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ON THE PSEUDO-INTERIOR OF MENGER SPONGE

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It is proven in [1] that the pseudo-interior of the Menger universal space μ^1 is homeomorphic to the Nöbeling space N_1^3. Note that a complete description of topological properties of μ^1 and N_1^3 in [1] is given. This note contains a simple proof that the pseudo interior of μ^1 and N_1^3 are homeomorphic.

1. Introduction.

1.1. Let $\mathbb{N}_+ = \mathbb{N} \cup \{0\}$ and $I = [0,1]$. Also, the relation $X \approx Y$ means that topological spaces X and Y are homeomorphic. Recall next some well-known facts. Denote by $\mathcal{C}$ Cantor ternary set in the interval $[0,1]$. Recall that $\mathcal{C}$ consists of those points $x \in [0,1]$ which can be written in ternary system by using only digits 0 and 2. For example, $\frac{1}{3}$ can be written as $0.1_{(3)}$. At the same time $\frac{1}{3} = 0.022\ldots_{(3)}$, so $\frac{1}{3} \in \mathcal{C}$. Generally speaking, the Cantor set contains a subset $\partial_p(\mathcal{C})$, traditionally called the pseudo-boundary of $\mathcal{C}$. The pseudo boundary consists of all points of $\mathcal{C}$ which can be presented in two ways in a ternary system. For example, $\frac{1}{3} \in \partial_p(\mathcal{C})$. Reasonably the set $\mathcal{C} \backslash \partial_p(\mathcal{C}) \approx int_p\ (\mathcal{C})$ is referred to as a pseudo interior of $\mathcal{C}$.

Claim 1.1. *Another way to describe the Cantor set is*

$$\mathcal{C} = [0,1] \backslash \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right)$$

Proof. The interval $\left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right)$ is the middle open third of the interval $\left[\frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right]$. Now, for every $n \in \mathbb{N}$ we divide the interval $[0,1]$ by 3^n parts and remove the middle open third from each of them. Of course, some summands are contained in others. Note however that this does not confuse the construction of the Cantor set. For example supposing $n = 2$ we have $\bigcup\limits_{k=0}^{2} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right) = \left(\frac{1}{9}, \frac{2}{9} \right) \cup \left(\frac{4}{9}, \frac{5}{9} \right) \cup \left(\frac{7}{9}, \frac{8}{9} \right)$

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and obviously $\left(\frac{4}{9},\frac{5}{9}\right) \subset \left(\frac{3}{9},\frac{6}{9}\right) = \left(\frac{1}{3},\frac{2}{3}\right)$ – a summand which appears for $k = 0$. However, the union $\bigcup_{n=1}^{2}\bigcup_{k=0}^{2}\left(\frac{3k+1}{3^n},\frac{3k+2}{3^n}\right)$ remains as it should be:

$$\bigcup_{n=1}^{2}\bigcup_{k=0}^{2}\left(\frac{3k+1}{3^n},\frac{3k+2}{3^n}\right) = \left(\frac{1}{9},\frac{2}{9}\right) \cup \left(\frac{1}{3},\frac{2}{3}\right) \cup \left(\frac{7}{9},\frac{8}{9}\right). \qquad \square$$

It is not hard to see that

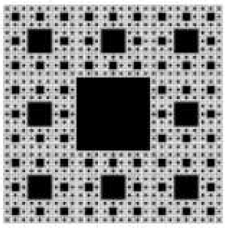
$$int_p\ (\mathcal{C}) = [0,1] \backslash \bigcup_{n=1}^{\infty}\bigcup_{k=0}^{3^{n-1}-1}\left[\frac{3k+1}{3^n},\frac{3k+2}{3^n}\right]$$

and this representation has the same "defects" as the one in the Claim 1.1.

1.2. Sierpinski carpet $\mathcal{S}$ [2] is a two-dimensional analog of the Cantor set. Shortly speaking it can be obtained by dividing the unit square into 9 squares and the removing the middle open square. Then continue the process of recursively removing open middle squares. To describe precisely $\mathcal{S}$ we need some notations. Let us put for $k, l \in \mathbb{N}_+$ and $n \in \mathbb{N}$ (as in Claim 1.1)

$$\mathcal{B} = \bigcup_{n=1}^{\infty}\bigcup_{k,l=0}^{3^{n-1}-1}\left(\frac{3k+1}{3^n},\frac{3k+2}{3^n}\right) \times \left(\frac{3l+1}{3^n},\frac{3l+2}{3^n}\right),$$

then $\mathcal{S} = I^2 \backslash \mathcal{B}$. The first several iterations look like in the next picture (it is not fatal if some square in the union is contained in a bigger one).



