonlinearity, for $d \geq 3$:

$$\begin{cases} i\partial_t u + \Delta_x u - Vu - c[|\cdot|^{d-\gamma} * |u|^p]|u|^{p-2}u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^d, \\ u(0,x) = u_0(x). \end{cases} \tag{1.1}$$

Here, $u = u(t,x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ and $c > 0$. We will require p and γ to satisfy the following conditions:

$$p > 1 + \frac{\gamma+2}{d}, \quad 2 \leq p < p^*(d), \quad p^*(d) = \begin{cases} +\infty & \text{if} \quad d = 1,2, \\ \dfrac{d+\gamma}{d-2} & \text{if} \quad d \geq 3. \end{cases} \tag{1.2}$$

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Moreover we assume that $V : \mathbb{R}^d \to \mathbb{R}$ is a non-negative Schwartz function[1] such that

$$\sup_{y\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{V(x)}{|x-y|^{d-2}} dx < +\infty, \qquad \frac{x}{|x|} \cdot \nabla V(x) \leq -k|\nabla V(x)| \leq 0, \tag{1.3}$$

with $k > 0$. Equation (1.1) is important in many models of mathematical physics. For instance, it was variously introduced in quantum mechanics in order to study the behaviour of the Bose–Einstein condensates, by considering the self-interactions of the such charged particles, as one can see in [5–7] and the references therein.

Pursued by that, at this point we can state our main result, that is

Theorem 1.1. *Assume $d \geq 3$ and let $u \in \mathcal{C}(\mathbb{R}, H^1(\mathbb{R}^d))$ be a global solution to* (1.1) *such that* (1.2), (1.3) *are satisfied. Then*

$$\lim_{t\to\pm\infty} \|u(t,x)\|_{L^q(\mathbb{R}^d)} = 0, \tag{1.4}$$

provided that $2 < q \leq \dfrac{2d}{d-2}$.

The Morawetz multiplier technique and the resulting estimates are a fundamental tool to study the properties of solutions to (1.1). These were obtained for the first time in [8] for the nonlinear Klein–Gordon equation with a general nonlinearity and successively used for proving the asymptotic completeness in [9] for the cubic nonlinear Schrödinger equation (NLS) in $\mathbb{R}^3$ and in [10] for the $L^2 - H^1$ intercritical NLS. Recently, a new approach based on the bilinear Morawetz inequalities eased the proof of the scattering. We refer to [11] and again to [10]. Regarding SCH, [12] and [13] applied the pseudo-conformal transform to study the scattering in spaces more regular than H^1 when $p = 2$. The associated Morawetz inequalities and the asymptotic completeness in the energy space were derived in [10]. In [14] the results were improved by a new Morawetz estimate. In the critical case [15] established scattering for general data with $d \geq 5$ via new localized Morawetz estimates. Scattering in the focusing case was achieved in [16] and [17] for small data and radial data. For large general data we mention mainly [1]. We refer also to [18] and [19] for the NLS in a general setting. Inspired by the last aforementioned papers, we present here the full decay property of the solutions to (1.1) which leads as a consequence to the scattering. We will use a technique that combines Morawetz inequalities, a localization argument, new interaction Morawetz estimates and interpolation and treats in an unified manner all space dimensions $d \geq 3$. Our result is new in the literature, we underline also that no radial assumption is made on the potential $V(x)$.

[1]The assumptions on the potential $V(x)$ can be relaxed. See for example [2–4] and the corresponding references.

2. Preliminaries. We denote

$$H^1(\mathbb{R}^d) = (1-\Delta_x)^{-\frac{1}{2}} L^2(\mathbb{R}^d), \quad H^1(\mathbb{R}^d) = H^1_x.$$

We indicate also by $f \in L^q(\mathbb{R}^d)$, for $1 \le q < \infty$, if

$$\|f\|^q_{L^q(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |f(x)|^q \, dx < \infty, \quad L^q(\mathbb{R}^d) = L^q_x,$$

with obvious modification when $q = \infty$. We recall some of the results concerning the well-posedness of (1.1) already available in the literature, such as [2, 16, 17] and the references therein. They can be summarized as

