# Serdica

## Mathematical Journal

# Сердика

## Математическо списание



For further information on
Serdica Mathematical Journal
which is the new series of
Serdica Bulgaricae Mathematicae Publicationes
visit the website of the journal http://www.math.bas.bg/~serdica
or contact: Editorial Office
Serdica Mathematical Journal
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: serdica@math.bas.bg

Serdica Math. J. **27** (2001), 67-90

**Serdica**
Mathematical Journal

Institute of Mathematics
Bulgarian Academy of Sciences

# FORMATION OF SINGULARITIES FOR WEAKLY NON-LINEAR $N \times N$ HYPERBOLIC SYSTEMS

Chiara Boiti, Renato Manfrin

*Communicated by I. D. Iliev*

Abstract. We present some results on the formation of singularities for $C^1$-solutions of the quasi-linear $N \times N$ strictly hyperbolic system $U_t + A(U)U_x = 0$ in $[0, +\infty) \times \mathbb{R}_x$. Under certain weak non-linearity conditions (weaker than genuine non-linearity), we prove that the first order derivative of the solution blows-up in finite time.

**Introduction.** In this paper we consider the quasi-linear strictly hyperbolic system

$$U_t + A(U)U_x = 0 \qquad \text{in } [0, +\infty) \times \mathbb{R}_x\,,$$

where $A(U)$ is an $N \times N$ matrix with $C^1(\mathbb{R}^N)$-entries.

We look for sufficient conditions to obtain blow-up for the $C^1$-solution of the associated Cauchy problem with small compactly supported initial data.

2000 *Mathematics Subject Classification:* 35L65, 35L67.
*Key words:* Blow-up, hyperbolic systems.

Denoting by $\lambda_j(U)$, for $j = 1, \dots, N$, the distinct real eigenvalues of the matrix $A(U)$, and by $r_j(U)$ and $l_j(U)$ the relative right and left eigenvectors with $\|r_j\| = 1$ and $l_i \cdot r_j = \delta_{ij}$, we shall consider the following non-linear condition:

(0.1) $\nabla\lambda_j(U_j(s)) \cdot r_j(U_j(s)) \neq 0$ on a dense subset of $\mathbb{R}_s$, for $j = 1$ or $N$

along the $j$-th characteristic trajectory $U_j(s)$ defined by

$$\begin{cases} \dfrac{d}{ds}U_j(s) = r_j(U_j(s)) \qquad j = 1, \dots, N \\ U_j(0) = 0\,. \end{cases}$$

This condition is weaker than the condition of *genuine non-linearity*

$$\nabla\lambda_j(U) \cdot r_j(U) \neq 0 \qquad \forall U \in \mathbb{R}^N, \quad \forall j = 1, \dots, N$$

introduced by Lax in [8].

Remark, moreover, that our condition (0.1) seems to be "quite sharp" to obtain blow-up, since in [16] the existence of a unique global $C^1$-solution of the Cauchy problem for small initial data under the *weakly linearly degenerate condition*

$$\nabla\lambda_j(U_j(s)) \cdot r_j(U_j(s)) \equiv 0 \qquad \forall j = 1, \dots, N$$

is proved.

Under the above assumption (0.1), we obtain the formation of singularities for the $C^1$-solution of the Cauchy problem

$$\begin{cases} U_t + A(U)U_x = 0 \\ U(0,x) = \varepsilon U_o(x) \in C^1_o(\mathbb{R}) \end{cases}$$

for $\varepsilon$ small enough, if the following condition on the initial data is satisfied:

$$l_j(0) \cdot U_o'(x) \not\equiv 0 \tag{0.2}$$

for $j = 1$ or $N$ (according to which $j = 1$ or $N$ makes condition (0.1) valid).

If we consider, as in [16] and [17], the example of the one-dimensional gas dynamics, we find that our condition (0.2) is in this case necessary and sufficient to obtain blow-up (see §2, Example 2.4).

Finally, (see Theorem 3.1 below), we give also some sufficient conditions for the formation of singularities in the case of general (not necessarily small) initial data, but only in the case of $3 \times 3$ systems.

**1. Preliminaries.** Given the system

$$U_t + A(U)U_x = 0 \qquad \text{in } [0,+\infty) \times \mathbb{R}_x\,, \tag{1.1}$$

we shall assume in the following that the matrix $A(U)$ has $C^1(\mathbb{R}^N)$ entries and $N$ real distinct eigenvalues

$$\lambda_1(U) < \lambda_2(U) < \ldots < \lambda_N(U) \qquad \forall U \in \mathbb{R}^N.$$

We can then choose $N$ right eigenvectors $r_1(U), \ldots, r_N(U)$:

$$A(U)r_i(U) = \lambda_i(U)r_i(U) \qquad \forall U \in \mathbb{R}^N,$$

and $N$ left eigenvectors $l_1(U), \ldots, l_N(U)$:

$$l_i(U)A(U) = \lambda_i(U)l_i(U) \qquad \forall U \in \mathbb{R}^N,$$

normalized such that

$$\|r_i(U)\|_{\mathbb{R}^N} = 1 \quad \text{and} \quad l_i(U) \cdot r_j(U) = \delta_{ij} \qquad \forall U \in \mathbb{R}^N,$$

for all $i, j = 1, \ldots, N$.

**Definition 1.1.** *We call $i$-th* characteristic trajectory $U_i(s)$ *the solution of the Cauchy problem*

$$\begin{cases} \dfrac{d}{ds}U_i(s) = r_i(U_i(s)) \\ U_j(0) = 0\,, \end{cases} \tag{1.2}$$

*for $i = 1, \ldots, N$.*

**Definition 1.2.** *Given, for some $T > 0$, a solution $U(t,x) \in C^1([0,T) \times \mathbb{R}_x)^N$ of the hyperbolic system* (1.1)*, and given $p = (t_p, x_p) \in [0,T) \times \mathbb{R}_x$, we denote by $x_i(t,p)$ the solution of the Cauchy problem*

$$\begin{cases} \dfrac{d}{dt}x_i(t,p) = \lambda_i(U(t, x_i(t,p))) \\ x_i(t_p, p) = x_p \end{cases} \tag{1.3}$$

*for $i = 1, \ldots, N$. The application*

$$t \mapsto \gamma_i(t,p) = (t, x_i(t,p))$$

*is then called the $i$-th* characteristic curve *passing through $p$, and we denote its trace on the $(t,x)$-plane by $\Gamma_i(p)$:*

$$\Gamma_i(p) = \{(t,x) \in [0,T) \times \mathbb{R} : \ x = x_i(t,p)\}\,.$$

*When $p=(0,y)$ we shall also write $x_i(t,y)$, $\gamma_i(t,y)$ and $\Gamma_i(y)$ instead of, respectively, $x_i(t,p)$, $\gamma_i(t,p)$ or $\Gamma_i(p)$.*

We now follow F. John, considering however $C^1$-solutions (instead of $C^2$-solutions) $U(t,x)$ of (1.1), and defining, for $i=1,\dots,N$,

$$w_i(t,x)=l_i(U(t,x))\cdot U_x(t,x)\,.$$

By the choice of the eigenvectors of $A(U)$, we immediately obtain that

$$U_x=\sum_{i=1}^N w_i r_i, \qquad U_t=-\sum_{i=1}^N \lambda_i w_i r_i\,.$$

We shall need in the sequel the following *John's formula* for a solution $U\in C^1([0,T)\times\mathbb{R}_x)^N$ (for the proof in the case of $C^1$-solutions see [12]):

$$\begin{aligned}&w_i(\gamma_i(t_1,p))-w_i(\gamma_i(t_0,p))=\\ &\qquad=\int_{t_0}^{t_1}\sum_{j,k=1}^N \gamma_{ijk}(U(\gamma_i(\tau,p)))w_j(\gamma_i(\tau,p))w_k(\gamma_i(\tau,p))d\tau\end{aligned}\tag{1.4}$$

for all $t_0,t_1\in[0,T)$ and $i=1,\dots,N$, where $\gamma_{ijk}=\gamma_{ijk}(U(t,x))$ are given by:

$$\gamma_{ijk}=(\lambda_j-\lambda_k)l_i Dr_k\{r_j\}-\delta_{ik}\nabla\lambda_i\cdot r_j\tag{1.5}$$

(here $Dr_k\{r_j\}$ denotes the differential of $U\mapsto r_k(U)$ applied to $r_j$).

