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RECENT DEVELOPMENTS IN ANOMALOUS DIFFUSIONS

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In this paper we review briefly some of the results in the area of anomalous diffusions which are related to the anomalous aggregation phenomenon. Loosely, speaking this phenomenon occurs when a particle moves in a milieu with obstacles which alter so much its otherwise Markovian or diffusive motion that instead of free movement throughout the environment the particle tends to spend a predominant proportion of time in the vicinity of the strongest traps. This type of behaviour is observed in systems such as human cells, polluted rivers, motion in porous media, etc. We also offer some historic account on the appearance of anomalous diffusion in science and the main contributions in this contemporary area of mathematics. We also present our published results with Bruno Toaldo (Turin, Italy) on the topic of anomalous aggregation in this very active field of research. We also discuss some open problems and future directions for research which present a formidable technical challenge.

1. Introduction: history, background and definitions. Stochastic processes are an important tool for modelling the world around us. Beginning with the fundamental papers of Einstein [7] on the kinetic theory of molecules and of Bachelier [2] on the probabilistic evaluation of stock prices on financial markets, the Brownian motion has been widely used for modelling in physics, biology, economics, financial mathematics and other areas of science. One of the most significant achievements of the application of the Brownian motion is the approximation of the Avogadro's number by Perrin [13], who has been awarded the Nobel prize in part for this seminal work. Besides this the Black-Scholes model, based on the Brownian motion, is at the core of modern financial mathematics and thanks to uniqueness of the martingale measure it serves as the guiding point in the advanced regulations of these markets. There are numerous other applications of this stochastic process which we leave without any discussion.

The versatility of the Brownian motion for modelling originates from its universality – if a large number of similar objects (molecules/particles) affect cumulatively and in an independent way another object (larger particle/pollen) then its behaviour (motion) is very well approximated in the limit by one and the same stochastic process (Brownian motion) which is in addition analytically tractable. Despite the universality of the Brownian motion the world is far too complex to be captured well by a single type of a stochastic process. Therefore, researchers have come to consider Lévy processes - stochastic processes that possess the same fundamental properties as the Brownian motion, that

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is stationarity and independence of the increments. Lévy processes offer great richness for modelling by uniting in their domain compound Poisson processes, stable processes, subordinators and for this reason they are used in all areas where the Brownian motion itself has been employed, see the discussion in [1]. All stochastic processes that extend the Brownian motion, but preserve its fundamental properties, are Markovian, that is their future evolution depends only on their present state. However, the majority of real world complex systems are expected to have a long-term memory. Since the treatise of Richardson [14] on turbulent diffusion experimental results have amply demonstrated that the Brownian assumption is inadequate for many systems. The clearest manifestation of non-Brownian behaviour is that, whilst for t units of time the Brownian motion is expected to diffuse roughly \sqrt{t} units of distance away from the origin, many observed complex systems exhibit average displacements of the type t^α , $\alpha \neq 1/2$, or even more exotic ones, see the excellent reviews of Metzler and Klafter [11, 12]. Stochastic processes that account for long-term time dependence and other characteristics that may potentially violate the requirements of Markov processes and model displacements non-typical for the Brownian motion are collectively known as *anomalous diffusion*.

Anomalous diffusion is a rich class of stochastic processes which cannot only account for different properties of a complex system such as time-dependence, diversity and interactions between individual elements, but also incorporates external forces, potentials and boundary values. For this reason anomalous diffusion is adopted virtually everywhere in modern science, see [8, 11, 12]. Typically, upon consideration of physical or other laws, an integro-differential equation is proposed to describe the quantities related to the behaviour of a system. If a stochastic process can be associated to this equation and thereby represents the trajectories of the evolution of the system then these quantities are of the form

$$\mathbb{P}(X_t \in dx) = p(t, x)dx \text{ and } \mathbb{E}(u(X_t)|X_0 = x),$$

where $X = (X_t)_{t \geq 0}$ is the stochastic process and the function u is a given observable. The stochastic process, usually anomalous diffusion, is then investigated with the aim to answer questions related to the asymptotic behaviour, the convergence to equilibrium, the expected displacement, the evaluation of observables and the finer probabilistic structure in the paths of the system.