Proposition 2.1. *Let $d \ge 1$, assume (1.2) and the first condition in (1.3) are satisfied. Then for all $u_0 \in H^1_x$ there exists a unique global solution $u \in \mathcal{C}(\mathbb{R}, H^1_x)$ to (1.1). In addition,*

$$\|u(t)\|_{L^2_x} = \|u_0\|_{L^2_x}, \quad E(u(t)) = E(u_0), \tag{2.1}$$

with

$$E(u(t)) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla_x u(t,x)|^2 \, dx + \int_{\mathbb{R}^d} V(x)|u(t,x)|^2 \, dx$$
$$+ \frac{1}{2p}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|u(t,x)|^p |u(t,y)|^p}{|x-y|^{d-\alpha}} \, dx \, dy.$$

Proof. The proof of the proposition is standard and can be obtained by energy method (Theorem 3.3.9 and Remark 3.3.12 in [20]) combined with the Gagliardo–Nirenberg inequality

$$\int_{\mathbb{R}^d} (|x|^{-(d-\gamma)} * |u|^p)|u|^p dx \lesssim \|u\|^{2p}_{L_x^{\frac{2pd}{d+\gamma}}} \lesssim \|u\|^{2p}_{H^1_x},$$

for $p \in \left[\frac{d+\gamma}{d}, \frac{d+\gamma}{d-2}\right]$ ($p \in \left[\frac{d+\gamma}{d}, \infty\right)$, if $d = 1, 2$) and the defocusing character of (1.1). □

3. Morawetz identities and inequalities. We introduce also some further notations. Given a function $f \in H^1(\mathbb{R}^d, \mathbb{C})$, we denote by

$$m_f(t,x) := |f(t,x)|^2, \qquad j_f(t,x) := \text{ Im } \left[\overline{f}(t,x)\nabla_x f(t,x)\right],$$

the mass density and the momentum density, respectively. Our first contribution reads then as

Lemma 3.1. *Let $d \ge 1$ and $u \in \mathcal{C}(\mathbb{R}, H^1_x)$ be a global solution to (1.1) such that (1.2), (1.3) are satisfied. Moreover, let $\phi = \phi(x) : \mathbb{R}^d \to \mathbb{R}$ a sufficiently regular and decaying function, and denote by*

$$\mathcal{V}(t) := \int_{\mathbb{R}^d} \phi(x)\, m_u(t,x)\, dx.$$

Then the following identities hold:

$$\dot{\mathcal{V}}(t) = \int_{\mathbb{R}^d} \phi(x)\dot{m}_u(t,x)\,dx = 2\int_{\mathbb{R}^d} j_u(t,x)\cdot\nabla_x\phi(x)\,dx \tag{3.1}$$

and

$$\begin{aligned}\ddot{\mathcal{V}}(t) &= \int_{\mathbb{R}^d} \phi(x)\ddot{m}_u(t,x)\,dx = -\int_{\mathbb{R}^d} \Delta_x^2\phi(x)m_u(t,x)\,dx \\ &+ 4\int_{\mathbb{R}^d} \nabla_x u(t,x)D_x^2\phi(x)\cdot\nabla_x\overline{u}(t,x)\,dx \\ &+ c\frac{2(p-2)}{p}\int_{\mathbb{R}^d} \Delta_x\phi(x)\Big[|x|^{-(d-\gamma)} * |u(t,x)|^p\Big]|u(t,x)|^p\,dx \\ &- 2\int_{\mathbb{R}^d} \nabla_x\phi(x)\cdot\nabla_x V(x)m_u(t,x)\,dx \\ &- \frac{4}{p}c\int_{\mathbb{R}^d} \nabla_x\phi(x)\cdot\nabla_x\Big[|x|^{-(d-\gamma)} * |u(t,x)|^p\Big]|u(t,x)|^p\,dx,\end{aligned} \tag{3.2}$$

where $D_x^2\phi \in \mathcal{M}_{d\times d}(\mathbb{R}^d)$ is the Hessian matrix of ϕ and $\Delta_x^2\phi = \Delta_x(\Delta_x\phi)$ the Bi-Laplacian operator.

