МАТЕМАТИКА И МАТЕМАТИЧЕСКО ОБРАЗОВАНИЕ, 2021
MATHEMATICS AND EDUCATION IN MATHEMATICS, 2021
Proceedings of the Fiftieth Spring Conference
of the Union of Bulgarian Mathematicians
2021

EVOLUTION FLOW AND GROUND STATES FOR FRACTIONAL SCHRÖDINGER–HARTREE EQUATIONS*

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We consider the fractional Schrödinger–Hartree type equations and focus our study on one particular case: the half-wave equation with nonlocal Hartree type interaction terms. The results we present can be divided in the following main topics:

a) existence, asymptotic properties of ground states and their linear stability/instability;

b) existence or explosion phenomena of the evolution flow with large data below/above the ground state barrier for the corresponding Cauchy problem for the half-wave equation;

c) uniqueness of the ground states for the Schrödinger–Hartree type equations;

d) blow-up for mass-critical nonlinear Schrödinger (NLS) equation with non-local Hartree type interaction terms

1. General Introduction. We consider the Cauchy problem for the fractional nonlinear Schrödinger–Hartree equation

$$\begin{cases} iu_t + (-\Delta)^\beta u - \mu v|u|^{p-2}u - \kappa u|u|^{q-1} = 0, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^d \\ (-\Delta)^{\alpha/2} v = |u|^p, \\ u(0,x) = u_0(x). \end{cases} \tag{1}$$

with self-interacting term $u|u|^{q-1}$. Here, the operator $(-\Delta)^\beta$ is the fractional power $\beta \in (0,1]$ of the Laplace operator. The parameters μ, κ are non-negative ones.

2. Existence of ground states and their linear stability/instability.

2.1. The classical Hartree-Choquard-Pekar model. We start with the case $\kappa = 0$, $\mu = 1$ and we will be interested in the properties of ground states $u(t,x) = e^{i\omega t}\phi(x)$, with $\phi > 0$. Clearly, $\phi = \phi_{p,\omega}$ will then satisfy the profile equation

$$(-\Delta)^\beta \phi - c_{d,\gamma}(|\cdot|^{-\gamma} * |\phi|^p)|\phi|^{p-2}\phi = \omega\phi, \quad x \in \mathbb{R}^d, \ \gamma = d - \alpha. \tag{2}$$

The equation (2) is (a fractional) version of the well-known Choquard equation.

As one expects, most of the work was done in the classical context, $\beta = 1$, for the Hartree-Choquard-Pekar system (for $\alpha \in (0,d)$)

$$iu_t - \Delta u - I_\alpha[|u|^p]|u|^{p-2}u = 0, (t,x) \in \mathbb{R} \times \mathbb{R}^d. \tag{3}$$

*The author was supported in part by INDAM, GNAMPA – Gruppo Nazionale per l'Analisi Matematica, la Probabilita e le loro Applicazioni, by Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, by Top Global University Project, Waseda University.

2020 Mathematics Subject Classification: 35A15, 35B44, 35C07.

Key words: half-wave equation, blow-up solution, ground states.

The standing wave solutions of the form $e^{-it}v$ satisfy

$$-\Delta v + v - I_\alpha[|v|^p]|v|^{p-2}v = 0. \tag{4}$$

The question for existence of localized solutions for (4) has been well-studied (cf. [13], [7]).

2.2. Linear stability/instability for the fractional Choquard equation. Our results concern both the fractional model (2) and the more classical version (3). More precisely, we are interested in the existence properties of solitary waves for (4), that is whether and under what conditions, one obtains *nice* ground state solutions of (4). To this end, introduce the optimization problem

$$\begin{cases} E(u) := \frac{1}{2}\||\nabla|^\beta u\|^2_{L^2(\mathbb{R}^d)} - \frac{c}{2p}\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|u(x)|^p|u(y)|^p}{|x-y|^\gamma}dxdy \to \min \\ \text{subject to } \int_{\mathbb{R}^d}|u(x)|^2dx = \lambda, u \in H^\beta(\mathbb{R}^d). \end{cases} \tag{5}$$

At least formally, one can see that the associated Euler-Lagrange equation is exactly (2). Also, the expression $E(u)$ does not necessarily make sense for all $u \in H^\beta(\mathbb{R}^d)$, but it will under certain restrictions on β, p. In [8] the following result is obtained.