In particular, it will be useful to notice that

$$\begin{cases}\gamma_{ijj}\equiv 0 & \text{for } i\neq j\\ \gamma_{iii}=-\nabla\lambda_i\cdot r_i\,.\end{cases}$$

Let us finally recall two lemmas from [12] that we shall need in the following:

**Lemma 1.3.** *Let $U(t,x)\in C^1([0,T)\times\mathbb{R}_x)^N$ be the solution, for some $T>0$, of the hyperbolic Cauchy problem*

$$\begin{cases}U_t+A(U)U_x=0\\[2ex] U(0,x)=U_o(x)\in C_o^1(\mathbb{R}_x)^N\,.\end{cases}\tag{1.6}$$

*Assume, moreover, that for some real $\alpha<\beta$ and $i_o\in\{1,\dots,N\}$*

$$w_i(0,x)=0\qquad \forall x\in[\alpha,\beta],\ i\in\{1,\dots,N\}\setminus\{i_o\}\,.$$

*Then, for $i \neq i_o$ we have that $w_i(t,x) \equiv 0$ in the region*

$$\Omega_\alpha^\beta = \{(t,x):\ 0 \le t < T,\ x_m(t,\alpha) \le x \le x_\ell(t,\beta)\}\,,$$

*where*

$$\ell = \min\{1 \le i \le N:\ i \neq i_o\}, \quad m = \max\{1 \le i \le N:\ i \neq i_o\}\,.$$

**Lemma 1.4.** *Let $U(t,x) \in C^1([0,T) \times \mathbb{R}_x)^N$ be the solution, for some $T > 0$, of the hyperbolic Cauchy problem* (1.6)*.*

*Assume, moreover, that, for fixed $\beta \in \mathbb{R}$, $i_o \in \mathbb{N}$, with $1 < i_o < N$, and $p = (t_p, x_p) \in ([0,T) \times \mathbb{R}_x) \setminus \Gamma_{i_o}(\beta)$, the graphs $\Gamma_{i_o-1}(p)$ and $\Gamma_{i_o+1}(p)$ intersect $\Gamma_{i_o}(\beta)$.*

*It follows that if*

$$w_i(t, x_{i_o}(t,\beta)) = 0 \qquad \forall t \in [0,T),\ i \in \{1,\dots,N\} \setminus \{i_o\}$$

*then also*

$$w_i(t,x) \equiv 0 \qquad \text{in } \Omega(i_o,\beta,p) \qquad \text{for } i \neq i_o\,,$$

*where $\Omega(i_o,\beta,p)$ is the region bounded by $\Gamma_{i_o}(\beta)$, $\Gamma_{i_o-1}(p)$ and $\Gamma_{i_o+1}(p)$.*

**2. Blow-up for $N \times N$ hyperbolic systems with small initial data.** Using the notation introduced in the previous section, we now prove:

**Theorem 2.1.** *Let us consider the hyperbolic Cauchy problem in $[0,+\infty) \times \mathbb{R}_x$:*

$$\begin{cases} U_t + A(U)U_x = 0 \\ U(0,x) = \varepsilon U_o(x)\,, \end{cases} \tag{2.1}$$

*where $\varepsilon \in \mathbb{R}$, $U_o \in C_o^1(\mathbb{R}_x)^N$, and $A(U)$ is an $N \times N$ matrix with $C^1(\mathbb{R}^N)$ entries and $N$ distinct real eigenvalues*

$$\lambda_1(U) < \lambda_2(U) < \ldots < \lambda_N(U) \qquad \forall U \in \mathbb{R}^N. \tag{2.2}$$

*Let us assume, moreover, that*

$$\frac{d}{ds}\lambda_i(U_i(s)) \neq 0 \qquad \text{on a dense subset of } \mathbb{R}_s \tag{2.3}$$

*and*

$$l_i(0) \cdot U_o'(x) \not\equiv 0 \tag{2.4}$$

*both for* $i = 1$ *or for* $i = N$.

*Then, for* $\varepsilon \neq 0$ *small enough, the* $C^1$*-solution of the Cauchy problem* (2.1) *must develop some singularities in finite time.*

Proof. Let $\operatorname{supp} U_o \subset [\alpha, \beta]$ with $\alpha < \beta$.

Let us first recall that by the local existence theorem for hyperbolic systems (cf. [4]) there exists $T_\varepsilon > 0$ such that the Cauchy problem (2.1) admits a unique solution $U \in C^1([0, T_\varepsilon) \times \mathbb{R}_x)^N$, and $T_\varepsilon \to +\infty$ as $\varepsilon \to 0$. More precisely, by direct inspection of the proof, we know that $T_\varepsilon \geq \dfrac{C}{|\varepsilon|}$ provided $\varepsilon \neq 0$ is small enough.

Besides, for every fixed $T > 0$ there exist $c = c(T) > 0$ and $\delta = \delta(T) > 0$ such that for $|\varepsilon| \leq \delta$ the solution exists in $[0, T] \times \mathbb{R}$ and

$$\|U\|_{C^1([0,T]\times\mathbb{R}_x)^N} \leq c|\varepsilon| \,. \tag{2.5}$$

For the next we now need to fix some $T_o > 0$ so that using the estimate (2.5), for $\varepsilon$ small enough, the graphs $\Gamma_i(\alpha)$ and $\Gamma_j(\beta)$ of the characteristic curves $\gamma_i(t, \alpha)$ and $\gamma_j(t, \beta)$ intersect in the strip $[0, T_o] \times \mathbb{R}$ if $i > j$.

Remark that since we have compactly supported initial data, by the finite speed of propagation property

$$U(t, x) \equiv 0 \qquad \text{for} \quad x \leq \alpha + \lambda_1(0)t \quad \text{and for} \quad x \geq \beta + \lambda_N(0)t, \qquad t \geq 0\,.$$

Therefore, if $U(t, x) \in C^1([0, T_\varepsilon) \times \mathbb{R}_x)^N$ is the (local) solution of the Cauchy problem (2.1), then $U(t, x)$ is uniformly bounded on every strip $[0, T] \times \mathbb{R}_x$ with $0 < T < T_\varepsilon$, and this implies that for every $p = (t_p, x_p) \in [0, T_\varepsilon) \times \mathbb{R}_x$ the characteristic curves $\gamma_i(t, p)$ are defined for all $t \in [0, T_\varepsilon)$.

Let us set

$$\lambda = \min_{1 \leq i \leq N-1} (\lambda_{i+1}(0) - \lambda_i(0))\,,$$

and fix then

$$T_o = 2\,\frac{\beta - \alpha}{\lambda}\,.$$

By definition of $\lambda$, since the eigenvalues $\lambda_i(U)$ are continuous functions of $U \in \mathbb{R}^N$, we can find $\rho > 0$ such that

$$\|U\|_{\mathbb{R}^N},\ \|V\|_{\mathbb{R}^N} \leq \rho \ \Rightarrow\ \lambda_{i+1}(U) - \lambda_i(V) \geq \frac{\lambda}{2} \ \ \forall i = 1, \ldots, N-1\,. \tag{2.6}$$

By (2.5) we can then find $c = c(T_o) > 0$ and $\delta = \delta(T_o) > 0$ such that

$$\text{if } |\varepsilon| \leq \varepsilon_0 = \min\left\{\delta, \frac{\rho}{c}\right\},\ \text{ then } \|U\|_{C^0([0,T_o]\times\mathbb{R})^N} \leq \|U\|_{C^1([0,T_o]\times\mathbb{R})^N} \leq c|\varepsilon| \leq \rho, \tag{2.7}$$

and hence by (2.6):

$$\lambda_{i+1}(U(t_1,x_1)) - \lambda_i(U(t_2,x_2)) \geq \frac{\lambda}{2} \ \forall (t_1,x_1),(t_2,x_2) \in [0,T_o]\times\mathbb{R}\,, \tag{2.8}$$

for all $i = 1, \dots, N-1$. Then we can easily see that in this strip $[0,T_o]\times\mathbb{R}$ all the graphs $\Gamma_i(\alpha)$ and $\Gamma_j(\beta)$ intersect if $i > j$. Indeed, by (2.8) it follows that for all $t \in [0,T_o]$ we have $x_j(t,\beta) - x_i(t,\alpha) \leq \beta - \alpha - \frac{\lambda}{2}t$. This will be useful in the sequel.