Proof. We prove the identities for a smooth rapidly decreasing solution $u = u(t,x)$, letting the general case $u \in \mathcal{C}(\mathbb{R}, H_x^1)$ to a density argument. We give some details for obtaining (3.2) and we shall drop the variable t for simplicity. An integration by parts and (1.1) give

$$\begin{aligned}&2\partial_t\int_{\mathbb{R}^d} j_u(x)\cdot\nabla_x\phi(x)\,dx \\ &= 2\,\mathrm{Re}\int_{\mathbb{R}^d} i\partial_t u(x)(\Delta_x\phi(x)\bar{u}(x) + 2\nabla_x\phi(x)\cdot\nabla_x\bar{u}(x))\,dx \\ &= 2\,\mathrm{Re}\int_{\mathbb{R}^d} (-\Delta_x u(x) + \big(c[|x|^{-\tilde{\gamma}} * |u(x)|^p]|u(x)|^{p-2} + V(x)\big)u(x))(\Delta_x\phi(x)\bar{u}(x) \\ &\qquad + 2\nabla_x\phi(x)\cdot\nabla_x\bar{u}(x))\,dx\end{aligned}$$

with $\tilde{\gamma} = d - \gamma$. We have the following identity

$$\begin{aligned}&2\,\mathrm{Re}\int_{\mathbb{R}^d} (-\Delta_x u(x) + V(x)u(x))(\Delta_x\phi(x)\bar{u}(x) + 2\nabla_x\phi(x)\cdot\nabla_x\bar{u}(x))\,dx \\ &= -\int_{\mathbb{R}^d} \Delta_x^2\phi(x)\,|u(x)|^2\,dx + 4\int_{\mathbb{R}^d} \nabla_x u(x)D_x^2\phi(x)\nabla_x\bar{u}(x)\,dx \\ &- 2\int_{\mathbb{R}^d} \nabla_x\phi(x)\cdot\nabla_x V(x)m_u(x)\,dx.\end{aligned} \tag{3.3}$$

Moreover, one gets

$$\begin{aligned}
&\operatorname{Re}\int_{\mathbb{R}^d}[|x|^{-\tilde{\gamma}}*|u(x)|^p]|u(x)|^p\Delta_x\phi(x)\,dx\\
&\qquad+2\operatorname{Re}\int_{\mathbb{R}^d}[|x|^{-\tilde{\gamma}}*|u(x)|^p]|u(x)|^{p-2}u(x)\nabla_x\phi(x)\cdot\nabla_x\bar{u}(x)\,dx\\
&=\operatorname{Re}\int_{\mathbb{R}^d}[|x|^{-\tilde{\gamma}}*|u(x)|^p]|u(x)|^p\Delta_x\phi(x)\,dx\\
&\qquad+\frac{2}{p}\operatorname{Re}\int_{\mathbb{R}^d}[|x|^{-\tilde{\gamma}}*|u(x)|^p]\nabla_x\phi(x)\cdot\nabla_x|u(x)|^p\,dx.
\end{aligned}$$

Then, by an integration by parts of the second term in the last line above, one retrieves

$$\begin{aligned}
&2\operatorname{Re}\int_{\mathbb{R}^d}[|x|^{-\tilde{\gamma}}*|u(x)|^p]|u(x)|^p\Delta_x\phi(x)\,dx\\
&\qquad+4\operatorname{Re}\int_{\mathbb{R}^d}[|x|^{-\tilde{\gamma}}*|u(x)|^p]|u(x)|^{p-2}u(x)\nabla_x\phi(x)\cdot\nabla_x\bar{u}(x)\,dx\\
&=\frac{2(p-2)}{p}\int_{\mathbb{R}^d}\Delta_x\phi(x)\Big[|x|^{-(d-\gamma)}*|u(x)|^p\Big]|u(x)|^p\,dx\\
&\qquad-\frac{4}{p}\int_{\mathbb{R}^d}\nabla_x\phi(x)\cdot\nabla_x\Big[|x|^{-(d-\gamma)}*|u(x)|^p\Big]|u(x)|^p\,dx.
\end{aligned}\tag{3.4}$$

Combining now the identities (3.3) and (3.4), we arrive finally at (3.2). □

At this point we can prove the following

Lemma 3.2. *Assume $d\geq 3$ and let $u\in\mathcal{C}(\mathbb{R},H^1_x)$ be a global solution to* (1.1) *such that* (1.2), (1.3) *are satisfied. Then it holds that, for $J\subseteq\mathbb{R}$,*

$$-2\int_J\int_{\mathbb{R}^d}\frac{x}{|x|}\cdot\nabla_xV(x)m_u(t,x)\,dx\,dt\leq\|u_0\|^2_{H^1_x}.\tag{3.5}$$