Theorem 1. *Let $\beta \in (0,1], \gamma \in (0,d)$ and $p > 1$. Assume in addition the relationship*

$$0 < (p-2)d + \gamma < 2\beta. \tag{6}$$

Then, there exists a solution of (2), $\phi \in H^\beta(\mathbb{R}^d)$, namely a solution of the constrained minimization problem (5). Moreover, ϕ is bell-shaped.

Further, we have the following stability result obtained in [8].

Theorem 2. *Let $p > 2$. Then, the ground states ϕ constructed in Theorem 1 are spectrally stable as solutions of (2).*

The waves constructed in Theorem 1 are the minimizers of the problem $\inf_{\|u\|_{L^2}=\lambda} E(u)$ (dubbed "normalized solutions" in [13]), where the energy functional is given by

$$E(u) := \frac{1}{2}\int_{\mathbb{R}^d}[||\nabla|^\beta u(x)|^2 + |u(x)|^2]dx - \frac{c_{d,\gamma}}{2p}\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{|u(x)|^p|u(y)|^p}{|x-y|^\gamma}dxdy.$$

They turn out to be spectrally stable, per the claim of Theorem 2. It so happens *these are all the stable solitary waves there are*, at least in the classical case $\beta = 1$, as we discuss now.

3. Evolution flow for semilinear half-wave equation. Our next step is the analysis of the evolution flow for the semi linear half-wave equation

$$\begin{cases} i\partial_t u = \sqrt{-\Delta}u - u|u|^{p-1}, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^d \\ u(0,x) = f(x) \in H^s(\mathbb{R}^d). \end{cases} \tag{7}$$

Since now on $H^s(\mathbb{R}^d)$ and $\dot{H}^s(\mathbb{R}^d)$ denote respectively the usual inhomogeneous and homogeneous Sobolev spaces in $\mathbb{R}^d$, endowed with the norms $\|(1-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^d)}$ and $\|(-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^d)}$. We shall also refer to $H^s_{rad}(\mathbb{R}^d)$ as to the set of functions belonging to $H^s(\mathbb{R}^d)$ which are radially symmetric.

The first question is to define the evolution flow for (7) in appropriate Sobolev spaces. Of special importance is the fact that we have 2 conservation laws: the conservation of the mass, using L^2 norm and the conservation of the energy expressed in terms of $H^{1/2}$ norm. We recall that two values of the nonlinearity p are quite relevant: the nonlinearity

$u|u|^{2/(n-1)}$, which is $H^{1/2}$-critical, and the nonlinearity $u|u|^{2/n}$, which is L^2-critical.

Our main result about the Cauchy problems (7) guarantees the local existence of the evolution flow in H^1_{rad} for $p \in (1, 1+\frac{2}{d-1})$ and global existence in the same Sobolev norm for smaller range for p.

Theorem 3 (see [1])**.** *Let $d \geq 2$, $p \in \left(1, 1+\frac{2}{d-1}\right)$. Then for every $R > 0$ there exists $T = T(R) > 0$ and a Banach space X_T such that:*

- *$X_T \subset \mathcal{C}([0,T]; H^1_{rad}(\mathbb{R}^d))$;*
- *for any $f(x) \in H^1_{rad}(\mathbb{R}^d)$ with $\|f\|_{H^1(\mathbb{R}^d)} \leq R$, there exists a unique solution $u(t,x) \in X_T$ of (7).*

Assume moreover that $p \in \left(1, 1+\frac{2}{d}\right)$, then the solution is global in time.

Next we shall analyze the issue of standing waves. We recall that standing waves are special solutions to (7) with a special structure, namely $u(t,x) = e^{i\omega t}v(x)$, where $\omega \in \mathbb{R}$ plays the role of the frequency. Indeed $u(t,x)$ is a standing wave solution if and only if $v(x)$ satisfies

$$(-\Delta)^{1/2}v + \omega v - v|v|^p = 0 \quad \text{in } \mathbb{R}^d. \tag{8}$$

It is worth mentioning that, following the pioneering paper [3], it is well understood how to build up solitary waves for half-wave equation.