In this short paper we are interested in a particular problem, namely the anomalous aggregation phenomenon, that is of current interest in the area of anomalous diffusions. To describe it we introduce some basic notation. Let $M = (M_t)_{t \geq 0}$ be a continuous state Markov process in \mathbb{R}^d , $d \geq 1$, and $\sigma = (\sigma_s)_{s \geq 0}$ be an almost surely increasing real-valued additive process, e.g. an increasing Lévy process (subordinator), which may or may not be dependent on M . Then, if $L(v) = \inf \{s \geq 0 : \sigma_s > v\}$ is the right-inverse of σ , a semi-Markov process is defined as $X = (X_t)_{t \geq 0} = (M_{L(t)})_{t \geq 0}$. Since $L = (L(v))_{v \geq 0}$ has levels of constancies such processes are ideal for modelling of the following anomalous diffusive motion: let a particle move in an environment, whose motion, if unhindered, would be free and would unfold according to M ; however, in the environment there are obstacles that temporarily confine the particle according to the length of levels of constancies of L after which it is released; then the motion is described by X . Note that in this example independence of M and σ corresponds to homogeneity of the environment whereas heterogeneity, that is space-dependent obstacles, requires dependence between

position (M) and holding times ($L = \sigma^{-1}$). When M is a strong Markov process and σ is independent of it subordinator the deep connection between time fractional equations of the type of (1.1) (analytical viewpoint) and X (probabilistic viewpoint) is well-understood, see Chen [6]. In particular, when $M = B$ is a Brownian motion and σ is an α -stable subordinator (it has the scaling property $(\sigma_{as})_{s \geq 0} \stackrel{w}{=} a^{\frac{1}{\alpha}} (\sigma_s)_{s \geq 0}$, $a > 0$, $\alpha \in (0, 1)$), then

$$q := q_t(x) = \mathbb{E} [u(X_t) | X_0 = x]$$

is the strong solution to the time fractional equation

$$(1.1) \quad \partial_t^\alpha q = \frac{1}{2} \Delta q,$$

where Δ is the space Laplacian, ∂_t^α is the fractional derivative in time and $q_0 = u$ is an initial condition, see [3] for more information. For general X to model the anomalous diffusion of systems in heterogeneous environments the assumption of independence between M and σ ought to be relaxed. However, for X to have the capacity to model complex real phenomena, which are often described by integro-differential equations stemming from laws of nature, the essential connection between the probabilistic and analytic perspectives has to be preserved and the probabilistic properties of X need to be properly elucidated. The latter has been one of the main goals of the paper [15] on which the current article is built up and it is achieved by extending (1.1) to the heterogeneous case. Once this is established in [15] we set out to investigate the anomalous aggregation phenomenon which seems to have been firstly recognized and studied by Fedotov [5, 9] in relation to the motion of a complex molecule within a cell. Mostly, these investigations have been numerical with strong heuristic analytical considerations as to why the aggregation phenomenon can be rigorously proved. In [15] we prove that when $M = B$, that is the free motion is Brownian, there are necessary and sufficient conditions for the aforementioned phenomenon to occur.