Further put

$$\overline{\mathcal{B}} = \bigcup_{n=1}^{\infty}\bigcup_{k,l=0}^{3^{n-1}-1}\left[\frac{3k+1}{3^n},\frac{3k+2}{3^n}\right] \times \left[\frac{3l+1}{3^n},\frac{3l+2}{3^n}\right],$$

then the set $\mathcal{S}\backslash\overline{\mathcal{B}}$ is called a pseudo interior $int_p\mathcal{S}$ of $\mathcal{S}$ and reasonably $\mathcal{S}\backslash int_p\mathcal{S}$ is referred to as a pseudo boundary $\partial_p\mathcal{S}$ of $\mathcal{S}$.

1.3. Menger sponge μ^1 is a subset of the three-dimensional cube I^3 for which the three projections on the coordinate planes are equal to the Sierpinski carpet: $\pi_{xy}\left(\mu^1\right) = \pi_{yz}\left(\mu^1\right) = \pi_{zx}\left(\mu^1\right) \equiv \mathcal{S}$. The picture below shows the first several steps for forming Menger sponge.

Denote next by $\mathcal{S}_{xy}$, $\mathcal{S}_{yz}$ and $\mathcal{S}_{zx}$ copies of $\mathcal{S}$ placed in the coordinate planes xOy, yOz

and zOx. Then it follows from the above description that:

$$\mu^1 = \pi_{xy}^{-1}(\mathcal{S}_{xy}) \bigcup \pi_{yz}^{-1}(\mathcal{S}_{yz}) \bigcup \pi_{zx}^{-1}(\mathcal{S}_{zx}).$$

Now we can define a pseudo boundary $\partial_p\mu^1$ of μ^1 as

$$\partial_p\mu^1 = \pi_{xy}^{-1}(\partial_p\mathcal{S}_{xy}) \bigcup \pi_{yz}^{-1}(\partial_p\mathcal{S}_{yz}) \bigcup \pi_{zx}^{-1}(\partial_p\mathcal{S}_{zx}).$$

As above the set $\mu^1 \backslash \partial_p\mu^1 \backslash int_p\mu^1$ is by definition the pseudo interior of μ^1. Note that as above

$$int_p\mu^1 = \pi_{xy}^{-1}(int_p\mathcal{S}_{xy}) \bigcup \pi_{yz}^{-1}(int_p\mathcal{S}_{yz}) \bigcup \pi_{zx}^{-1}(int_p\mathcal{S}_{zx}).$$

1.4. Now let us recall shortly what the Nöbeling space N_1^3 is. According to [3] it is the set of all points $(x, y, z) \in [0,1]^3 := I^3$ such that no more than one of the coordinates is rational. We use in this note that

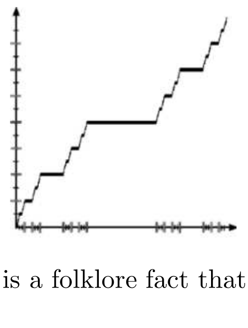
$$N_1^3 = I^3 \backslash \left(Q_{xy} \bigcup Q_{xz} \bigcup Q_{yz}\right)$$

where $Q_{xy} = \left\{(x, y, z) \in I^3 | \ x \text{ and } y \text{ are rationals}\right\}$. The sets Q_{xz} and Q_{yz} are determined in a similar way. We are now ready to consider the main result of this publication, namely that $int_p\mu^1 \approx N_1^3$.

2. The main result. Further we shall use the Cantor function (the so called "Devil's Staircase"). The Cantor function: κ:[0,1] $\to$ [0,1] can be defined as

$$\kappa(x) = \begin{cases} \sum\limits_{n=1}^{\infty} \dfrac{a_n}{2^n} & \text{if } x = \sum\limits_{n=1}^{\infty} \dfrac{2a_n}{3^n} \in \mathcal{C}; \text{ where } a_n \in \{0,1\} \\ \underbrace{\sup}_{y \le x; y \in \mathcal{C}} \kappa(y) & \text{if } x \in [0,1] \backslash C \end{cases}.$$

Here is the graph Γ_κ of the function κ:



which explains the name of κ. It is a folklore fact that κ is (not strongly) increasing.

Claim 2.1. *The function κ is a constant in any interval* $\left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n}\right)$, *where* $0 \le k \le 3^{n-1} - 1$.