Let us now take $|\varepsilon| \leq \varepsilon_0$, assume by contradiction that the solution $U \in C^1([0,T_o]\times\mathbb{R})^N$ is in fact defined and $C^1$ on the whole $[0,+\infty)\times\mathbb{R}$, and prove that it must develop, on the contrary, some singularities at some finite time $\overline{T} > T_o$.

Let us first recall (cf. [5]) that in the regions $C_2$ and $D_{N-1}$, where

$$C_i = \{(t,x) \in [0,+\infty)\times\mathbb{R} :\ x \leq x_i(t,\alpha)\} \qquad i = 1,\dots,N \tag{2.9}$$

$$D_i = \{(t,x) \in [0,+\infty)\times\mathbb{R} :\ x \geq x_i(t,\beta)\} \qquad i = 1,\dots,N\,, \tag{2.10}$$

the solution $U(t,x)$ is a simple wave, i.e.:

$$w_2(t,x) \equiv \cdots \equiv w_N(t,x) \equiv 0 \qquad \text{in } C_2 \tag{2.11}$$

and

$$w_1(t,x) \equiv \cdots \equiv w_{N-1}(t,x) \equiv 0 \qquad \text{in } D_{N-1}\,. \tag{2.12}$$

Let us assume conditions (2.3) and (2.4) to be satisfied for $i = N$ and let us then prove the formation of singularities in the region $D_{N-1}$ (analogously, if conditions (2.3) and (2.4) are satisfied for $i = 1$, then the blow-up occurs in $C_2$).

By (2.12)

$$U_x(t,x) = w_N(t,x) r_N(U(t,x)) \qquad \text{in } D_{N-1}$$

and hence, denoting by $\dfrac{d}{d_i t}$, for $i \in \{1,\dots,N\}$, the derivation in the direction of the $i$-th characteristic curve, in $D_{N-1}$ we have:

$$\begin{aligned} \frac{d}{d_N t}U(t,x) &= U_t(t,x) + \lambda_N(U(t,x))U_x(t,x) \\ &= U_t(t,x) + \lambda_N(U(t,x)) w_N(t,x) r_N(U(t,x)) \\ &= U_t(t,x) + A(U(t,x))U_x(t,x) \equiv 0\,. \end{aligned}$$

This means that for all $p = (t_p, x_p)$ in $D_{N-1}$

$$U(t, x_N(t,p)) \equiv U(t_p, x_p) \qquad \forall t \geq t_p$$

and hence

$$x_N(t,p) = x_p + \lambda_N(U(t_p, x_p))(t - t_p) \qquad \forall p \in D_{N-1},\ t \geq t_p\,,$$

i.e. $\Gamma_N(p)$ is a straight line for each $p \in D_{N-1}$.

We shall prove the formation of singularities by showing that some of these graphs $\Gamma_N(y) = \Gamma_N(0,y)$, for $y \in [\alpha,\beta]$, intersect in $D_{N-1}$, contradicting the uniqueness property of solutions of the Cauchy problem (1.3) if $U \in C^1([0,+\infty)\times\mathbb{R}_x)^N$.

Since the $\Gamma_N(y)$'s are straight lines in $D_{N-1}$, the intersection depends on their slope and hence on their angular coefficient along $\Gamma_{N-1}(\beta)$.

By the above arguments it's now clear that to prove our theorem it's enough to show that $t \mapsto \lambda_N(U(t, x_{N-1}(t,\beta)))$ is not monotone decreasing for $t \in [0,+\infty)$.

To this aim we first remark that

$$\begin{cases} \dfrac{d}{dt}U(t,x_{N-1}(t,\beta)) = U_t(t,x_{N-1}(t,\beta)) + \lambda_{N-1}(U(t,x_{N-1}(t,\beta)))U_x(t,x_{N-1}(t,\beta)) \\ \qquad = [\lambda_{N-1}(U(t,x_{N-1}(t,\beta))) - \lambda_N(U(t,x_{N-1}(t,\beta)))]\cdot \\ \qquad\quad \cdot w_N(t,x_{N-1}(t,\beta))r_N(U(t,x_{N-1}(t,\beta))) \\ U(0,x_{N-1}(0,\beta)) = U(0,\beta) = 0\,, \end{cases}$$

and hence

$$U(t,x_{N-1}(t,\beta)) = U_N(\phi_N(t)) \qquad \forall t \geq 0\,, \tag{2.13}$$

where $\phi_N(t)$ is the solution of the Cauchy problem

$$\begin{cases} \phi_N'(t) = [\lambda_{N-1}(U_N(\phi_N(t))) - \lambda_N(U_N(\phi_N(t)))]w_N(t,x_{N-1}(t,\beta)) \\ \phi_N(0) = 0\,. \end{cases} \tag{2.14}$$

Indeed, this can be easily checked by computing the derivative of $U_N(\phi_N(t))$ and noticing then that $U(t,x_{N-1}(t,\beta))$ and $U_N(\phi_N(t))$ satisfy the same Cauchy problem.

But to prove that $\lambda_N(U_N(\phi_N(t)))$ is not monotone decreasing, it's enough to show that $\phi_N(t)$ is not monotone. As a matter of fact, if $\phi_N(t)$ is not monotone we can then find $0 \leq t_1 < t_2 < t_3$ such that

$$\phi_N(t_1) = \phi_N(t_3) < \phi_N(t_2)$$

or

$$\phi_N(t_1) = \phi_N(t_3) > \phi_N(t_2)\,.$$

This implies that

$$\lambda_N(U_N(\phi_N(t_1))) = \lambda_N(U_N(\phi_N(t_3)))\,.$$

But $\lambda_N(U_N(\phi_N(t)))$ is not constant in $[t_1, t_3]$, since by assumption (2.3) we have that $\dfrac{d}{ds}\lambda_N(U_N(s))$ does not vanish identically in the interval of extremities $\phi_N(t_1)$ and $\phi_N(t_2)$.

To prove, finally, that $\phi_N(t)$ is not monotone, it's enough to show, by (2.2) and (2.14), that $w_N(t, x_{N-1}(t,\beta))$ changes sign for some $t > 0$.

To this aim we first begin by estimating the sign of $w_N(t, x_N(t,y))$ for $t > 0$ and $y \in [\alpha, \beta]$. This will give also the estimate of $w_N(t, x_{N-1}(t,\beta))$ when the characteristic curve $\gamma_N(t,y)$ intersects the characteristic curve $\gamma_{N-1}(t,\beta)$.

For this reason we can restrict here to consider $0 \le t \le T_o$, since all graphs $\Gamma_N(y)$, for $y \in [\alpha, \beta]$, intersect $\Gamma_{N-1}(\beta)$ for $0 \le t \le T_o$, by the choice of $T_o$.

By John's formula (1.4):

$$\begin{aligned} & w_N(t, x_N(t,y)) = \\ (2.15)\quad & = w_N(0,y) + \int_0^t \sum_{j,k=1}^{N} \gamma_{Njk}(U(\tau, x_N(\tau,y)))\, w_j(\tau, x_N(\tau,y))\, w_k(\tau, x_N(\tau,y))\, d\tau, \end{aligned}$$

where

$$w_N(0,y) = \varepsilon l_N(0) \cdot U_o'(y) + O(\varepsilon^2)\,. \tag{2.16}$$

Let us now estimate the integral in (2.15).

Since the $\gamma_{ijk}$'s, given by (1.5), are continuous functions of $U$, we can find, because of (2.7), a real constant $M > 0$ such that

$$|\gamma_{ijk}(U(\tau, x_N(\tau,y)))| \le M \qquad \forall \tau \in [0, T_o],\ y \in \mathbb{R},\ i,j,k = 1, \ldots, N\,. \tag{2.17}$$

Let us set

$$w(t) = \max_{1 \le i \le N} \sup_{x \in \mathbb{R}} |w_i(t,x)|\,.$$

This is a well defined bounded and continuous function of $t \in [0, T_o]$ (because of (2.7)), since $U(t,x) \equiv 0$ for $x \le \alpha + \lambda_1(0)t$ or $x \ge \beta + \lambda_N(0)t$.