2. Description of the main results and discussion. We start with the extension of (1.1). For this purpose we rigorously introduce the stochastic process M , which is defined as the sextuplet $M = (\Omega, \mathcal{F}, \mathcal{F}_y, M_y, \theta_y, P^x)$ and assumed to be a Hunt process on $(\mathbb{R}, \mathcal{B}(\mathbb{R}^d))$, \mathcal{F}_y , i.e. it is right-continuous, $y \mapsto M_y$ is a.s. right-continuous, M is normal and strong Markov with respect to \mathcal{F}_y and quasi-left-continuous on $[0, \infty)$. Hereafter, we shall further consider only the case when M_y is a Feller process, and thus it possesses a semigroup of operators $\{T_y\}_{y \geq 0}$ defined by $(T_y u)(x) = \mathbb{E}^x u(M_y)$ satisfying $T_y : C_0(\mathbb{R}^d) \mapsto C_0(\mathbb{R}^d)$, where $C_0(\mathbb{R}^d)$ denotes the space of continuous functions on \mathbb{R}^d vanishing at infinity, and being strongly continuous in the sup-norm $\|\cdot\|$, i.e. $\|T_y u - u\| \rightarrow 0$, as $y \rightarrow 0$. The process $(\Omega, \mathcal{F}, \mathcal{F}_y, M_y, \sigma_y, \theta_y, P^x)$ will be an additive process with σ_y one-dimensional, strictly increasing and constructed as follows: let $D \in \mathbb{R}^+ \times \mathbb{R}^d$ be a Borel set and let us introduce

$$(2.1) \quad \mu_M(D) = \ell(\{y \geq 0 : (y, M(y)) \in D\}),$$

where ℓ is the Lebesgue measure. For Borel sets $D = A \times S$ the measure μ computes the amount of time $\ell(A)$ spent by M_y in the set $S \in \mathcal{B}(\mathbb{R}^d)$. For fixed A we define by $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ the occupation measure

$$(2.2) \quad \mu_{M,A}(S) := \mu_M(A \times S).$$

Then we have the identity

$$(2.3) \quad \int_A u(M(y)) \ell(dy) = \int_{\mathbb{R}^d} u(x) \mu_{M,A}(dx)$$

which holds for every (measurable) non-negative function u on \mathbb{R}^d . Therefore, we may assume without a loss of generality, that for fixed $A = [0, y]$, it holds that

$$(2.4) \quad \mathbb{E}^x \left[e^{-\lambda\sigma(y)} \mid M(w), w \leq y \right] = e^{-\int_0^\infty (1-e^{-\lambda s}) \int_{\mathbb{R}^d} \nu(ds, w) d\mu_{M,[0,y]}(\omega)(dw)},$$

where, for any $w \in \mathbb{R}^d$, $\nu(\cdot, w)$ is the Lévy measure of some non-decreasing Lévy process (subordinator), and thereby it is supported on $(0, \infty)$ with the following integrability condition

$$(2.5) \quad \int_0^\infty (s \wedge 1) \nu(ds, w) < \infty$$

being fulfilled, for any $w \in \mathbb{R}^d$. Henceforth, if $\mu_{M,A}(dw)$ is absolutely continuous with respect to ℓ one furthermore deduces that

$$(2.6) \quad \mathbb{E}^x \left[e^{-\lambda\sigma(y)} \mid M(w), w \leq y \right] = e^{-\int_0^\infty (1-e^{-\lambda s}) \int_{\mathbb{R}^d} \nu(ds, w) l_{M,[0,y]}(w) dw},$$

where $l_{M,[0,y]}(w)$ is the Radon-Nycodim derivative (local time of M at w). Certainly, one can opt for a version of the local time such that $l_{X,[0,y]}(w, \omega)$ is a well defined r.v. for every ω so $l_{X,[0,y]}(w, \omega)$ is measurable ($\mathbb{R}^+ \times \Omega \mapsto \mathbb{R}^d$). We employ the following notation

$$(2.7) \quad \mathbb{E}^x \left[e^{-\lambda\sigma(y)} \mid M(w), w \leq y \right] = e^{-\int_0^y f(\lambda, M_w(\omega)) dw}$$

where the functions

$$(2.8) \quad [0, \infty) \times \mathbb{R}^d \mapsto f(\lambda, x) = \int_0^\infty (1 - e^{-\lambda s}) \nu(ds, x)$$

are such that $\lambda \mapsto f(\lambda, x)$ is a family of Bernstein functions parametrized by $x \in \mathbb{R}^d$. We emphasize that $f(\lambda, x)$ can be regarded as the Laplace exponents of the subordinators representing the increments of σ when $M_w = x$, see the monograph of Schilling et al. [16] for a detailed account of the functional-analytic properties of Bernstein functions.