Proof. Note that $\frac{3k+1}{3^n} = \frac{k}{3^{n-1}} + \frac{1}{3^n}$ and $\frac{3k+2}{3^n} = \frac{k}{3^{n-1}} + \frac{2}{3^n}$. Since $\frac{1}{3^n} = \sum_{k=n+1}^{\infty} \frac{2}{3^k}$ we have $\kappa\left(\frac{3k+1}{3^n}\right) = \kappa\left(\frac{3k+2}{3^n}\right)$. Therefore if $\frac{3k+1}{3^n} < x < \frac{3k+2}{3^n}$ we have

$$\kappa\left(\frac{3k+1}{3^n}\right) \le \quad \kappa(x) \le \ \kappa\left(\frac{3k+2}{3^n}\right). \qquad \square$$

Consider next the function $f : I^2 \to I^2$ defined by the rule $f(x,y) = (\kappa(x), \kappa(y))$ and denote by g the restriction of f on $int_p\mathcal{S}$. Put $K = f(int_p\mathcal{S})$.

Theorem 2.2. *The function $g : int_p\ (\mathcal{S}) \to K$ is a homeomorphism.*

Proof. It is easy to see that g is a continuous inclusion. Also g is onto by definition. Hence g is convertible; put $h = g^{-1}$. Note that if $(\xi, \eta) \in K$ and $\xi = \sum_{n=1}^{\infty} \frac{\alpha_n}{2^n}$; $\eta = \sum_{n=1}^{\infty} \frac{\beta_n}{2^n}$ then $h(\xi,\eta) = \left(\sum_{n=1}^{\infty} \frac{2\alpha_n}{3^n}, \sum_{n=1}^{\infty} \frac{2\beta_n}{3^n}\right)$. Now suppose that (x,y) and (u,v) are points of K. Then $x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}$; $y = \sum_{n=1}^{\infty} \frac{b_n}{2^n}$ and $u = \sum_{n=1}^{\infty} \frac{c_n}{2^n}$, $v = \sum_{n=1}^{\infty} \frac{d_n}{2^n}$ where $a_n, b_n, c_n, d_n \in \{0,1\}$. Note that $h(x,y) - h(u,v) = 2\sqrt{\left(\sum_{n=p}^{\infty} \frac{\alpha_n - c_n}{3^n}\right)^2 + \left(\sum_{n=p}^{\infty} \frac{b_n - d_n}{3^n}\right)^2}$. Now put $\lambda := \lambda(x,u) = \min\{n \in \mathbb{N} |\ a_n \ne c_n\}$ and $\mu := \mu(y,v) = \min\{n \in \mathbb{N} |\ b_n \ne d_n\}$. Thus for $p \in \mathbb{N}$ with $\lambda(x,u) \ge p$ and $\mu(y,v) \ge p$ we have

$$h(x,y) - h(u,v) = 2\sqrt{\left(\sum_{n=p}^{\infty} \frac{\alpha_n - c_n}{3^n}\right)^2 + \left(\sum_{n=p}^{\infty} \frac{b_n - d_n}{3^n}\right)^2}.$$

Next we have $2\sum_{n=p}^{\infty} \frac{|a_n - c_n|}{3^n} \le \frac{2}{3^p} + \sum_{n=p+1}^{\infty} \frac{2}{3^n} = \frac{1}{3^{p-1}}$, and similarly, $2\sum_{n=p}^{\infty} \frac{|b_n - d_n|}{3^n} \le \frac{1}{3^{p-1}}$. Finally, one obtains $h(x,y) - h(u,v) \le 2\sqrt{\frac{2}{3^{p-1}}}$. Now we can finish the proof: let $\varepsilon > 0$ be an arbitrary positive number. Next, choose $p \in \mathbb{N}$ in such a way that $2\sqrt{\frac{2}{3^{p-1}}} < \varepsilon$. Then if $\min\{\lambda(x,u)\,,\ \mu(y,v)\} > p$ which is equivalent to $h(x,y) - h(u,v) < \sqrt{\frac{1}{2^{p-2}}}$ we have $h(x,y) - h(u,v) < \varepsilon$. In other words, the function h is uniformly continuous. $\square$

Corollary 2.3. *The set $f\left(I^2 \backslash \mathcal{S}\right)$ is a countable dense set L which consists of all binary fractions of the type $\frac{l}{2^p}$ and $I^2 \backslash L = K$ is homeomorphic to $int_p\mathcal{S}$.*

Proof. The function f is constant of the type $\frac{l}{2^p}$ on any closed square in $I^2 \backslash int_p\mathcal{S}$

and is a homeomorphism on $int_p\mathcal{S}$. According to Theorem 2.2 K is homeomorphic to $int_p\ (\mathcal{S})$. $\square$

Thus, it was proven above that the pseudo-interior of the Sierpinski carpet is homeomorphic to $I^2\backslash K$. *Note that L is a dense countable subset of the square I^2*. We use here (because of the limited size of the paper) a result from [4, pp. 140-141, Theorem 7.2]:

Lemma 2.4 ([4])**.** *Let A and B are dense countable subsets of the unit square I^2. There exists a homeomorphism $g_{A,B} : I^2 \to I^2$ such that:*

(a) $g_{A,B}(A) = B$ and

(b) $g_{A,B} = id|_{\partial(I^2)}$, in other words, the restriction of g on the boundary $\partial\left(I^2\right)$ of the square is an identity.