Therefore, as in (2.15), we find that

$$|w_i(t, x_i(t,y))| \le w(0) + \int_0^t \sum_{j,k=1}^{N} |\gamma_{ijk}(U(\tau, x_i(\tau,y)))| w^2(\tau)\, d\tau \qquad \forall i = 1, \ldots, N$$

and hence, from (2.17):

$$0 \le w(t) \le w(0) + N^2 M \int_0^t w^2(\tau)d\tau\,.$$

Therefore

$$w(t) \le \frac{w(0)}{1 - N^2 M w(0) t} \qquad \forall\, 0 \le t < \min\left\{T_o\,, \frac{1}{N^2 M w(0)}\right\}.$$

Besides, we easily have, as in (2.16), that $w(0) \le |\varepsilon|\overline{L}$ for some $\overline{L} > 0$, and hence

$$0 \le w(t) \le \frac{|\varepsilon|\overline{L}}{1 - |\varepsilon| N^2 M \overline{L} t} \qquad \forall\, 0 \le t < \min\left\{T_o\,, \frac{1}{|\varepsilon| N^2 M \overline{L}}\right\}.$$

Then

$$\left|\int_0^t \sum_{j,k=1}^N \gamma_{Njk}(U(\tau, x_N(\tau,y))) w_j(\tau, x_N(\tau,y)) w_k(\tau, x_N(\tau,y)) d\tau\right|$$

$$\le \; N^2 M \int_0^t \frac{\varepsilon^2 \overline{L}^2}{(1 - |\varepsilon| N^2 M \overline{L}\tau)^2} d\tau$$

$$= \; N^2 M \frac{\varepsilon^2 \overline{L}^2}{1 - |\varepsilon| N^2 M \overline{L} t}\, t \qquad \forall\, 0 \le t < \min\left\{T_o\,, \frac{1}{|\varepsilon| N^2 M \overline{L}}\right\},$$

and choosing

$$|\varepsilon| \le \varepsilon_1 < \min\left\{\varepsilon_0\,, \frac{1}{N^2 M \overline{L} T_o}\right\},$$

we have that

$$\left|\int_0^t \sum_{j,k=1}^N \gamma_{Njk}(U(\tau, x_N(\tau,y))) w_j(\tau, x_N(\tau,y)) w_k(\tau, x_N(\tau,y)) d\tau\right|$$

$$\le \; \varepsilon^2 \frac{N^2 M \overline{L}^2 T_o}{1 - \varepsilon_1 N^2 M \overline{L} T_o} \qquad \forall\, 0 \le t \le T_o\,. \tag{2.18}$$

Hence, from (2.15), (2.16) and (2.18):

$$w_N(t, x_N(t,y)) = \varepsilon l_N(0) \cdot U_o'(y) + O(\varepsilon^2) \;\forall\, 0 \le t \le T_o,\; y \in [\alpha, \beta],\; |\varepsilon| \le \varepsilon_1. \tag{2.19}$$

Since

$$l_N(0) \cdot U_o'(y) = \frac{d}{dy}(l_N(0) \cdot U_o(y))$$

is the derivative of a function with compact support in $[\alpha, \beta]$, it must change sign in the interval $[\alpha, \beta]$. Therefore, by (2.19), $w_N(t, x_{N-1}(t, \beta))$ must change sign in the interval $[0, T_o]$, provided $\varepsilon$ is small enough.

This concludes the proof of Theorem 2.1. $\square$

**Remark 2.2.** The assumption (2.3) seems "not too far" from being sharp, since Theorem 4.1 of [16] shows that if

$$\frac{d}{ds}\lambda_i(U_i(s)) = 0 \qquad \forall s \in \mathbb{R},\ \forall i = 1, \dots, N\,,$$

then for $\varepsilon$ sufficiently small the Cauchy problem (2.1) admits a unique global $C^1$-solution.

**Remark 2.3.** If we assume

$$\frac{d}{ds}\lambda_i(U_i(s)) \neq 0 \qquad \forall s \in \mathbb{R}, \tag{2.20}$$

for $i = 1$ or $N$, instead of (2.3) (which is still a little bit weaker than the genuine non-linearity condition), then from the proof of Theorem 2.1 we can also give an estimate of the life-span $T_\varepsilon$ of the solution of the form:

$$\exists c_1, c_2 > 0 \quad \text{such that} \quad \frac{c_1}{|\varepsilon|} \leq T_\varepsilon \leq \frac{c_2}{|\varepsilon|}. \tag{2.21}$$

Indeed,

$$\begin{aligned}
&\frac{d}{dt}\lambda_N(U(t, x_{N-1}(t, \beta)))\\
&= [\lambda_{N-1}(U(t, x_{N-1}(t, \beta))) - \lambda_N(U(t, x_{N-1}(t, \beta)))]w_N(t, x_{N-1}(t, \beta)) \cdot\\
&\quad \cdot \nabla\lambda_N(U(t, x_{N-1}(t, \beta))) \cdot r_N(U(t, x_{N-1}(t, \beta)))\,.
\end{aligned}$$

Since it is not restrictive to assume $\varepsilon > 0$ in (2.1), we can then choose $0 < \varepsilon < \varepsilon_1$ sufficiently small, so that assumption (2.20) implies

$$\nabla\lambda_N(U(t, x_{N-1}(t, \beta))) \cdot r_N(U(t, x_{N-1}(t, \beta))) \geq \eta > 0 \qquad \forall t \in [0, T_o] \tag{2.22}$$

or

$$\nabla\lambda_N(U(t, x_{N-1}(t, \beta))) \cdot r_N(U(t, x_{N-1}(t, \beta))) \leq -\eta < 0\ \forall t \in [0, T_o] \tag{2.23}$$

for some $\eta > 0$, because of (2.7).

Let us assume, for instance, that (2.22) is satisfied. Then we fix $[y_2, y_1] \subset [\alpha, \beta]$ such that

$$l_N(0) \cdot U_o'(y) \leq -\delta < 0 \qquad \forall y \in [y_2, y_1]\,, \tag{2.24}$$

for some $\delta > 0$ (if (2.23) is satisfied, then we fix $[y_2, y_1] \subset [\alpha, \beta]$ where $l_N(0) \cdot U_o'(y) \geq \delta > 0$); this is possible since $l_N(0) \cdot U_o(y)$ has compact support in $[\alpha, \beta]$ and is not identically zero because of (2.4).

From (2.19) and (2.24) we deduce that there exists $\delta' > 0$ such that

$$w_N(t, x_N(t, y)) \leq -\varepsilon\delta' \qquad \forall t \in [0, T_o],\ y \in [y_2, y_1]$$

if $\varepsilon$ is small enough.

Denoting by $(t_1, x_1)$ and $(t_2, x_2)$ the intersection points of, respectively, $\Gamma_N(y_1)$ and $\Gamma_N(y_2)$ with $\Gamma_{N-1}(\beta)$, we have that $0 \leq t_1 < t_2 \leq T_o$ (since $\Gamma_N(y_1)$ and $\Gamma_N(y_2)$ cannot intersect for $0 \leq t \leq T_o$) and

$$w_N(t, x_{N-1}(t, \beta)) \leq -\varepsilon\delta' < 0 \qquad \forall t \in [t_1, t_2]\,.$$

Finally, by (2.8), we obtain the estimate

$$\frac{d}{dt}\lambda_N(U(t, x_{N-1}(t, \beta))) \geq \varepsilon\frac{\delta'\eta\lambda}{2} > 0 \qquad \forall t \in [t_1, t_2]\,. \tag{2.25}$$

This implies that the life-span $T_\varepsilon$ of the solution is bounded from above by the time $\overline{T}$ when the straight line $\Gamma_N(y_1)$ intersects the straight line $\Gamma_N(y_2)$ (in $D_{N-1}$).