Proof. We choose $\psi=\psi(x,y)=|x-y|$ and set $\phi(x)=\psi(x,0)$. This gives

$$\begin{aligned}
&\nabla_x\psi=\frac{x-y}{|x-y|},\quad \Delta_x^2\psi=\begin{cases}-\dfrac{(d-1)(d-3)}{|x-y|^3} & \text{if}\quad d\geq 4,\\ -2\delta_{x=y}\quad (=D_x^2\psi) & \text{if}\quad d=3,\end{cases}\\
&\nabla_xu(t,x)D_x^2\phi(x)\cdot\nabla_x\overline{u}(t,x)\geq 0.
\end{aligned}\tag{3.6}$$

By integrating the Morawetz identity (3.2) w.r.t. the time variable on the interval $J=[t_1,t_2]$, with $t_1,t_2\in\mathbb{R}$, one obtains the following chain of inequalities

$$2\left[\int_{\mathbb{R}^d} j_u(t,x)\cdot\nabla_x\phi(x)\,dx\right]_{t=t_1}^{t=t_2} \geq -2\int_{t_1}^{t_2}\int_{\mathbb{R}^d}\Delta_x^2\phi(x)m_u(t,x)\,dx\,dt$$

$$-2\int_{t_1}^{t_2}\int_{\mathbb{R}^d}\nabla_x\phi(x)\cdot\nabla_x V(x)m_u(t,x)\,dx\,dt$$

$$-\frac{4}{p}c\int_{t_1}^{t_2}\int_{\mathbb{R}^d}\nabla_x\phi(x)\cdot\nabla_x\left[|x|^{-(d-\gamma)}*|u(t,x)|^p\right]|u(t,x)|^p\,dx\,dt$$

$$\gtrsim -2\int_{t_1}^{t_2}\int_{\mathbb{R}^d}\Delta_x^2\phi(x)m_u(t,x)\,dx\,dt \tag{3.7}$$

$$-2\int_{t_1}^{t_2}\int_{\mathbb{R}^d}\nabla_x\phi(x)\cdot\nabla_x V(x)m_u(t,x)\,dx\,dt$$

$$+\frac{2}{p}(d-\alpha)c\int_{t_1}^{t_2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{|x-z|^{d-\gamma+2}}|u(t,x)|^p|u(t,z)|^p(x-z)$$

$$\cdot\left(\frac{x}{|x|}-\frac{z}{|z|}\right)dx\,dz\,dt.$$

By the inequality

$$(x-z)\cdot\left(\frac{x-y}{|x-y|}-\frac{z-y}{|z-y|}\right)\geq 0, \tag{3.8}$$

here with $y=0$, we can drop the last term and the l.h.s of (3.7), by means of the Cauchy–Schwartz inequality, can be bounded as

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^d}\left|\frac{x}{|x|}\cdot\nabla_x V(x)\right|m_u(t,x)\,dx\,dt$$

$$\lesssim\left[\int_{\mathbb{R}^d} j_u(t,x)\cdot\nabla_x\psi(x,y)\,dx\right]_{t=t_1}^{t=t_2}\lesssim\|u_0\|_{H_x^1}^2<\infty,$$

since the H_x^1-norm is a conserved quantity. This, by (3.6), gives the proof of (3.5). □

4. Interaction Morawetz identities and inequalities. This section is devoted to displaying the identities and inequalities associated with the bilinear Morawetz action.

Lemma 4.1. *Let $u\in\mathcal{C}(\mathbb{R},H_x^1)$ be a global solution to* (1.1) *such that* (1.2), (1.3) *are satisfied, let $\phi=\phi(|x|):\mathbb{R}^d\to\mathbb{R}$ be a convex radial function, regular and decaying enough, let $\psi(x,y):=\phi(|x-y|):\mathbb{R}^{2d}\to\mathbb{R}$ and*

$$I(t):=\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\psi(x,y)m_u(t,x)m_u(t,y)\,dx\,dy.$$