2.1. Integro-differential equation. Let Π_t be the operator

$$(2.9) \quad (q_t u)(x) := \mathbb{E}[u(X(t)) \mid X(0) = x].$$

The first main result establishes a link between

$$(2.10) \quad t \mapsto q_t u,$$

for suitable functions u , and the solutions to

$$(2.11) \quad \frac{d}{dt} \int_0^t (\rho(s, \cdot) - \rho(0, \cdot)) \bar{\nu}(t-s, \cdot) ds = G\rho(t, \cdot),$$

where

$$(2.12) \quad (\mathfrak{D}_i \rho(t))(\cdot) := \frac{d}{dt} \int_0^t (\rho(s, \cdot) - \rho(0, \cdot)) \bar{\nu}(t-s, \cdot) ds$$

and, for any $s > 0$, $x \in \mathbb{R}^d$,

$$(2.13) \quad \bar{\nu}(s, x) := \nu((s, \infty), x)$$

and G is the generator of the already introduced Markov process M . Hereafter, we write $\rho(t)$ instead of $\rho(t, \cdot)$ or $\rho(t, x)$, when the dependence on the vector variable $x \in \mathbb{R}^d$ is

vitally used. We then have the result.

Theorem 2.1. *Assume that the strong Markov processes M_t and (M_t, σ_t) are Feller processes associated with the semigroups of operators T_t and P_t . Let A be the generator of P_t and assume that $C_c^\infty(\mathbb{R}^d) \subset \text{Dom}(G)$ as well as $C_c^\infty(\mathbb{R}^{d+1}) \subset \text{Dom}(A)$ and so G and A are pseudo-differential operators with bounded coefficients. Next, let $x \mapsto f(\lambda, x)$ be continuous and such that*

$$(2.14) \quad \sup_{x \in \mathbb{R}^d} \int_0^\infty (s \wedge 1) \nu(ds, x) < \infty.$$

If, furthermore, the process is uniformly continuous in x at zero, we have that the mapping

$$(2.15) \quad [0, \infty) \ni t \mapsto q(t) := (q_t u)(x)$$

is a mild solution of (2.11) for any $u \in C_0(\mathbb{R}^d)$.

We proceed with the anomalous aggregation results.

2.2. Anomalous aggregation phenomenon. In this section we study the asymptotic behaviour of the process $X(t) = B(L(t))$ where B is a one-dimensional standard Brownian motion. Therefore the generator is given by $G = \frac{1}{2} \partial_x^2$ and we assume that

$$(2.16) \quad \nu(ds, x) = \frac{\alpha(x) s^{-\alpha(x)-1}}{\Gamma(1 - \alpha(x))} ds, \quad \alpha : \mathbb{R} \mapsto (0, 1),$$

that is the family of measures is given by the Lévy measures of stable subordinators, albeit with variable index governed by the function α . Equation (2.11) takes the particular form

$$(2.17) \quad \frac{d^{\alpha(x)}}{dt^{\alpha(x)}} q(t, x) = \frac{1}{2} \partial_x^2 q(t, x).$$

We investigate rigorously the anomalous aggregation phenomenon for the semi-Markov process, that is the time-changed Brownian motion, which is linked to (2.17) by the results presented above. Generally speaking, under some relatively mild technical assumptions on $x \mapsto \alpha(x)$, our findings validate entirely the simulations in the work of Fedotov and Falconer [10]. To be more precise we consider the asymptotic behaviour of two quantities, that is

$$(2.18) \quad \frac{\int_0^t 1_{\{X(s) \in A\}} ds}{t} \quad \text{and} \quad \mathbb{P}(X(t) \in A),$$

where $A \subseteq \mathbb{R}$ is typically some neighbourhood of the set where α attains its global minima. Note that the first quantity measures the average time spend in A whereas the second one is the probability of finding X in A at time t . Depending on the behaviour of $\ell(A \cap [-x, x])$, as $x \rightarrow \infty$, we furnish criteria based on $\alpha^* = \min_{x \in \mathbb{R}} \alpha(x)$, $\alpha_I =$