To estimate this time $\overline{T}$ let us set, for simplicity,

$$\lambda_{N,i} = \lambda_N(U(t_i, x_{N-1}(t_i, \beta))) \qquad \text{for } i = 1, 2\,.$$

In $D_{N-1}$ the straight lines $\Gamma_N(y_i)$, for $i = 1, 2$, are thus described by the equations

$$x = \lambda_{N,i}(t - t_i) + x_i\,,$$

where $x_i = x_{N-1}(t_i, \beta)$. Therefore they intersect for

$$\overline{T} = \frac{x_1 - \lambda_{N,1}t_1 - x_2 + \lambda_{N,2}t_2}{\lambda_{N,2} - \lambda_{N,1}}\,.$$

Note that $\lambda_{N,2} > \lambda_{N,1}$ by the choice of $t_1$ and $t_2$.

For $\varepsilon$ sufficiently small, because of (2.7):

$$\lambda_{N,i} = \lambda_N(0) + O(\varepsilon) \qquad \text{for } i = 1, 2$$

and

$$x_i = x_N(t_i, y_i) = y_i + \lambda_N(0)t_i + O(\varepsilon) \qquad \text{for } i = 1, 2\,.$$

Then

$$x_1 - \lambda_{N,1}t_1 - x_2 + \lambda_{N,2}t_2 = y_1 - y_2 + O(\varepsilon)\,.$$

Moreover,

$$\begin{aligned}\lambda_{N,2} - \lambda_{N,1} &= \lambda_N(U(t_2, x_{N-1}(t_2,\beta))) - \lambda_N(U(t_1, x_{N-1}(t_1,\beta))) \\ &= \int_{t_1}^{t_2} \frac{d}{ds}\lambda_N(U(s, x_{N-1}(s,\beta)))ds \\ &= \varepsilon\Lambda(t_2 - t_1) + o(\varepsilon) \qquad (2.26)\end{aligned}$$

for some $\Lambda > 0$ and for $\varepsilon$ small enough, because of (2.25).

To estimate $t_2 - t_1$ let us still choose $\varepsilon$ sufficiently small so that all characteristic curves can be approximated by straight lines, and hence $t_1$ and $t_2$ can be approximatively defined by:

$$\beta + \lambda_{N-1}(0)t_i = y_i + \lambda_N(0)t_i + O(\varepsilon), \qquad i = 1, 2\,.$$

Then

$$\begin{aligned}t_2 - t_1 &= \frac{\beta - y_2}{\lambda_N(0) - \lambda_{N-1}(0)} - \frac{\beta - y_1}{\lambda_N(0) - \lambda_{N-1}(0)} + O(\varepsilon) \\ &= \frac{y_1 - y_2}{\lambda_N(0) - \lambda_{N-1}(0)} + O(\varepsilon)\,,\end{aligned}$$

and hence, from (2.26):

$$\lambda_{N,2} - \lambda_{N,1} = \varepsilon\frac{\Lambda(y_1 - y_2)}{\lambda_N(0) - \lambda_{N-1}(0)} + o(\varepsilon)\,.$$

Therefore

$$\overline{T} = \frac{y_1 - y_2 + O(\varepsilon)}{\dfrac{\varepsilon\Lambda(y_1 - y_2)}{\lambda_N(0) - \lambda_{N-1}(0)} + o(\varepsilon)} = \frac{\lambda_N(0) - \lambda_{N-1}(0)}{\varepsilon\Lambda} + O(1).$$

This gives a bound from above of the life-span $T_\varepsilon$ of the solution.

From the proof of Theorem VI of [4] we have a similar estimate also from below, so that we finally obtain (2.21). □

Let us now consider an example of application of Theorem 2.1:

**Example 2.4** (system of one-dimensional gas dynamics)**.** The following system in $\mathbb{R}_t \times \mathbb{R}_x$ is considered in gas dynamics:

$$\begin{cases}\partial_t u + \partial_x p = 0 \\ \partial_t p + a(p,S)\partial_x u = 0 \\ \partial_t S = 0\,,\end{cases} \qquad (2.27)$$

where $u$ is the velocity, $S$ the entropy, $p$ the pressure and

$$a(p,S) \in C^1(\mathbb{R}^2), \qquad a(p,S) > 0 \quad \forall p, S \in \mathbb{R}$$

is connected with the state equation of the gas $p = p(\tau, S)$ by the relation

$$a(p,S) = -\frac{\partial}{\partial \tau} p(\tau, S)\,, \tag{2.28}$$

where $\tau$ is the specific volume. Remark that condition $a(p,S) > 0$ makes system (2.27) strictly hyperbolic. Indeed, if we set $U(t,x) = {}^t(u(t,x), p(t,x), S(t,x))$, we can rewrite system (2.27) in the standard form $U_t + A(U)U_x = 0$ with

$$A(U) = \begin{pmatrix} 0 & 1 & 0 \\ a(p,S) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues of $A(U)$ are

$$\lambda_1(U) = -\sqrt{a(p,S)} < 0 = \lambda_2(U) < \sqrt{a(p,S)} = \lambda_3(U)\,.$$

The normalized right eigenvectors $r_i(U)$ and left eigenvectors $l_i(U)$ are given by:

$$\begin{cases} r_1(U) = \dfrac{1}{\sqrt{1+a(p,S)}}\,{}^t(1, -\sqrt{a(p,S)}, 0) \\ r_2(U) = {}^t(0,0,1) \\ r_3(U) = \dfrac{1}{\sqrt{1+a(p,S)}}\,{}^t(1, \sqrt{a(p,S)}, 0) \end{cases}$$

and

$$\begin{cases} l_1(U) = \dfrac{\sqrt{1+a(p,S)}}{2\sqrt{a(p,S)}}(\sqrt{a(p,S)}, -1, 0) \\ l_2(U) = (0,0,1) \\ l_3(U) = \dfrac{\sqrt{1+a(p,S)}}{2\sqrt{a(p,S)}}(\sqrt{a(p,S)}, 1, 0)\,. \end{cases}$$

The Cauchy problem considered in gas dynamics is the one obtained by associating to system (2.27) the initial data

$$u(0,x) = \overline{u}_o + \varepsilon u_o(x), \quad p(0,x) = \overline{p}_o + \varepsilon p_o(x), \quad S(0,x) = \overline{S}_o + \varepsilon S_o(x),$$

where $\overline{u}_o, \overline{p}_o, \overline{S}_o \in \mathbb{R}$ are constants, while $u_o(x), p_o(x), S_o(x) \in C_o^1(\mathbb{R}_x)$; $\varepsilon$ is a parameter to be chosen sufficiently small.

Setting

$$\overline{U}_o = {}^t(\overline{u}_o, \overline{p}_o, \overline{S}_o) \in \mathbb{R}^3 \quad \text{and} \quad U_o(x) = {}^t(u_o(x), p_o(x), S_o(x)) \in C_o^1(\mathbb{R}_x)^3 ,$$

the Cauchy problem (2.27) is then of the same form as (2.1) shifted by a constant $\overline{U}_o$:

$$\text{(2.29)} \qquad \begin{cases} U_t + A(U)U_x = 0 \\ U(0,x) = \overline{U}_o + \varepsilon U_o(x) . \end{cases}$$

Let us verify whether the assumptions of Theorem 2.1 are satisfied. Setting

$$U_j(s) = {}^t(u_j(s), p_j(s), S_j(s)) \qquad \text{for } j = 1, 2, 3,$$

we obtain, for $j = 1, 3$:

$$\nabla\lambda_j(U_j(s)) \cdot r_j(U_j(s)) = \frac{a_p(p_j(s), \overline{S}_o)}{2\sqrt{1 + a(p_j(s), \overline{S}_o)}} ,$$

since $S_j(s) \equiv \overline{S}_o$ here. Note also that $\nabla\lambda_2(U_2(s)) \cdot r_2(U_2(s)) \equiv 0$ since $\lambda_2(U) \equiv 0$.

Condition (2.3) of Theorem 2.1 is thus expressed here by:

$$\text{(2.30)} \qquad a_p(p_j(s), \overline{S}_o) \neq 0 \qquad \text{on a dense subset of } \mathbb{R}_s ,$$

for $j = 1$ or 3, which will be verified if

$$a_p(p, \overline{S}_o) \neq 0 \qquad \text{on a dense subset of } \mathbb{R}_p ,$$

by the strict monotony of $p_j(s)$.