Then the following hold:

$$\dot{I}(t) = 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} j_u(t,y)\cdot\nabla_x\psi(x,y)\, m_u(t,y)\, dx\, dy, \tag{4.1}$$

$$\begin{aligned}\ddot{I}(t) \geq & -2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Delta_x^2\psi(x,y) m_u(t,x) m_u(t,y)\, dx\, dy \\ & + N_{(p,\psi)}(t) + R_{(p,\psi)}(t) + R_\psi^V(t),\end{aligned} \tag{4.2}$$

where

$$\begin{aligned} N_{(p,\psi)}(t) & \\ &= \lambda\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \Delta_x\psi(x,y)\Big[|x|^{-(d-\gamma)} * |u(t,x)|^p\Big]|u(t,x)|^p m_u(t,y)\, dx\, dy,\end{aligned} \tag{4.3}$$

with $\lambda = \dfrac{4c(p-2)}{p}$,

$$\begin{aligned} R_{(p,\psi)}(t) & \\ &= \lambda^*\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla_x\psi(x,y)\cdot\nabla_x\Big[|x|^{-(d-\gamma)} * |u(t,x)|^p\Big]|u(t,x)|^p m_u(t,y)\, dx\, dy,\end{aligned} \tag{4.4}$$

with $\lambda^* = \dfrac{-8c}{p}$ *and*

$$R_\psi^V(t) = -4\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \nabla_x\psi(x,y)\cdot\nabla_x V(x) m_u(t,x) m_u(t,y)\, dx\, dy.$$

Proof. As in the previous theorem, we prove the identities for a smooth solution u, treating the general case $u \in \mathcal{C}(\mathbb{R}, H_x^1)$ by a standard density argument. Moreover, we will drop again the variable t for easiness. First, one has

$$\dot{I}(t) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \dot{m}_u(x) m_u(y)\psi(x,y)\, dx\, dy + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} m_u(x)\dot{m}_u(y)\psi(x,y)\, dx\, dy.$$

Then, due to the symmetry of $\psi(x,y) = \phi(|x-y|)$, we obtain (4.1) by (3.1) and Fubini's theorem. Analogously, we can differentiate again and get the identity

$$\begin{aligned}\ddot{I}(t) = &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \ddot{m}_u(x) m_u(y)\psi(x,y)\, dx\, dy \\ &+ \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} m_u(x)\ddot{m}_u(y)\psi(x,y)\, dx\, dy + 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \dot{m}_u(x)\dot{m}_u(y)\psi(x,y)\, dx\, dy.\end{aligned}$$

Then we write $\ddot{I}(t) := \widetilde{A} + \widetilde{B}$. By (3.2), an application of Fubini's theorem and using again the symmetry of $\psi(x, y)$, we are allowed to set

$$\begin{aligned}\widetilde{A} =& -2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} m_u(x)m_u(y)\Delta_x^2\psi(x,y)\,dx\,dy\\&+\lambda\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Delta_x\psi(x,y)\Big[|x|^{-(d-\gamma)}*|u(x)|^p\Big]|u(x)|^p m_u(y)\,dx\,dy\\&+\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Big(\lambda^*\nabla_x\psi(x,y)\cdot\nabla_x\Big[|x|^{-(d-\gamma)}*|u(x)|^p\Big]|u(x)|^p\\&\qquad\qquad -4\nabla_x\psi(x,y)\cdot\nabla_x V(x)m_u(x)\Big)\,m_u(y)\,dx\,dy,\end{aligned}$$

with λ, λ^* defined as in (4.3) and (4.4). In conclusion, we get

$$\begin{aligned}\widetilde{A} = 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Delta_x\psi(x,y)\nabla_x m_u(x)\cdot\nabla_y m_u(y)\,dx\,dy\\+N_{(p,\psi)}(t)+R_{(p,\psi)}(t)+R_\psi^V(t).\end{aligned}\tag{4.5}$$

Moreover by (3.1), (3.2) and Fubini's theorem we introduce

$$\begin{aligned}\widetilde{B} =&4\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\nabla_x u(x)D_x^2\psi(x,y)\nabla_x\overline{u}(x)m_u(y)\,dx\,dy\\&+4\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}m_u(x)\nabla_x u(x)D_y^2\psi(x,y)\nabla_y\overline{u}(y)\,dx\,dy\\&+8\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}j_u(x)D_{xy}^2\psi(x,y)\cdot j_u(y)\,dx\,dy\\=&4\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\overline{u}(y)\nabla_x u(x)D_x^2\psi(x,y)\nabla_x\overline{u}(x)u(y)\,dx\,dy\\&+4\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\overline{u}(x)\nabla_y u(y)D_x^2\psi(x,y)\nabla_y\overline{u}(y)u(x)\,dx\,dy\\&-8\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\text{Im}\,\left[\overline{u}(x)\nabla_x u(x)\right]D_{xy}^2\psi(x,y)\,\text{Im}\,\left[\overline{u}(y)\nabla_y u(y)\right]\,dx\,dy,\end{aligned}$$