In the nonlocal context in which we are interested in, the minimization problem analogue of the one studied in [3] for NLS is the following one:

$$\mathcal{J}_r^{hw} = \inf_{u \in S_r} \mathcal{E}_{hw}(u) \tag{9}$$

where

$$S_r = \left\{u \in H^{1/2}(\mathbb{R}^d) \text{ such that } \|u\|^2_{L^2(\mathbb{R}^d)} = r\right\}. \tag{10}$$

and

$$\mathcal{E}_{hw}(u) = \frac{1}{2}\|u\|^2_{\dot{H}^{1/2}(\mathbb{R}^n)} - \frac{1}{p+1}\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^n)} + \frac{1}{2}\|u\|_2^2.$$

Now we recall the definition of stability.

Definition 1. *Let $\mathcal{N} \subset H^1_{rad}(\mathbb{R}^d)$ be bounded in $H^{1/2}(\mathbb{R}^d)$. We say that $\mathcal{N}$ is* weakly orbitally stable *by the flow associated with half-wave equation if for any $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\text{dist}_{H^{1/2}}(u(0,.), \mathcal{N}) < \delta \text{ and } u(0,x) \in H^1_{rad}(\mathbb{R}^d) \Rightarrow$$

$$\Phi_t(u(0,.)) \text{ is globally defined and } \sup_t \text{dist}_{H^{1/2}}(\Phi_t(u(0,.)), \mathcal{N}) < \epsilon$$

where $\text{dist}_{H^{1/2}}$ denotes the usual distance with respect to the topology of $H^{1/2}$ and $\Phi_t(u(0,.)$ is the unique global solution associated with the Cauchy problem and with initial condition $u(0,x)$.

We can now state the next result, where we use the notations (9) and (10). We state it as a corollary since it is a classical consequence of the concentration-compactness argument in the spirit of [3] and Theorem 3, that guarantees a global dynamic for HW (half-wave).

Corollary 1. *Let $1 < p < 1 + \frac{2}{d}$ and $d \geq 1$. Then for every $r > 0$ we have:*

- $\mathcal{J}_r^s > -\infty$ *(resp.* $\mathcal{J}_r^{hw} > -\infty$*) and* $\mathcal{B}_r^s \neq \emptyset$ *(resp.* $\mathcal{B}_r^{hw} \neq \emptyset$*) where* $\mathcal{B}_r^{hw} := \{v \in S_r$ *such that* $\mathcal{E}_{hw}(v) = \mathcal{J}_r^{hw}\}$. *In particular for every* $v \in \mathcal{B}_r^s$ *(resp.* $v \in \mathcal{B}_r^{hw}$*) there exists* $\omega \in \mathbb{R}$ *such that* $\sqrt{-\Delta} v + \omega v - v|v|^{p-1} = 0$*;*
- *the set* $\mathcal{B}_r^{hw}$ *is* weakly orbitally stable *by the flow associated with HW.*

In order to state our last result about existence/instability of ground states for HW, we need to introduce also the following functional:

$$\mathcal{P}(u) = \frac{1}{2}\|u\|^2_{\dot{H}^{1/2}(\mathbb{R}^d)} - \frac{n(p-1)}{2(p+1)}\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^d)},$$

and the corresponding set:

$$\mathcal{M} = \big\{u \in H^{1/2}(\mathbb{R}^d) \text{ such that } \mathcal{P}(u) = 0\big\}. \tag{11}$$

It is well known that we have the following inclusion

$$\{w \in S_r \text{ such that } \mathcal{E}'_{hw}|_{S_r} = 0\} \subset \mathcal{M},$$

namely every critical point of the energy $\mathcal{E}_{hw}$ on the constraint S_r belongs to the set $\mathcal{M}$. It is worth mentioning that this fact is reminiscent of the Pohozaev identity, which is here adapted to the case of HW. The following minimization problem will be crucial in the sequel:

$$\mathcal{I}_r = \inf_{S_r \cap \mathcal{M}} \mathcal{E}_{hw}(u).$$

Theorem 4 (see [1])**.** *Let* $d \geq 2$ *and* $1 + \frac{2}{d} < p < 1 + \frac{2}{d-1}$*. Then for every* $r > 0$ *we have:*