$\lim_{x \rightarrow \infty} \alpha(x)$, $\alpha_J = \lim_{x \rightarrow -\infty} \alpha(x)$ which separates, apart from the critical case, the two-regime behaviour namely

$$\lim_{t \rightarrow \infty} \frac{\int_0^t 1_{\{X(s) \in A\}} ds}{t} \in \{0, 1\}.$$

When the function α reaches its minima on some union of intervals we have been able to determine whether $\lim_{t \rightarrow \infty} \mathbb{P}(X(t) \in A)$ converges to 0 or 1 thereby mathematically confirming the numerical findings in [10]. We wish to emphasize that the existence of a limit

for the first relation in (2.18) does not necessarily imply the existence of the limit for the second one. We strongly suspect that it is the case in this setting but we have not been able to settle it in complete generality. We also think that “aggregation phenomenon” can be established for other Feller processes, e.g. a stable process, and thus further investigations in this direction are needed.

We start with the first result for the anomalous aggregation phenomenon. Then the following theorem for the average time holds true.

Theorem 2.2. *Let $\alpha : \mathbb{R} \mapsto (0, 1)$, $\alpha^* = \min_{x \in \mathbb{R}} \alpha(x) > 0$, $\max_{x \in \mathbb{R}} \alpha(x) < 1$ and*

$$1 > \lim_{x \rightarrow \infty} \alpha(x) = \alpha_I > \alpha^*, \quad 1 > \lim_{x \rightarrow -\infty} \alpha(x) = \alpha_J > \alpha^*.$$

Also let there exist β_0 small enough such that for all $\beta_0 \geq \beta$, the set

$$A_\beta = \{x \in \mathbb{R} : \alpha(x) < \alpha^* + \beta < 1\}$$

is bounded and satisfies $0 < \ell(A_\beta) < \infty$ and also $\ell(\partial A_\beta) = 0$. Then,

(1) *if $2\alpha^* < \min\{\alpha_I, \alpha_J\}$ we have that for any $0 \leq \beta \leq \beta_0$*

$$(2.19) \quad \lim_{t \rightarrow \infty} \frac{\int_0^t \mathbf{1}_{\{X(s) \in A_\beta\}} ds}{t} = 1, \text{ a.s.};$$

(2) *and if $2\alpha^* > \min\{\alpha_I, \alpha_J\}$, for any $K > 0$,*

$$(2.20) \quad \lim_{t \rightarrow \infty} \frac{\int_0^t \mathbf{1}_{\{X(s) \in A_\beta^c \cap [-K, K]^c\}} ds}{t} = 1, \text{ a.s.}$$

The next results concerns the probability of finding $X(t) \in A$ for large times. It requires some additional technical assumptions.

Theorem 2.3. *Let $\alpha : \mathbb{R} \mapsto (0, 1)$ and $A_0 = \{x \in \mathbb{R} : \alpha(x) = \alpha^*\} = \bigcup_i I_i$, be bounded and where I_i are disjoint intervals. Let also $\ell(A_0) \in (0, \infty)$, $\ell(\partial A_0) = 0$ and for all small $\beta > 0$, $A_0 = A_\beta$, where $A_\beta = \{x \in \mathbb{R} : \alpha(x) < \alpha^* + \beta\}$. Finally, let $\alpha^* = \min_{x \in \mathbb{R}} \alpha(x) > 0$, $\max_{x \in \mathbb{R}} \alpha(x) < 1$ and*

$$1 > \lim_{x \rightarrow \infty} \alpha(x) = \alpha_I, \quad 1 > \lim_{x \rightarrow -\infty} \alpha(x) = \alpha_J.$$

Then if $2\alpha^ < \min\{\alpha_I, \alpha_J\}$ it holds true that*

$$(2.21) \quad \lim_{t \rightarrow \infty} \mathbb{P}(X(t) \in A_0) = 1.$$

These two fundamental results are the following results [15, Theorems 4.12-4.13]. Note that both of them deal when A, A_0 are bounded sets. This explains the condition $2\alpha^* < \dots$ as the expected time of the original Brownian motion B in those sets is of order $t^{1/2}$.