Let us now verify condition (2.4), for $i = 1$ or 3:

$$l_j(\overline{U}_o) \cdot U_o'(x) = \varepsilon \frac{\sqrt{1 + a(\overline{p}_o, \overline{S}_o)}}{2\sqrt{a(\overline{p}_o, \overline{S}_o)}} [\sqrt{a(\overline{p}_o, \overline{S}_o)} u_o'(x) \mp p_o'(x)] .$$

This means that if condition (2.4) was not satisfied, then we should have

$$\begin{cases} \sqrt{a(\overline{p}_o, \overline{S}_o)} u_o'(x) - p_o'(x) \equiv 0 \\ \sqrt{a(\overline{p}_o, \overline{S}_o)} u_o'(x) + p_o'(x) \equiv 0 , \end{cases}$$

and hence $u_o'(x) \equiv p_o'(x) \equiv 0$, i.e.

$$u(0,x) \equiv \overline{u}_o, \qquad p(0,x) \equiv \overline{p}_o\,.$$

But in this case the Cauchy problem (2.29) would have the global solution $U(t,x) \in C^1([0,+\infty) \times \mathbb{R})$ given by:

$$u(t,x) = \overline{u}_o, \quad p(t,x) = \overline{p}_o, \quad S(t,x) = \overline{S}_o + \varepsilon S_o(x)\,.$$

Note also that $l_2(0) \cdot U_o'(x) = \varepsilon S_o'(x)$ which can be identically zero or not, and, at the same time, the Cauchy problem (2.29) could have a global solution or blow-up may occur.

We can conclude that in this example of one-dimensional gas dynamics Theorem 2.1 is "quite sharp", in the sense that if condition (2.30) is satisfied, then condition (2.4) is necessary and sufficient to have blow-up in finite time (for $\varepsilon$ sufficiently small).

Remark that from (2.28), by the local invertibility theorem, we can compute:

$$a_p(p,S) = -\frac{p_{\tau\tau}(\tau,S)}{p_\tau(\tau,S)} = \frac{p_{\tau\tau}(\tau,S)}{a(p,S)}\,.$$

This shows that our condition (2.30) is weaker than the following condition of [17]:

$$\text{(2.31)} \qquad \frac{\partial^\alpha}{\partial\tau^\alpha} p(\overline{\tau}_o, \overline{S}_o) = 0 \quad \text{for } 1 < \alpha < \beta, \qquad \frac{\partial^\beta}{\partial\tau^\beta} p(\overline{\tau}_o, \overline{S}_o) \neq 0$$

for some integer $\beta \geq 2$, where $\overline{\tau}_o > 0$ is determined by $\overline{p}_o = p(\overline{\tau}_o, \overline{S}_o)$.

Let us finally remark that if

$$a_p(p_j(s), \overline{S}_o) \neq 0 \qquad \forall s \in \mathbb{R}\,,$$

for $j = 1$ or 3, instead of (2.30), then condition (2.20) is satisfied and hence, from Remark 2.3, we obtain an estimate of the life-span $T_\varepsilon$ of the solution of the form (2.21), i.e. $T_\varepsilon \sim \varepsilon^{-1}$, which is the same estimate obtained in [17] when (2.31) is satisfied with $\beta = 2$.

**3. Blow-up for $3 \times 3$ hyperbolic systems with general initial data.** In the case of $3 \times 3$ hyperbolic systems we are able to give some sufficient conditions for the formation of singularities, without the assumption of small initial data. With respect to Theorem 2.1, we can also eliminate condition (2.4), but we need to strengthen assumptions (2.2) and (2.3):

**Theorem 3.1.** *Let us consider the hyperbolic Cauchy problem in* $[0,+\infty)\times\mathbb{R}_x$

$$\begin{cases} U_t + A(U)U_x = 0 \\ U(0,x) = U_o(x)\,, \end{cases} \tag{3.1}$$

*where* $U_o \in C^1_o(\mathbb{R}_x)^3$, $U_o \not\equiv 0$, *and* $A(U)$ *is a* $3\times 3$ *matrix with* $C^1(\mathbb{R}^3)$ *entries and real distinct eigenvalues* $\lambda_j(U)$ *satisfying, for a suitable constant* $\Lambda > 0$:

$$\lambda_{i+1}(U) - \lambda_i(V) \geq \Lambda \qquad \forall U,V \in \mathbb{R}^3,\ i = 1,2\,. \tag{3.2}$$

*Let us assume, moreover, that*

$$\frac{d}{ds}\lambda_i(U_i(s)) \geq 0 \quad \forall s \in \mathbb{R} \qquad \text{and is different from zero on a dense subset of } \mathbb{R}_s,\ \text{for } i = 1,3 \tag{3.3}$$

$$\frac{d}{ds}\lambda_2(U_2(s)) \neq 0 \qquad \text{on a dense subset of } \mathbb{R}_s \tag{3.4}$$

$$\frac{d}{ds}\lambda_2(U_i(s)) \geq 0 \quad \forall s \in \mathbb{R} \qquad \text{and is different from zero on a dense subset of } \mathbb{R}_s,\ \text{for } i = 1,3\,. \tag{3.5}$$

*Then, the* $C^1$*-solution of the Cauchy problem* (3.1) *must develop some singularities in finite time.*

P r o o f. Let $\operatorname{supp} U_o \subset [\alpha,\beta]$, with $\alpha < \beta$, and, by contradiction, let $U(t,x)$ be a $C^1$ global solution of (3.1) in $[0,+\infty)\times\mathbb{R}_x$.

We use the same notation as in the proof of Theorem 2.1; in particular, the regions $C_i$ and $D_i$ are defined by (2.9) and (2.10).

We can first remark that condition (3.2) ensure us that all the graphs $\Gamma_i(\beta)$ and $\Gamma_j(\alpha)$ must intersect for $i<j$ at some positive time $T \leq \dfrac{\beta-\alpha}{\Lambda}$.

Arguing therefore as in the proof of Theorem 2.1, we can say that the graphs $\Gamma_3(p)$, for $p = (t_p, x_p) \in D_2$, are straight lines which cannot intersect, so that we must have:

$$\frac{d}{dt}\lambda_3(U(t,x_2(t,\beta))) \leq 0 \qquad \forall t \geq 0\,.$$

But $U_x(t,x) = w_3(t,x)r_3(U(t,x))$ in $D_2$, and hence

$$\begin{aligned} 0 \geq\ & \frac{d}{dt}\lambda_3(U(t,x_2(t,\beta))) \\ =\ & [\lambda_2(U(t,x_2(t,\beta))) - \lambda_3(U(t,x_2(t,\beta)))]w_3(t,x_2(t,\beta))\,\cdot \end{aligned} \tag{3.6}$$

$$\cdot\nabla\lambda_3(U(t,x_2(t,\beta)))\cdot r_3(U(t,x_2(t,\beta))).$$

Recall that, from (2.13) and (2.14),

$$U(t,x_2(t,\beta)) = U_3(\phi_3(t)) \qquad \forall t \geq 0$$

with

$$\phi_3'(t) = [\lambda_2(U(t,x_2(t,\beta))) - \lambda_3(U(t,x_2(t,\beta)))]w_3(t,x_2(t,\beta))\,. \tag{3.7}$$

Therefore, if by contradiction

$$w_3(\tilde{t},x_2(\tilde{t},\beta)) < 0 \qquad \text{for some } \tilde{t} > 0\,, \tag{3.8}$$

then there exists $\delta > 0$ such that

$$w_3(t,x_2(t,\beta)) < 0 \qquad \forall t \in (\tilde{t}-\delta,\tilde{t}+\delta) = \tilde{I}\,.$$

This implies that

$$\phi_3'(t) > 0 \qquad \forall t \in \tilde{I}\,, \tag{3.9}$$

and hence $\phi_3(\tilde{I})$ contains some interval $\overline{I}$. By assumption (3.3) there must then exist some $\overline{s} \in \overline{I} \subset \phi_3(\tilde{I})$ such that

$$\nabla\lambda_3(U_3(\overline{s}))\cdot r_3(U_3(\overline{s})) > 0\,.$$

Since $\overline{s} \in \phi_3(\tilde{I})$ there is $\tilde{t}' \in \tilde{I}$ such that $\phi_3(\tilde{t}') = \overline{s}$ and hence

$$\nabla\lambda_3(U(\tilde{t}',x_2(\tilde{t}',\beta)))\cdot r_3(U(\tilde{t}',x_2(\tilde{t}',\beta))) = \nabla\lambda_3(U_3(\phi_3(\tilde{t}')))\cdot r_3(U_3(\phi_3(\tilde{t}'))) > 0\,.$$

But this contradicts (3.6), because of (3.9) and (3.7), since $\tilde{t}' \in \tilde{I}$.