- $\mathcal{I}_r > -\infty$ *and* $\mathcal{A}_r \neq \emptyset$*, where*
$$\mathcal{A}_r := \{v \in S_r \cap \mathcal{M} \text{ such that } \mathcal{E}_{hw}(v) = \mathcal{I}_r\}.$$
Moreover any $v \in \mathcal{A}_r$ *satisfies*
$$\sqrt{-\Delta} v + \omega v - v|v|^{p-1} = 0$$
for a suitable $\omega \in \mathbb{R}$*;*
- *assume* $f(x) \in S_r \cap H^1_{rad}(\mathbb{R}^d)$ *satisfies* $\mathcal{E}_{hw}(f) < \mathcal{I}_r$ *and* $\mathcal{P}(f) < 0$*,* $d \geq 2$ *and* $u(t,x)$ *is solution to (7), then the following alternative holds: either the solution blows-up in finite time or* $\|u(t,x)\|_{\dot{H}^{1/2}(R^d)} \geq e^{at}$ *for suitable* $a > 0$*. In particular the set* $\mathcal{A}_r$ *is not* weakly orbitally stable *for the flow associated with HW.*

4. Uniqueness of ground states for Hartree–Choquard equation. Ground states for the classical Hartree–Choquard equation are minimizers of the Hamiltonian

$$H(\psi) = \frac{1}{2}\|\nabla\psi\|^2_{L^2(\mathbb{R}^3)} - \frac{1}{4}D(|\psi|^2, |\psi|^2), \tag{12}$$

where $D(f,g)$ is quadratic form associated with Coulomb energy functional, i.e.

$$D(f,g) = \int_{\mathbb{R}^3} I(f)(x)\overline{g(x)}dx \tag{13}$$

and

$$I(f)(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} f(y)\frac{dy}{|x-y|} \tag{14}$$

is the classical Riesz potential. For any $p \geq 2$ one can define a modified p-Hamiltonian as follows

$$H_p(\psi) = \frac{1}{2}\|\nabla(\psi|\psi|^{(2-p)/p})\|^2_{L^2(\mathbb{R}^3)} - \frac{1}{2p}D(|\psi|^2, |\psi|^2). \tag{15}$$

The ground states are solutions to the constraint minimization problem

$$\inf_{\{\psi \in H^1(\mathbb{R}^3); \|\psi\|^2_{L^2(\mathbb{R}^3)} = \lambda\}} H_p(\psi). \tag{16}$$

A simple substitution

$$\psi|\psi|^{(2-p)/p} = u$$

enables us to transform (16) into the problem to find minimizer of

$$\inf_{\{u \in \dot{H}^1(\mathbb{R}^3) \cap L^p(\mathbb{R}^3); \|u\|^p_{L^p} = \lambda\}} \mathcal{H}_p(u), \tag{17}$$

where

$$\mathcal{H}_p(u) = \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{R}^3)} - \frac{1}{2p}D(|u|^p, |u|^p). \tag{18}$$

Standard symmetrization argument (we mean Schwartz symmetrization) and Gagliardo–Nirenberg inequality

$$D(|u|^p, |u|^p) \leq C_{GN}\|\nabla u\|^{2p/(6-p)}_{L^2(\mathbb{R}^3)}\|u\|^{2p(5-p)/(6-p)}_{L^p(\mathbb{R}^3)}, \quad \forall p \in [1, 5], \tag{19}$$

imply the existence of positive radial decreasing minimizers of (17), but only for the range $2 \leq p < 3$.

In [9] we studied existence and uniqueness of ground states for larger interval $2 \leq p < 5$ and for this reason we can define the Weinstein functional (see [15])

$$W_p(u) = \frac{\|\nabla u(x)\|^{2p/(6-p)}_{L^2(\mathbb{R}^3)}\|u(x)\|^{2p(5-p)/(6-p)}_{L^p(\mathbb{R}^3)}}{D(|u|^p, |u|^p)} \tag{20}$$

and consider the associated minimization problem

$$W_p^{min} = \inf_{\{u \in \dot{H}^1(\mathbb{R}^3) \cap L^p(\mathbb{R}^3); u \neq 0\}} W_p(u). \tag{21}$$