One may consider the case when any of the sets A, A_0 is unbounded. In particular, we study this situation under the condition.

$$\lim_{x \rightarrow \infty} x^{-c} \ell(A \cap [-x, x]) = a \in (0, \infty), \quad c \in [0, 1)$$

and A unbounded. Consider $A_1 = A \cap [0, \infty)$, $A_2 = A \cap (-\infty, 0)$. Assume further that

$$(2.22) \quad \begin{aligned} \lim_{x \rightarrow \infty} x^{-c_1} \ell(A_1 \cap [0, x]) &= a_1 \in (0, \infty) \\ \lim_{x \rightarrow \infty} x^{-c_2} \ell(A_2 \cap [-x, 0]) &= a_2 \in (0, \infty) \end{aligned}$$

and without loss of generality that $1 > c_1 \geq c_2 \geq 0$. In this case, we say that A satisfies the growth Assumption (G). We are now in a position to state the main result in this case.

Theorem 2.4. *Let $\alpha : \mathbb{R} \mapsto (0, 1)$, $\max_{x \in \mathbb{R}} \{\alpha(x)\} < 1$ and suppose that $\alpha^* = \min_{x \in \mathbb{R}} \alpha(x) > 0$. Assume further that $A = \{x \in \mathbb{R} : \alpha(x) = \alpha^* < 1\}$ satisfies the growth Assumption (G) with $1 > c_1 \geq c_2 \geq 0$, see (2.22). Also let $\ell(\partial A) = 0$ and $A = A_\beta = \{x \in \mathbb{R} : \alpha(x) < \alpha^* + \beta < 1\}$ for all $\beta > 0$ small enough. Finally, set $\alpha_\circ = \min\{\alpha_I, \alpha_J\}$, where we have $\lim_{x \rightarrow \infty, x \notin A} \alpha(x) = \alpha_I$ and $\lim_{x \rightarrow -\infty, x \notin A} \alpha(x) = \alpha_J$. Then,*

(1) if $\frac{2\alpha^*}{1+c_1} < \alpha_\circ$ we have that

$$(2.23) \quad \begin{aligned} \lim_{t \rightarrow \infty} \frac{\int_0^t 1_{\{X(s) \in A\}} ds}{t} &= 1 \text{ a.s.}, \\ \lim_{t \rightarrow \infty} \mathbb{P}(X(t) \in A) &= 1; \end{aligned}$$

(2) if $\frac{2\alpha^*}{1+c_1} > \alpha_\circ$ then for any $K > 0$

$$(2.24) \quad \lim_{t \rightarrow \infty} \frac{\int_0^t 1_{\{X(s) \in [-K, K]^c \cap A^c\}} ds}{t} = 1 \text{ a.s.}$$

We see now that the condition depends in the growth of A , namely we have $\frac{2\alpha^*}{1+c_1} < \dots$ and $c_1 = 0$ in the case of bounded sets.

2.3. Brief discussion on open problems and lines of future research. Finally, we discuss some directions for future work. The natural step is to replace the Brownian motion in the aggregation phenomenon case with more general stochastic processes. We have attempted to use general stable Lévy process but even in this case the probabilistic techniques we dispose of are not strong enough to prove anything meaningful. This probably means that general Feller processes are beyond reach unless one finds an alternative approach perhaps from analytical perspective.