We shall use Lemma 2.4 for $A = L_{xy},\ L_{yz},\ L_{zx}$ – the copies of L in the coordinate planes. Also, we put $B = I^2 \cap \mathbb{Q}^2$ where $\mathbb{Q}$ is the set of rational numbers and $B = B_{xy},\ B_{yz},\ B_{zx}$ – the copies of B in the coordinate planes. Using Lemma 2.4 we obtain the homeomorphism $\mathcal{H} : I^2 \to I^2$ for which $\mathcal{H}(L) = B$.

Lemma 2.5. *The sets $B_{uv} \times I$ and Q_{uv} are homeomorphic. Here $uv \in \{xy, yz, zx\}$.*

Proof. We limited ourselves in the case $uv = xy$. The proof is almost evident:

$$B_{xy} \times I = \left\{((x,y),z) \ \mid\ (x,y) \in I^2 \cap \mathbb{Q}^2 \text{ and } z \in I\right\} =$$

$$\left\{(x,y,z) \in I^3 \mid x \text{ and } y \text{ are rationals and } z\ \in [0,1]\right\} = Q_{xy}. \qquad \square$$

Lemma 2.6. *For $uv \in \{xy, yz, zx\}$the sets $\pi_{uv}^{-1}(L_{uv})$ and $B_{uv} \times I$ are homeomorphic.*

Proof. As in Lemma 2.5 we consider here the case $uv = xy$. Then we use Lemma 2.4 to obtain a homeomorphism $\mathcal{H}_{xy} : I^2 \to I^2$ for which $\mathcal{H}_{xy}(L_{xy}) = B_{xy}$. Then $\pi_{xy}^{-1}(L_{xy}) \approx \pi_{xy}^{-1}(\mathcal{H}_{xy}(L_{xy})) = \pi_{xy}^{-1}(B_{xy}) \approx B_{xy} \times I \approx Q_{xy}$. $\square$

Theorem 2.7. *The pseudo interior of the Menger sponge is homeomorphic to the Nöbeling space N_1^3.*

Proof. Following the definition, we have consequently:

$$N_1^3 = I^3\backslash\left(Q_{xy} \cup Q_{xz} \cup Q_{yz}\right) \approx I^3\backslash\left(\pi_{xy}^{-1}(L_{xy}) \cup \pi_{xy}^{-1}(L_{xy}) \cup \pi_{xy}^{-1}(L_{xy})\right);$$

$$N_1^3 \approx I \times I^2\backslash\left(I \times (L_{xy}) \cup^I \times (L_{xy}) \cup I \times (L_{xy})\right);$$

$$N_1^3 \approx I \times \left(\left(I^2\backslash(L_{xy})\right) \cup \left(I^2\backslash(L_{xy})\right) \cup \left(I^2\backslash(L_{xy})\right)\right);$$

$$N_1^3 \approx I \times \left(\left(I^2\backslash(L_{xy})\right) \cup \left(I^2\backslash(L_{xy})\right) \cup \left(I^2\backslash(L_{xy})\right)\right).$$

It follows from Corollary 2.3 that

$$N_1^3 \approx I \times \left(int_p\mathcal{S}_{xy} \cup int_p\mathcal{S}_{yz} \cup int_p\mathcal{S}_{zx}\right);$$

$$N_1^3 \approx I \times int_p\mathcal{S}_{xy} \cup I \times int_p\mathcal{S}_{yz} \cup I \times int_p\mathcal{S}_{zx}.$$

Therefore $N_1^3 \approx \pi_{xy}^{-1}(int_p\mathcal{S}_{xy}) \cup \pi_{yz}^{-1}(int_p\mathcal{S}_{yz}) \cup \pi_{zx}^{-1}(int_p\mathcal{S}_{zx}) \underbrace{=}_{\text{def}} int_p\mu^1$. $\square$

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ЗА ПСЕВДО-ВЪТРЕШНОСТТА НА КУБА НА МЕНГЕР

Владимир Тодоров

Публикацията съдържа сравнително лесно доказателство на това, че псевдо-вътрешността на куба на Менгер (понякога наричан *Менгеров сюнгер*) е хомеоморфна на пространството на Ньобелинг. Това е „лесна“ реплика на публикация на Кавамура, Левин и Тимчатин, където са доказани доста повече неща от теорията на размерностите. Доказателствата са достъпни и за мотивирани ученици от 11.–12. клас, ако се обяснят на повече от шест страници.