The existence of ground states is guaranteed by the following

Theorem 5 ([9])**.** *Assuming $2 \leq p < 5$, there is a minimizer $u \in \dot{H}^1(\mathbb{R}^3) \cap L^p(\mathbb{R}^3)$ of W_p, such that u is solution of*

$$-\Delta u + |u|^{p-2}u = I(|u|^p)|u|^{p-2}u \tag{22}$$

and satisfies the Pohozaev's normalization conditions

$$\frac{\|u\|^p_{L^p}}{5-p} = \|\nabla u\|^2_{L^2} = \frac{D(|u|^p, |u|^p)}{6-p} = k, \tag{23}$$

for some $k > 0$. In addition, there exists $x_0 \in \mathbb{R}^3$, $z \in \mathbb{C}$ with $|z| = 1$ and a decreasing function $Q : \mathbb{R}_+ \to \mathbb{R}_+$, so that $u(x) = zQ(|x - x_0|)$.

Our second result in [9] treats the uniqueness of minimizers Q of W_p satisfying (23), i.e.

$$Q \in \mathcal{G} = \{u \in \dot{H}^1_{rad} \cap L^p_{rad}; W_p^{min} = \inf_{\{u \in \dot{H}^1(\mathbb{R}^3) \cap L^p(\mathbb{R}^3); u \neq 0\}} W_p(u)\}$$

and such that (23) is fulfilled.

Theorem 6. *For any $2 \leq p < 5$ and any two radial positive minimizers $Q_1, Q_2 \in \mathcal{G}$,* *that satisfy* (23)*, we have* $Q_1 \equiv Q_2$.

There are different method to prove the uniqueness of positive radial minimizes of nonlinear elliptic equations with local type nonlinearities. The methods of Gidas, Ni and Nirenberg, McLeod and Serin and the subsequent refinements due to Kwong are also based on Sturm oscillation argument and therefore they work effectively for local type nonlinearities. In our case the nonlinearities involve the nonlocal Riesz potential and therefore we have met essential difficulties to follow this strategy.

The main idea is to use the asymptotic behaviour

$$I(|Q|^p)(x) = \frac{\|Q\|_{L^p}^p}{4\pi|x|} + o\left(|x|^{-1}\right), \quad x \to \infty \tag{24}$$

and obviously we gain control on the asymptotics of Riesz potential at infinity, since the L^p norm is a conserved Pohozhaev quantity. Applying then the Gronwall argument, we can conclude that the function

$$\varphi(r) = |Q_1(r) - Q_2(r)| + |I(|Q_1|^p)(r) - I(|Q_2|^p)(r)|$$

is identically zero for $r \in [0; \infty)$.

5. Mass critical blow up for Choquard equation with self-interacting term. Finally, we consider the case $\kappa = 1$ and $\mu \geq 0$ in (1), i.e we study the nonlinear Schrödinger equation with mass-critical nonlinearities

$$\begin{cases} i\partial_t u - \Delta u - |u|^{\frac{4}{3}}u - \mu\left(|x|^{-1} * |u|^{\frac{7}{3}}\right)|u|^{\frac{1}{3}}u = 0, \ t \in \mathbb{R}, \, x \in \mathbb{R}^3, \\ u(0,x) = u_0(x), \end{cases} \tag{25}$$

where $u = u(t,x)$ is complex-valued function in time-space $\mathbb{R} \times \mathbb{R}^3$. This model corresponds to a critical non-local perturbation of the classical mass critical problem $\mu = 0$ which still has the scaling symmetry of the problem.

The results in this section are obtained in collaboration with Yuan Li.

We can set

$$I(u)(x) = \left(|x|^{-1} * u\right),$$

and then we have the following system

$$\begin{cases} i\partial_t u - \Delta u - |u|^{\frac{4}{3}}u - \mu I(|u|^{\frac{7}{3}})|u|^{\frac{1}{3}}u = 0, \ t \in \mathbb{R}, \, x \in \mathbb{R}^3, \\ -\Delta I(|u|^{\frac{7}{3}}) = 4\pi|u|^{\frac{7}{3}}, \\ u(0,x) = u_0(x) \in H^1(\mathbb{R}^3). \end{cases} \tag{26}$$

Let us review some basic facts about the Cauchy problem. From Cazenave given $u_0 \in H^1(\mathbb{R}^3)$, there exists a unique maximal solution $u \in C([0,T); H^1(\mathbb{R}^3))$ to (25) and there holds the blowup alternative:

$$T < +\infty \ \text{implies} \ \lim_{t \to T} \|u(t)\|_{H^1} = +\infty. \tag{27}$$