3. Sketches of the proofs. We begin with a short outline of the proof of Theorem 2.1. Set

$$v(t, x) = \int_0^t q(s, x) ds.$$

We aim to show that v is a solution to the integrated version of (2.11). The proof uses the Laplace transforms

$$\hat{v}(\lambda, x) = \int_0^\infty e^{-\lambda s} v(s, x) ds; \quad \hat{q}(\lambda, x) = \int_0^\infty e^{-\lambda s} q(s, x) ds.$$

Then it is established that

$$\lambda(f(\lambda, \cdot) - G)\hat{q}(\lambda, x) = f(\lambda, x)u(x).$$

Using that

$$G\hat{v}(\lambda, x) = \frac{1}{\lambda}G\hat{q}(\lambda, x)$$

and plugging this above one gets

$$G\hat{v}(\lambda, x) = \frac{f(\lambda, x)}{\lambda} \hat{q}(\lambda, x) - \frac{f(\lambda, x)}{\lambda^2} u(x)$$

with

$$\frac{f(\lambda, x)}{\lambda} = \int_0^\infty e^{-\lambda t} \bar{\nu}(t, x) dt.$$

Then (2.11) follows by identifying the Laplace transforms.

We proceed with the brief exposition of the proofs for the theorems dealing with the aggregation phenomenon. First we set up some notation. For any set $A \subseteq \mathbb{R}$ we use

$$(3.1) \quad H_t(A) = \int_0^t 1_{\{B_s \in A\}} ds = \mu_{B, [0, t]}(A),$$

which for brevity we shall use $H_t := H_t(A)$ when A is clear. Then, if $l(\partial A) = 0 < l(A)$ then it holds, without a loss of generality, that

$$(3.2) \quad \sigma(s) = \sigma_1(H_s) + \sigma_2(s - H_s),$$

where σ_1, σ_2 are two independent increasing processes constructed from σ as follows

$$(3.3) \quad \begin{aligned} \sigma_1(H_s) &= \sum_{v \leq s} (\sigma(v) - \sigma(v-)) 1_{\{B_v \in A\}}; \quad \sigma_2(s - H_s) \\ &= \sum_{v \leq s} (\sigma(v) - \sigma(v-)) 1_{\{B_v \notin A\}}. \end{aligned}$$

Denote next $A^+ = A \cap \mathbb{R}^+, A^- = A \cap \mathbb{R}^-$ and assume without loss of generality that $A = A^+$. Also, we introduce

$$(3.4) \quad G(t) := \int_0^t l(A \cap [0, x]) dx \quad \text{and} \quad D(s) = \inf \{t > 0 : G(t) > s\}.$$

We use τ for the inverse local time at zero of the Brownian motion B . It is well-known that τ is a stable subordinator of index $1/2$. From [4, Chapter 9] we know that

$$(3.5) \quad \chi(t) := H_{\tau(t)} = \int_0^{\tau(t)} 1_{\{B(s) \in A\}} ds$$

is a driftless subordinator with Lévy measure say Π_χ and Laplace exponent $\Phi_\chi(u) = -\log \mathbb{E} \left[e^{-u\chi(1)} \right], u \geq 0$. Then, after some preliminary results, we show that

Proposition 3.1. *If $\chi(1)$ has a finite mean or the Laplace exponent Φ_χ is regularly varying at zero of index $\alpha \in (0, 1)$, then, for any $\varepsilon > 0$ small enough a.s.*

$$(3.6) \quad \lim_{t \rightarrow \infty} \frac{H_{\tau(t)}}{H_{t^{2+\varepsilon}}} = 0; \quad \lim_{t \rightarrow \infty} \frac{H_{\tau(t)}}{H_{t^{2-\varepsilon}}} = \infty.$$

The last proposition is useful as it connects the behaviour of H at fixed times to that of $H_{\tau(t)}$ which is a tractable subordinator.