where we took advantage of the symmetry of $D^2\psi$ to eliminate the real part condition in the first two summands of the identity above. At this point, by using some rearrangements and the dispersion properties of the solution u, we find out that

$$\widetilde{B} = 2\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}D_x^2\phi(|x-y|)\left[G_1(x,y)\overline{G}_1(x,y)+G_2(x,y)\overline{G}_2(x,y)\right]\,dx\,dy,$$

with $G_1(x,y), G_2(x,y)$ defined as follows:

$$G_1(x,y) := u(x)\nabla_y \bar{u}(y) + \bar{u}(y)\nabla_x u(x) \text{ and } G_2(x,y) := u(x)\nabla_y u(y) - u(y)\nabla_x u(x).$$

Therefore $\widetilde{B} \geq 0$, due to the fact that ϕ is a convex function. The above argument implies, in combination with (4.5), the proof of (4.2). □

4.1. A nonlinear Morawetz inequality. We have the following proposition, that is a consequence of inequality (4.2),

Proposition 4.1. *Let $u \in \mathcal{C}(\mathbb{R}, H^1_x)$ be a global solution to* (1.1) *such that* (1.2), (1.3) *are satisfied. Moreover, let $\mathcal{Q}^d_{\tilde{x}}(r) = \tilde{x} + [-r,r]^d$, with $r > 0$ and $\tilde{x} \in \mathbb{R}^d$. Then one gets: for $d \geq 4$,*

$$\int_{\mathbb{R}} \int_{(\mathcal{Q}^d_{\tilde{x}}(r))^2} |u(t,x)|^2 |u(t,y)|^2 \, dx \, dy \, dt < \infty, \tag{4.6}$$

where $(\mathcal{Q}^d_{\tilde{x}}(r))^2 = \mathcal{Q}^d_{\tilde{x}}(r) \times \mathcal{Q}^d_{\tilde{x}}(r)$, and for $d = 3$,

$$\int_{\mathbb{R}} \int_{\mathcal{Q}^3_{\tilde{x}}(r)} |u(t,x)|^4 \, dx \, dt < \infty. \tag{4.7}$$

Proof. Let us choose $\psi(x,y) = |x-y|$ and deal with the case $d \geq 4$. We have that $R_{(p,|x-y|)}(t) \geq 0$. In fact we rewrite (4.4) as

$$R_{(p,|x-y|)}(t) = \frac{8}{p}(d-\alpha) \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{|x-z|^{d-\gamma+2}} |u(t,x)|^p |u(t,z)|^p K(t,x,z) \, dx \, dz,$$

where

$$K(t,x,z) = (x-z) \cdot \int_{\mathbb{R}^d} m_u(t,y) \left(\frac{x-y}{|x-y|} - \frac{z-y}{|z-y|} \right) dy \geq 0,$$

again by inequality (3.8). Thus (4.2) reduces to

$$-\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Delta^2_x \psi(x,y) m_u(t,x) m_u(t,y) \, dx \, dy + N_{(p,|x-y|)}(t) \leq \ddot{I}(t) - R^V_{|x-y|}(t). \tag{4.8}$$

By (4.1), (3.6) and integrating (4.8) w.r.t. the time variable again on $[t_1, t_2]$, one obtains

$$\begin{aligned} 2 \left[\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} j_u(t,x) \cdot \nabla_x \psi(x,y) m_u(t,y) \, dx \, dy \right]_{t=t_1}^{t=t_2} + \int_{t_1}^{t_2} |R^V_\psi(t)| \, dt \\ \geq -2 \int_{t_1}^{t_2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Delta^2_x \psi(x,y) m_u(t,x) m_u(t,y) \, dx \, dy \, dt \end{aligned} \tag{4.9}$$

$$+\int_{t_1}^{t_2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Delta_x\psi(x,y)\Big[|x|^{-(d-\gamma)}*|u(t,x)|^p\Big]|u(t,x)|^p m_u(t,y)\,dx\,dy\,dt$$
$$\gtrsim\int_{t_1}^{t_2}\int_{(\mathcal{Q}^d_{\tilde{x}}(2r))^2}|u(t,x)|^2|u(t,y)|^2\,dx\,dy\,dt$$
$$+\lambda\int_{t_1}^{t_2}\int_{(\mathcal{Q}^d_{\tilde{x}}(2r))^3}|u(t,x)|^p|u(t,y)|^2|u(t,z)|^p\,dx\,dy\,dz\,dt,$$