Furthermore, the H^1 flow admits the conservation laws:

Mass:

$$M(u)(t) = \int |u(t,x)|^2 dx = M(u_0). \tag{28}$$

Energy:

$$
\begin{aligned}
E_\mu(u)(t) =& \frac{1}{2}\int |\nabla u(x,t)|^2 dx - \frac{3}{10}\int |u(x,t)|^{\frac{10}{3}} dx \\
& - \frac{3\mu}{14}\int I\left(|u(x,t)|^{\frac{7}{3}}\right)|u(x,t)|^{\frac{7}{3}} dx = E_\mu(u_0).
\end{aligned} \tag{29}
$$

First, we recall the structure of the mass critical problem. In this case, the scaling symmetry

$$u_a(t,x) = a^{\frac{3}{2}} u(a^2 t, ax)$$

acts on the set of solutions and leaves the mass invariant

$$\|u_a(t,\cdot)\|_{L^2} = \|u(a^2 t,\cdot)\|_{L^2}.$$

5.1. The case $\mu = 0$. A criterion of global-in-time existence for H^1 initial data is derived by using the Gagliardo-Nirenberg inequality with the best constant

$$\|u\|_{L^{\frac{10}{3}}}^{\frac{10}{3}} \leq C\|u\|_{L^2}^{\frac{4}{3}}\|\nabla u\|_{L^2}^2,$$

where $C = \frac{5}{3}\frac{1}{\|Q\|_{L^2}^{\frac{4}{3}}}$, and Q is the unique up to symmetries solution to the positive ground state equation

$$-\Delta Q + Q - |Q|^{\frac{4}{3}} Q = 0, \ Q(x) > 0, \ Q \in H^1(\mathbb{R}^3).$$

Thus, for all $u \in H^1(\mathbb{R}^3)$, we have

$$E_0(u) \geq \frac{1}{2}\|\nabla u\|_{L^2}^2\left[1 - \left(\frac{\|u\|_{L^2}}{\|Q\|_{L^2}}\right)^{\frac{4}{3}}\right], \tag{30}$$

which together with the conservation of mass, energy and the blowup criterion (27) implies the global existence of solution with initial data $\|u_0\|_{L^2} < \|Q\|_{L^2}$.

At the mass critical level $\|u_0\|_{L^2} = \|Q\|_{L^2}$, the pseudo-conformal symmetry of (25) yields an explicit minimal blowup solution:

$$S(t,x) = \frac{1}{|t|^{\frac{3}{2}}} Q\left(\frac{x}{t}\right) e^{-i\frac{|x|^2}{4t}} e^{\frac{i}{t}}, \ \|S(t)\|_{L^2} = \|Q\|_{L^2}, \ \|\nabla S(t)\|_{L^2} \overset{t\to 0^-}{\sim} \frac{1}{|t|}.$$

Merle obtained the classification in the energy space of minimal blowup elements; the only H^1 finite time blowup solution with mass $\|u\|_{L^2} = \|Q\|_{L^2}$ is given by above up to the symmetries of the flow.

Note that the minimal blow up dynamic can be extended to the super critical mass case $\|u_0\|_{L^2} > \|Q\|_{L^2}$ and that corresponds to an unstable threshold dynamics between global in time scattering solutions and finite time blow up solutions in the stable blow up regime

$$\|\nabla u(t)\|_{L^2} \sim \sqrt{\frac{\log|\log|T^* - t||}{T^* - t}}, \text{ as } t \sim T^*.$$

Results about the existing literature for the L^2 critical blow up problem, could be found in [12] and references therein.