Then (3.8) cannot be satisfied, i.e.

$$w_3(t,x_2(t,\beta)) \geq 0 \qquad \forall t \geq 0\,. \tag{3.10}$$

Analogously, $\Gamma_1(p)$ are straight lines for $p = (t_p,x_p) \in C_2$ which cannot intersect, so that for all $t \geq 0$

$$\begin{aligned} 0 \leq\ & \frac{d}{dt}\lambda_1(U(t,x_2(t,\alpha))) \\ =\ & [\lambda_2(U(t,x_2(t,\alpha))) - \lambda_1(U(t,x_2(t,\alpha)))]w_1(t,x_2(t,\alpha))\,\cdot \\ & \cdot\nabla\lambda_1(U(t,x_2(t,\alpha)))\cdot r_1(U(t,x_2(t,\alpha))) \end{aligned}$$

and hence

$$w_1(t,x_2(t,\alpha)) \geq 0 \qquad \forall t \geq 0\,. \tag{3.11}$$

Let us now show that we cannot have, at the same time,

$$w_3(t, x_2(t, \beta)) \equiv 0 \qquad \forall t \geq 0 \tag{3.12}$$

and

$$w_1(t, x_2(t, \alpha)) \equiv 0 \qquad \forall t \geq 0\,. \tag{3.13}$$

If this was the case, indeed, we should have:

$$\begin{cases} \dfrac{d}{dt}U(t, x_2(t, \beta)) = U_t(t, x_2(t, \beta)) + \lambda_2(U(t, x_2(t, \beta)))U_x(t, x_2(t, \beta)) \\ \qquad\qquad\qquad\qquad = [\lambda_2(U(t, x_2(t, \beta))) - \lambda_3(U(t, x_2(t, \beta)))]\cdot \\ \qquad\qquad\qquad\qquad\quad \cdot w_3(U(t, x_2(t, \beta)))r_3(U(t, x_2(t, \beta))) \equiv 0 \\ U(0, \beta) = 0 \end{cases}$$

and, analogously,

$$\begin{cases} \dfrac{d}{dt}U(t, x_2(t, \alpha)) \equiv 0 \\ U(0, \alpha) = 0\,. \end{cases}$$

Therefore

$$U(t, x_2(t, \beta)) \equiv 0 \qquad \forall t \geq 0 \tag{3.14}$$

and

$$U(t, x_2(t, \alpha)) \equiv 0 \qquad \forall t \geq 0\,. \tag{3.15}$$

Since

$$\begin{aligned} \frac{d}{dt}U(t, x_3(t, y)) = {} & U_t(t, x_3(t, y)) + \lambda_3(U(t, x_3(t, y)))U_x(t, x_3(t, y)) \\ = {} & 0 \qquad \text{in } D_2\,, \end{aligned}$$

then (3.14) implies that $U(t, x) \equiv 0$ in $D_2$, and hence

$$w_i(t, x) \equiv 0 \qquad \text{in } D_2,\ \forall i = 1, 2, 3\,.$$

Then from Lemma 1.4 we deduce that

$$w_i(t, x) \equiv 0 \qquad \text{in } D_1,\ \text{for } i = 1, 3\,. \tag{3.16}$$

Analogously, since

$$\frac{d}{dt}U(t, x_1(t, y)) = 0 \qquad \text{in } C_2 ,$$

from (3.15) we have that $U(t, x)$ and hence $w_i(t, x)$ are identically zero in $C_2$, for $i = 1, 2, 3$, and hence from Lemma 1.4 again:

$$w_i(t, x) \equiv 0 \qquad \text{in } C_3, \text{ for } i = 1, 3 . \tag{3.17}$$

Using now Lemma 1.3 for $0 \le t \le \bar{t}$, where $\bar{t}$ is the time when $\Gamma_3(\alpha)$ intersects $\Gamma_1(\beta)$, from (3.16) and (3.17) we obtain that

$$w_1(t, x) \equiv w_3(t, x) \equiv 0 \qquad \forall(t, x) \in [0, +\infty) \times \mathbb{R} . \tag{3.18}$$

This means that

$$U_x(t, x) = w_2(t, x)r_2(U(t, x)) \qquad \forall(t, x) \in [0, +\infty) \times \mathbb{R} \tag{3.19}$$

and hence

$$\begin{aligned} \frac{d}{dt}U(t, x_2(t, y)) &= U_t(t, x_2(t, y)) + \lambda_2(U(t, x_2(t, y)))U_x(t, x_2(t, y)) \\ &= 0 \qquad \forall(t, y) \in [0, +\infty) \times \mathbb{R} , \end{aligned}$$

i.e.

$$U(t, x_2(t, y)) = U(0, y) \qquad \forall(t, y) \in [0, +\infty) \times \mathbb{R}$$

and the graphs $\Gamma_2(y)$ are straight lines.

Since we are assuming by contradiction to have a global $C^1$ solution, these straight lines can never intersect, and therefore we must have:

$$\lambda_2(U(0, y)) \equiv \lambda_2(U_o(y)) \equiv \lambda_2(0) \qquad \forall y \in \mathbb{R} .$$

This implies that

$$\begin{aligned} 0 \equiv \frac{d}{dt}\lambda_2(U(0, y)) &= \nabla\lambda_2(U_o(y)) \cdot U_o'(y) = \nabla\lambda_2(U_o(y)) \cdot w_2(0, y)r_2(U_o(y)) \\ &= w_2(0, y)\nabla\lambda_2(U_2(\phi_2(y))) \cdot r_2(U_2(\phi_2(y))) , \end{aligned} \tag{3.20}$$

where

$$\phi_2(y) = \int_\alpha^y w_2(0, x)dx ,$$

since $U_2(\phi_2(y))$ and $U(0, y)$ satisfy the same Cauchy problem, because of (3.19).

We can then argue as in the proof of (3.10), showing that (3.4) and (3.20) imply that

$$w_2(0,y) \equiv 0 \qquad \forall y \in \mathbb{R}\,.$$

If there would exist, indeed, some $\tilde{y} \in \mathbb{R}$ such that $w_2(0,\tilde{y}) \neq 0$, then we should find, because of (3.4), some $\tilde{y}' \in \mathbb{R}$ such that

$$w_2(0,\tilde{y}')\nabla\lambda_2(U_2(\phi_2(\tilde{y}'))) \cdot r_2(U_2(\phi_2(\tilde{y}'))) \neq 0\,,$$

contradicting (3.20).