with $(\mathcal{Q}^d_{\tilde{x}}(r))^3=\mathcal{Q}^d_{\tilde{x}}(r)\times\mathcal{Q}^d_{\tilde{x}}(r)\times\mathcal{Q}^d_{\tilde{x}}(r)$ and where in the last line of the previous inequality we made use of

$$\inf_{x,y,z\in\mathcal{Q}^d_{\tilde{x}}(r)}\left(\frac{1}{|x-y|},\frac{1}{|z-y|}\right)=\inf_{x,y,z\in\mathcal{Q}^d_{0}(r)}\left(\frac{1}{|x-y|},\frac{1}{|z-y|}\right)\gtrsim 1,$$

for any $\tilde{x}\in\mathbb{R}^d$. By (1.3) and an application of (3.5), the l.h.s of (4.9) can be now bounded as

$$2\left[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}j_u(t,x)\cdot\nabla_x\psi(x,y)m_u(t,y)\,dx\,dy\right]_{t=t_1}^{t=t_2}$$
$$+2\int_{t_1}^{t_2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left|\frac{x}{|x|}\cdot\frac{x-y}{|x-y|}\nabla_x V(x)\right|m_u(t,x)m_u(t,y)\,dx\,dy\,dt$$
$$\lesssim\|u_0\|^4_{H^1_x}+\|u_0\|^2_{L^2_x}\|u_0\|^2_{H^1_x}<\infty,$$

for any $t_1,t_2\in\mathbb{R}$, since the H^1_x-norm is a conserved quantity. We get (4.6) allowing $t_1\to-\infty, t_2\to\infty$. We finally attain (4.7) by just repeating the same steps as above and noticing that

$$-\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\Delta_x^2\psi(x,y)m_u(t,x)m_u(t,y)\,dx\,dy\,dt=2\int_{\mathbb{R}^d}|u(t,x)|^4\,dx\,dt,$$

because $d=3$ and by (3.6). □

5. The decay of solutions. In this section we prove the main Theorem 1.1 if $2<q<\dfrac{2d}{d-2}$.

Proof. We deal with $p>2$ and $d\geq 4$. The case $p=2$ can be handled in a similar manner. It is sufficient to prove the property (1.4) for a suitable $2<q<\dfrac{2d}{d-2}$, since the thesis for the general case can be obtained by the conservation laws (2.1) and interpolation. More precisely it is sufficient to show that

$$\lim_{t\to\pm\infty}\|u(t,x)\|_{L_x^{2+4/d}}=0. \tag{5.1}$$

Then the decay of the L^q_x-norm for all $2 < q < \dfrac{2d}{d-2}$ follows by combining (5.1) with the bound

$$\sup_{t\in\mathbb{R}} \|u(t,x)\|_{H^1_x} < \infty. \tag{5.2}$$

We write the following localized Gagliardo–Nirenberg inequality (see [1])

$$\|\varphi\|^{\frac{2d+4}{d}}_{L_x^{\frac{2d+4}{d}}} \le C \left(\sup_{x\in\mathbb{R}^d} \|\varphi\|_{L^2(\mathcal{Q}_x(1))}\right)^{\frac{4}{d}} \|\varphi\|^2_{H^1_x}, \tag{5.3}$$

where $\mathcal{Q}_x(1)$ is the unit cube in $\mathbb{R}^d$ centred in x. Next, assume by contradiction that (5.1) is false, then by (5.2) and by (5.3) we deduce the existence of a sequence $(t_n, x_n) \in \mathbb{R}\times\mathbb{R}^d$ with $|t_n| \to \infty$ and $\epsilon_0 > 0$ such that

$$\inf_n \|u(t_n,x)\|^2_{L^2(\mathcal{Q}_{x_n}(1))} = \epsilon_0^2. \tag{5.4}$$

For simplicity we can assume that $t_n \to \infty$ (the case $t_n \to -\infty$ can be treated by a similar argument). Notice that by (3.1) in conjunction with (5.2) we get