5.2. The case $\mu > 0$. We consider the model

$$i\partial_t u - \Delta u - |u|^{\frac{4}{3}} u - \mu I(|u|^{\frac{7}{3}})|u|^{\frac{1}{3}} u = 0. \tag{31}$$

We state our first result that shows the existence of solitary waves with small mass. The smallness of the mass shall be connected with where Q the unique radial positive ground state solution of equation

$$-\Delta Q + Q = |Q|^{\frac{4}{3}} Q \tag{32}$$

and the best constant C_* in the Gagliardo-Nirenberg's inequality

$$\|I(|u|^{7/3})|u|^{7/3}\|_{L^1} \leq C_* \|\nabla u\|_{L^2}^2 \|u\|_{L^2}^{8/3}.$$

Theorem 7 (Solitary waves with small mass)**.** *Let $\mu > 0$ be small enough. For all $a \in \left(0, \left(\frac{2}{1+\sqrt{1+8\mu C_* \|Q\|_{L^2}^{\frac{8}{3}}}}\right) \|Q\|_{L^2}^2\right)$ there exists a positive Schwartz radially symmetric solution of*

$$\Delta Q_\mu - Q_\mu + Q_\mu^{\frac{7}{3}} + \mu I(|Q_\mu|^{\frac{7}{3}}) Q_\mu^{\frac{4}{3}} = 0, \ \|Q_\mu\|_{L^2} = a.$$

Define the linear operator $L_{+,\mu}$ and $L_{-,\mu}$ associated to Q_μ by

$$L_{+,\mu}\xi = -\Delta\xi + \xi - \frac{7}{3} Q_\mu^{\frac{4}{3}} \xi - \frac{7}{3}\mu I\left(|Q_\mu|^{\frac{4}{3}}\xi\right) Q_\mu^{\frac{4}{3}} - \frac{4}{3}\mu I\left(|Q_\mu|^{\frac{7}{3}}\right) Q_\mu^{\frac{1}{3}}\xi, \tag{33}$$

$$L_{-,\mu}\xi = -\Delta\xi + \xi - Q_\mu^{\frac{4}{3}}\xi - \mu I\left(|Q_\mu|^{\frac{7}{3}}\right) Q_\mu^{\frac{1}{3}}\xi, \tag{34}$$

acting on $L^2(\mathbb{R}^3)$ with form domain $H^1(\mathbb{R}^3)$. We have the following nondegeneracy result.

$$\ker L_{+,\mu} = \{0\} \text{ when } L_{+,\mu} \text{ is restricted to } L^2_{rad}(\mathbb{R}^3),$$
$$\ker L_{-,\mu} = \{Q_\mu\}.$$

Comments on this result

(1) Existence. From the standard variational argument, we can easily obtain the existence.

(2) Nondegeneracy of $L_{-,\mu}$. From the Sturm argument, we can obtain the $\ker L_{-,\mu} = \{Q_\mu\}$. Here we do not need to assume that the parameter μ is small enough.

(3) Nondegeneracy of $L_{+,\mu}$. This case is very difficult and we only consider the radial case. Here we develop a novel perturbation approach, together with the nondegeneracy property of linear operators $L_{+,0}$ and $L_{-,0}$ to prove this result. On the other hand, we can easily obtain that $Q_\mu \to Q_0$ in $H^1(\mathbb{R}^3)$, but this is a very rough estimate. In this case, this rough estimate is not sufficient. Here we obtain the following estimate

$$\|Q_\mu - Q\|_{H^2} \lesssim \mu.$$

Remark 1. (i) In the above Theorem 6, we only obtain the radial non-degeneracy property.

(ii) We can deduce from Theorem 7 that $L_{+,\mu}[\nabla Q_\mu] = 0$, but it is not clear that $\ker L_{+,\mu} = \text{span}\{\nabla Q_\mu\}$. This property of the wave Q_μ is often referred to as *non-degeneracy.*

A second main result is the existence of a minimal mass blowup solution for (31).

Theorem 8 (Existence of minimal mass blowup elements)**.** *Let $u_0 \in H^1_{rad}(\mathbb{R}^3)$ and $\mu > 0$ is small enough. For $E_\mu(u_0) \in \mathbb{R}_+^*$, there exists $t^* < 0$ and a radial minimal mass solution $u \in \mathcal{C}\left([t^*, 0); H^1(\mathbb{R}^3)\right)$ of equation* (31) *with*

$$\|u\|_{L^2} = \|Q_\mu\|_{L^2}, \ E_\mu(u) = E_\mu(u_0),$$

which blows up at time $T = 0$. More precisely, it holds that

$$u(t,x) - \frac{1}{\lambda^{\frac{3}{2}}(t)} Q_\mu \left(\frac{x}{\lambda(t)} \right) e^{i\gamma(t)} \to 0 \ \textit{in}\ L^2(\mathbb{R}^3)\ \textit{as}\ t \to 0^-,$$