Then $w_2(0,y) \equiv 0$ together with (3.18) imply that $\partial_x U(0,x) = U_o'(x) = 0$ for all $x \in \mathbb{R}$. This contradicts the assumption $U_o \not\equiv 0$, and therefore we cannot have both (3.12) and (3.13). Then, from (3.10) and (3.11), one at least of the two following strict inequalities must be satisfied:

$$w_3(t, x_2(t,\beta)) > 0 \qquad \text{in some } [t_1,t_2] \subset [0,+\infty) \tag{3.21}$$

or

$$w_1(t, x_2(t,\alpha)) > 0 \qquad \text{in some } [t_1,t_2] \subset [0,+\infty). \tag{3.22}$$

But

$$\begin{aligned}\frac{d}{dt}\lambda_2(U(t,x_2(t,\beta))) = \; & [\lambda_2(U(t,x_2(t,\beta))) - \lambda_3(U(t,x_2(t,\beta)))]w_3(t,x_2(t,\beta)) \cdot \\ & \cdot\nabla\lambda_2(U(t,x_2(t,\beta))) \cdot r_3(U(t,x_2(t,\beta))) \\ \le \; & 0 \qquad \forall t \ge 0 \qquad (3.23)\end{aligned}$$

by (3.5) and (3.10), and analogously

$$\begin{aligned}\frac{d}{dt}\lambda_2(U(t,x_2(t,\alpha))) = \; & [\lambda_2(U(t,x_2(t,\alpha))) - \lambda_1(U(t,x_2(t,\alpha)))]w_1(t,x_2(t,\alpha)) \cdot \\ & \cdot\nabla\lambda_2(U(t,x_2(t,\alpha))) \cdot r_1(U(t,x_2(t,\alpha))) \\ \ge \; & 0 \qquad \forall t \ge 0 \qquad (3.24)\end{aligned}$$

by (3.5) and (3.11).

Moreover, from (3.21) and (3.22), arguing as in the proof of (3.10), we have that at least one of the two preceding inequalities, (3.23) or (3.24) must be strict for some $t \ge 0$, because of (3.5). This means that the graphs $\Gamma_2(\alpha)$ and $\Gamma_2(\beta)$ must intersect somewhere, contradicting the existence of a global $C^1$-solution of the Cauchy problem (3.1). We must then have blow-up in finite time. □

**Remark 3.2.** Conditions (3.3) and (3.4) are weaker then the genuine non-linearity condition. Assumption (3.5) is, on the contrary, an extra condition.

Let us now consider an example of application of Theorem 3.1:

**Example 3.3** Let us consider the $3 \times 3$ matrix of $U = {}^t(U_1, U_2, U_3)$:

$$A(U) = \begin{pmatrix} U_1^3 & 0 & U_2^2 + 2 \\ 0 & U_1^3 + U_2^2 + 1 & 0 \\ 0 & 0 & U_1^3 + U_2^2 + 2 \end{pmatrix}.$$

The matrix $A(U)$ has eigenvalues

$$\lambda_1(U) = U_1^3\,, \quad \lambda_2(U) = U_1^3 + U_2^2 + 1\,, \quad \lambda_3(U) = U_1^3 + U_2^2 + 2\,.$$

They satisfy condition(3.2) with $\Lambda = 1$.

We can find the right eigenvectors

$$r_1(U) = {}^t(1,0,0), \quad r_2(U) = {}^t(0,1,0), \quad r_3(U) = \frac{1}{\sqrt{2}}\,{}^t(1,0,1),$$

and then compute

$$U_1(s) = {}^t(s,0,0), \quad U_2(s) = {}^t(0,s,0), \quad U_3(s) = \frac{1}{\sqrt{2}}\,{}^t(s,0,s) \qquad \text{for } s \in \mathbb{R}\,.$$

Let us verify that conditions (3.3), (3.4) and (3.5) are satisfied:

$$\frac{d}{ds}\lambda_1(U_1(s)) = \nabla\lambda_1(U_1(s)) \cdot r_1(U_1(s)) = 3s^2$$

$$\frac{d}{ds}\lambda_3(U_3(s)) = \nabla\lambda_3(U_3(s)) \cdot r_3(U_3(s)) = \frac{3}{2\sqrt{2}}\,s^2$$

$$\frac{d}{ds}\lambda_2(U_2(s)) = \nabla\lambda_2(U_2(s)) \cdot r_2(U_2(s)) = 2s$$

$$\frac{d}{ds}\lambda_2(U_1(s)) = \nabla\lambda_2(U_1(s)) \cdot r_1(U_1(s)) = 3s^2$$

$$\frac{d}{ds}\lambda_2(U_3(s)) = \nabla\lambda_2(U_3(s)) \cdot r_3(U_3(s)) = \frac{3}{2\sqrt{2}}\,s^2\,.$$

Since all assumptions of Theorem 3.1 are satisfied, the solution of the associated Cauchy problem

$$\begin{cases} U_t + A(U)U_x = 0 \\ U(0,x) = U_o(x) \in C_o^1(\mathbb{R})^3, \quad U_o \not\equiv 0 \end{cases}$$

must develop some singularities in finite time.

Remark that this system is not genuinely non-linear, since all $\frac{d}{ds}\lambda_i(U_i(s))$ vanish for $s = 0$. Not even the assumptions of the theorems of [12] are satisfied in this example, since $U = 0$ is not a minimum point nor a maximum point for any of the eigenvalues $\lambda_1(U), \lambda_2(U), \lambda_3(U)$.

## REFERENCES

[1] S. Alinhac. Blowup for Nonlinear Hyperbolic Equations. Birkhäuser, Boston, 1995.

[2] C. Boiti, R. Manfrin. Some Results of Blow-Up and Formation of Singularities for Non-Strictly Hyperbolic Equations. *Funkcial. Ekvac.* **43**, *1* (2000), 87–119.

[3] F. Colombini, D. Del Santo. Development of singularities for nonlinear hyperbolic systems with periodic data. *Ann. Univ. Ferrara, Sez. VII – Sc. Mat.* Suppl. vol. **XLI** (1995), 15–28.

[4] P. Hartman, A. Winter. Hyperbolic partial differential equations. *Amer. J. Math.* **74** (1952), 834–864.

[5] F. John. Formation of singularities in one dimensional nonlinear wave propagation. *Comm. Pure Appl. Math.* **27** (1974), 377–405.

[6] F. John. Nonlinear Wave Equations, Formation of Singularities. University Lecture Series, Amer. Math. Soc., Providence, 1990.

[7] S. Klainerman, A. Majda. Formation of singularities for wave equations including the nonlinear vibrating string. *Comm. Pure Appl. Math.* **33** (1980), 241–263.

[8] P. D. Lax. Hyperbolic systems of conservation laws II. *Comm. Pure Appl. Math.* **10** (1957), 537–566.

[9] T. P. Liu. Development of singularities in the nonlinear waves for quasilinear hyperbolic partial differential equations. *J. Differential Equations* **33** (1979), 92–111.

[10] A. Majda. Compressible Fluid Flow and Systems of Conservation Laws in Several Variables. Springer-Verlag, New-York, 1984.

[11] R. Manfrin. Formation of singularities for certain hyperbolic systems in one space dimension. Preprint D.C.A. Istituto Univ. di Architettura di Venezia, January 2000.

[12] R. Manfrin. On the breakdown of solutions to $N \times N$ quasilinear hyperbolic systems. Preprint D.C.A. Istituto Univ. di Architettura di Venezia, May 2000.

[13] P. Popivanov, S. Slavova. Blow up of the solutions for quasilinear hyperbolic systems and application to the nonlinear vibrating string equation. In: Proc. of the 9th International Colloquium on Differential Equations (Ed. D. Bainov) Plovdiv, Bulgaria, August 18–23, 1998; *Utrecht VSP.* (1999), 331–335.

[14] L. Ta-tsien. Global regularity and formation of singularities of solutions to first order quasilinear hyperbolic systems. *Proc. Roy. Soc. Edinburgh Sect A* **87** (1981), 255–261.

[15] L. Ta-tsien, D. Kong. Blow-up of periodic solutions to quasilinear hyperbolic systems. *Nonlinear Anal.* **26** (1996), 1779–1789.

[16] L. Ta-tsien, Y. Zhou, D. Kong. Weak linear degeneracy and global classical solutions for general quasilinear hyperbolic systems. *Comm. Partial Differential Equations* **19** (1994), 1263–1317.

[17] L. Ta-tsien, Y. Zhou, D. Kong. Global classical solutions for general quasilinear hyperbolic systems with decay initial data. *Nonlinear Anal.* **28** (1997), 1299–1332.

*Chiara Boiti*
*Dipartimento di Matematica*
*Via Machiavelli, 35*
*44100 Ferrara, Italy*
*e-mail:* boiti@dm.unife.it

*Renato Manfrin*
*D.C.A. - Ist. Univ. di Architettura*
*Tolentini, S. Croce n. 191*
*30135 Venezia, Italy*
*e-mail:* manfrin@iuav.unive.it

*Received January 18, 2001*