$$\sup_{n,t} \Big|\frac{d}{dt}\int \chi(x-x_n)|u(t,x)|^2\,dx\Big| < \infty,$$

where $\chi(x)$ is a smooth and non-negative cut-off function such that $\chi(x) = 1$ for $x \in \mathcal{Q}_0(1)$ and $\chi(x) = 0$ for $x \notin \mathcal{Q}_0(2)$. Consequently, by the fundamental theorem of calculus we deduce

$$\left|\int_{\mathbb{R}^d} \chi(x-x_n)|u(\sigma,x)|^2dx - \int_{\mathbb{R}^d} \chi(x-x_n)|u(t,x)|^2dx\right| \le \widetilde{C}|t-\sigma|,$$

for a $\widetilde{C} > 0$ which is independent of n. By combining this fact with (5.4) and the structure of χ, we get the existence of $T > 0$ such that

$$\inf_n \left(\inf_{t\in(t_n,t_n+T)} \|u(t,x)\|^2_{L^2(\mathcal{Q}_{x_n}(2))}\right) \gtrsim \epsilon_1^2,$$

for some $\epsilon_1 > 0$. Observe also that since $t_n \to \infty$ we can assume (modulo subsequence) that the intervals $(t_n, t_n + T)$ are disjoint. In particular we have, for $d \ge 4$,

$$\begin{aligned}\sum_n T\epsilon_1^4 &\lesssim \sum_n \int_{t_n}^{t_n+T} \int_{(\mathcal{Q}^d_{x_n}(2))^2} |u(t,x)|^2|u(t,y)|^2\,dx\,dy\,dt\\ &\lesssim \int_{\mathbb{R}} \sup_{\tilde{x}\in\mathbb{R}^d} \int_{(\mathcal{Q}^d_{\tilde{x}}(2))^2} |u(t,x)|^2|u(t,y)|^2\,dx\,dy\,dt,\end{aligned}$$

and hence we get a contradiction since the left hand side is divergent and the right hand side is bounded as in (4.6). It remains to shed light on the case $d = 3$. Arguing as above we get

$$\sum_n T\epsilon_1^4 \lesssim \int_{\mathbb{R}} \sup_{\tilde{x}\in\mathbb{R}^d} \int_{\mathcal{Q}^3_{\tilde{x}}(2)} |u(t,x)|^4 \, dx \, dt,$$

which brings again to a contradiction by (4.7). The proof is completed. □

6. Scattering. As a straightforward application we complete the proof of Theorem 1.1. More precisely we achieve

Proposition 6.1. *Assume* $d \geq 3$ *and let* $u \in \mathcal{C}(\mathbb{R}, H^1_x)$ *be a global solution to* (1.1) *such that* (1.2), (1.3) *are satisfied. Then*

- (asymptotic completeness) *There exists* $u_0^{\pm} \in H^1_x$ *such that*

  (6.1) $$\lim_{t\to\pm\infty} \left\| u(t,\cdot) - e^{it(\Delta_x - V)} u_0^{\pm}(\cdot) \right\|_{H^1_x} = 0.$$

- (existence of wave operators) *For every* $u_0^{\pm} \in H^1_x$ *there exists unique initial data* $u_0 \in H^1(\mathbb{R}^d)$*, such that the global solution to* (1.1) $u \in \mathcal{C}(\mathbb{R}, H^1_x)$ *satisfies* (6.1).

Moreover

(6.2) $$\lim_{t\to\pm\infty} \|u(t,x)\|_{L_x^{\frac{2d}{d-2}}} = 0.$$

Proof. The proof of the asymptotic completeness and existence of wave operators follows the same lines of the one appearing in [1] and comes out from classical arguments (see also Theorems 7.8.1 and 7.8.4 in [20]), thus we skip it. Once (6.1) is achieved, the Sobolev embedding combined with the dispersive estimate for the free propagator

$$\begin{aligned} \|u(t)\|_{L_x^{\frac{2d}{d-2}}} &\lesssim \left\| u(t) - e^{it\Delta_x} u_0^{\pm} \right\|_{H^1_x} + \left\| e^{it\Delta_x} u_0^{\pm} \right\|_{L_x^{\frac{2d}{d-2}}} \\ &\lesssim \left\| u(t) - e^{it\Delta_x} u_0^{\pm} \right\|_{H^1_x} + \frac{1}{t} \left\| u_0^{\pm} \right\|_{L_x^{\frac{2d}{d+2}}}, \end{aligned}$$

valid for any $t \in \mathbb{R}$ and (6.1), allow also to conclude that (6.2) holds true. □

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