where

$$\lambda(t) = \lambda^* t + \mathcal{O}(t^3),\ \ \gamma(t) = \frac{1}{\lambda^* |t|} + \mathcal{O}(t),$$

with some constant $\lambda^* > 0$, and the blowup speed is given by

$$\|\nabla u(t)\|_{L^2} \sim \frac{C(u_0)}{|t|},\ \textit{as}\ t \to 0^-.$$

REFERENCES

[1] J. Bellazzini, V. Georgiev, E. Lenzmann, N. Visciglia. Long time dynamics for semi-relativistic NLS and half wave in arbitrary dimension. *Math. Ann.*, **371**, No 1–2 (2018), 707–740.

[2] J. Bellazzini, V. Georgiev, E. Lenzmann, N. Visciglia. On traveling solitary waves and absence of small data scattering for nonlinear half-wave equations. *Comm. Math. Phys.*, **372**, No 2 (2019), 713–732.

[3] T. Cazenave, P. L. Lions. Orbital Stability of Standing Waves for Some Nonlinear Schrodinger Equations. *Comm. Math. Phys.*, **85** (1982), 549–561.

[4] J. Chen, B. Guo. Strong instability of standing waves for a nonlocal Schrödinger equation. *Phys. D*, **227** (2007) 142–148.

[5] Z. Cheng, Z. Shen, M. Yang. Instability of standing waves for a generalized Choquard equation with potential. *J. Math. Phys.* **58** (2017), 011504.

[6] R. Frank, E. Lenzmann. Uniqueness of non-linear ground states for fractional Laplacians in **R**. *Acta Math.* **210**, No 2 (2013), 261–318.

[7] Ch. Genev, G. Venkov. Soliton and blow-up solutions to the time-dependent Schrodinger-Hartree equation. *Discrete Contin. Dyn. Syst. Ser. S* , **5**, No 5 (2012), 903–923.

[8] V. Georgiev, A. Stefanov. On the classification of the spectrally stable standing waves of the Hartree problem. *Phys. D*, **370** (2018), 29–39.

[9] V. Georgiev, M. Tarulli, G. Venkov. George Existence and uniqueness of ground states for p-Choquard model. *Nonlinear Anal.*, **179** (2019), 131–145.

[10] E. H. Lieb. Existence and uniqueness of the minimizing solution of Choquard's nonlinear equation. *Stud. Appl. Math.* **57**, No. 2 (1976/1977), 93–105.

[11] Li Ma, Lin Zhao. Classification of positive solitary solutions of the nonlinear Choquard equation. *Arch. Ration. Mech. Anal.* **195**, No 2 (2010), 455–467.

[12] F. Merle, P. Raphaël, J. Szeftel. The instability of Bourgain-Wang solutions for the L^2 critical NLS. Amer. *J. Math.*, **135** (2013), 967–1017.

[13] V. Moroz, J. Van Schaftingen. Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics. *J. Funct. Anal.*, **265**, no. 2 (2013), 153–184.

[14] P. Tod, I. M. Moroz. An analytical approach to the Schrödinger-Newton equations. *Nonlinearity*, **12**, No 2 (1999), 201–216.

[15] M. Weinstein. Modulational stability of ground states of nonlinear Schrödinger equations. *SIAM J. Math. Anal*, **16** No. 3, (1985) 472–491.

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ЕВОЛЮЦИОНЕН ПОТОК И ОСНОВНИ СЪСТОЯНИЯ ЗА ДРОБНОТО УРАВНЕНИЕ НА ШРЬОДИНГЕР–ХАРТРИ

Владимир Георгиев

Изучаваме уравненията на Шрьодингер–Хартри с дробна степен на оператора на Лаплас. Концентрираме нашите изследвания върху следните главни точки:

а) съществуване и асимптотика на солитонните решения;

б) съществуване и асимптотика на еволюционния поток на задачата на Коши за уравнението на половин вълна;

в) единственост на солитонни решения на уравненията на Шрьодингер–Хартри;

г) избухване на решенията на масово критичното поле на Шрьодингер–Хартри.