In the case of bounded sets we use specific versions of the quantities above, i.e. Because neither of the asymptotic relations in (2.18) depends on finite time horizon we can without loss of generality assume that $A \subseteq \mathbb{R}^+$ (the Brownian motion would pass below A for a finite period of time) and $\lim_{x \rightarrow \infty} l(A \cap [0, x]) = a \in (0, \infty)$. In this case in the notation of

[4, Chapter 9] as $t \rightarrow \infty$

$$G(t) = \int_0^t \ell(A \cap [0, x]) dx \sim at$$

and thus as $t \rightarrow \infty$

$$D(t) = \inf \{s > 0 : G(s) > t\} \sim \frac{t}{a},$$

see (3.4). Then, according to [4, Chapter 9, Corollary 9.4 (ii)] we have that

$$\int_0^t \bar{\Pi}_\chi(x) dx \asymp \frac{t}{D(t)} \approx a,$$

where $\bar{\Pi}_\chi(x) = \int_x^\infty \Pi_\chi(dy)$. Therefore, we are able to get the result.

Corollary 3.2. *If $A \subseteq \mathbb{R}^+$ and $\lim_{x \rightarrow \infty} \ell(A \cap [0, x]) = a \in (0, \infty)$ then a.s. for any $\varepsilon > 0$*

$$(3.7) \quad \lim_{t \rightarrow \infty} \frac{H_t}{t^{\frac{1}{2}-\varepsilon}} = \infty; \quad \lim_{t \rightarrow \infty} \frac{H_t}{t^{\frac{1}{2}+\varepsilon}} = 0.$$

Next, we specify the asymptotic behaviour of $\sigma_1(H_t)$ or $\sigma_2(t - H_t)$ and compare it to σ_1, σ_2 at deterministic times. From Corollary 3.2 we arrive at the following result.

Corollary 3.3. *It holds true that, for any $\varepsilon > 0$,*

$$(3.8) \quad \lim_{t \rightarrow \infty} \frac{\sigma_1(H_t)}{\sigma_1(t^{\frac{1}{2}+\varepsilon})} = 0; \quad \lim_{t \rightarrow \infty} \frac{\sigma_1(H_t)}{\sigma_1(t^{\frac{1}{2}-\varepsilon})} = \infty, \text{ a.s.},$$

provided there exists $\alpha \in (0, 1)$ such that for any $\varepsilon_1 > 0$ small enough a.s.

$$(3.9) \quad \overline{\lim}_{t \rightarrow \infty} \frac{\sigma_1(t)}{\sigma^\alpha(t)} < \infty$$

$$\underline{\lim}_{t \rightarrow \infty} \frac{\sigma_1(t)}{\sigma^{\alpha+\varepsilon_1}(t)} > 0,$$

where σ^β stands for a suitable stable subordinator of index $\beta \in (0, 1)$ defined on the same path space as σ_1 .

These are the main ingredients of the proof which is however quite a bit involved and we refer the interested reader to the paper [15]. Basically the idea is to find the main asymptotic terms for the expected time and it this is done through suitable embedded subordinators. The proof for the asymptotic behaviour of

$$\mathbb{P}(X_t \in A)$$

is even more involved and this requires the additional technical assumptions which are stated in the main theorem. We have tried hard to relax them but to no avail.

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НОВИ РЕЗУЛТАТИ В ОБЛАСТТА НА АНОМАЛНИТЕ ДИФУЗИИ

Младен Савов

В тази статия разглеждаме накратко някои резултати в областта на аномалните дифузии, които са свързани с феномена на аномалната агрегация. Най-общо казано, това явление възниква, когато частица се движи в среда с препятствия, които променят дотолкова нейното иначе Марковско или дифузионно движение, че вместо да се придвижва свободно в средата, частицата прекарва основна част от времето в областите на най-силните капани. Това поведение се наблюдава в системи като човешките клетки, замърсени реки, движение в пореста среда и т.н. В тази статия ние също предлагаме исторически обзор на появата на аномалните дифузии в науката и най-основните приноси в тази активна област. Също така разглеждаме резултатите добити с Бруно Тоалдо (Торино, Италия), които касаят аномалната агрегация. В допълнение споменаваме и някои отворени проблеми и посоки за бъдещи изследвания.