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THE DOITCHINOV COMPLETION OF A REGULAR PARATOPOLOGICAL GROUP

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In memory of Professor D. Doitchinov

Abstract. We show that the two-sided quasi-uniformity $\mathcal{U}_B$ of a regular paratopological group $(G,\cdot)$ is quiet. The Doitchinov completion $(\widehat{G},\widehat{\mathcal{U}_B})$ of $(G,\mathcal{U}_B)$ can be considered a paratopological group containing G as a doubly dense subgroup whenever G is Abelian. Furthermore $\widehat{\mathcal{U}_B}$ is the two-sided quasi-uniformity of $(\widehat{G},\cdot)$. These results generalize in an appropriate way important results about topological groups to regular paratopological groups. A counterexample dealing with the non-Abelian case is presented.

Furthermore we give conditions, depending on quasi-uniform completeness properties, under which a paratopological group is a topological group.

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1. Introduction. It is known that each 2-Hausdorff paratopological group can be 2-densely embedded into a 2-Hausdorff paratopological group whose two-sided quasi-uniformity is bicomplete. However bicompleteness is a rather weak completeness property for quasi-uniform spaces and often satisfied in spaces that one would prefer to see incomplete. In [4, 5, 6] D. Doitchinov introduced a completeness theory for so-called quiet quasi-uniform spaces that is very well-behaved and extends the completion theory of uniform spaces in a natural way. It was asked in [11, Problem 4] whether there exist natural applications of this theory to topological algebra.

In this note we wish to show that this is indeed the case. We shall show that the two-sided quasi-uniformity of a regular T_0-paratopological group is quiet and that its Doitchinov completion yields a paratopological group whenever the product of any two Cauchy filter pairs is a Cauchy filter pair. While the latter condition holds in any Abelian paratopological group, we show by an example that it is not satisfied in paratopological groups in general.

The paper ends with some results and open questions concerning the property of left K-completeness in paratopological groups.

2. Basic facts and preliminary results. Let $(G, \cdot)$ be a group. As usual, e denotes the identity element of G and, for each $x \in G$, x^{-1} denotes the inverse element of x. For $x, y \in G$ we shall write xy instead of $x \cdot y$ and G instead of $(G, \cdot)$ if no confusion arises. For $A, B \subseteq G$ we write $AB = \{ab : a \in A, b \in B\}$ and $A^{-1} = \{a^{-1} : a \in A\}$.

A *paratopological group* is a pair (G, τ) where G is a group and τ is a topology on G such that the function $\Phi : (G \times G, \tau \times \tau) \to (G, \tau)$ defined by $\Phi(x, y) = xy$ is continuous. If in addition the function $\theta : (G, \tau) \to (G, \tau)$ defined by $\theta(x) = x^{-1}$ is continuous, then (G, τ) is called a *topological group*. If (G, τ) is a paratopological group, then so is (G, τ^{-1}) where $\tau^{-1} = \{A \subseteq G : A^{-1} \in \tau\}$; τ^{-1} is called the *conjugate topology* of τ and (G, τ, τ^{-1}) is called a *parabitopological group* [16]. Of course, $\tau = \tau^{-1}$ iff G is a topological group.

The bitopological space (G, τ, τ^{-1}) is called 2-*Hausdorff* provided that the topology $\tau \vee \tau^{-1}$ is a Hausdorff topology.

Example 1. Let $+$ be the usual addition on the reals $\mathbf{R}$ and let $\mathcal{S}$ be the Sorgenfrey topology on $\mathbf{R}$ (i.e. the basic open sets are of the form $[x, a[$ with $x < a$). Then $(\mathbf{R}, \mathcal{S})$ is a paratopological group.

A *quasi-uniformity* on a set X is a filter $\mathcal{U}$ on $X \times X$ such that (a) each member of $\mathcal{U}$ is a reflexive relation on X, and (b) if $U \in \mathcal{U}$ then $V \circ V \subseteq U$ for some $V \in \mathcal{U}$. The pair $(X, \mathcal{U})$ is called a *quasi-uniform space*. Note that for any quasi-uniformity $\mathcal{U}$ the filter of inverse relations $\mathcal{U}^{-1}$ is also a quasi-uniformity. Furthermore $\mathcal{U}^* := \mathcal{U} \vee \mathcal{U}^{-1}$ is a uniformity.

The topology $\tau(\mathcal{U}) = \{H \subseteq X :$ for each $x \in H$ there is $U \in \mathcal{U}$ with $U(x) \subseteq H\}$ is called the *topology induced* by $\mathcal{U}$. (Here $U(x) = \{y \in X : (x, y) \in U\}$.) A topological space (X, τ) *admits* $\mathcal{U}$ provided that τ is the topology induced by $\mathcal{U}$.

Example 2. [15] Equip the set $X = \mathbf{Q}^\omega$ with the topology $\tau(\mathcal{U})$ where $\mathcal{U}$ denotes the quasi-uniformity generated by the following base $\{U_n : n \in \omega\}$. For each $n \in \omega$ and $x \in X$ the set $U_n(x)$ is the set of all $x' \in X$ which satisfy: (1) x and x' agree on the initial segment n, and (2) if $\Delta := \Delta(x, x')$ is the first coordinate in which x and x' differ, then $x(\Delta) \leq x'(\Delta) \leq x(\Delta) + 2^{-n}$. It is readily checked that $(X, +)$ is a paratopological group where the addition of two elements of X is the obvious addition of functions.

A subset A of a bitopological space (X, τ_1, τ_2) is called *doubly dense* (resp. 2-*dense*) if A is dense both in (X, τ_1) and (X, τ_2) (resp. dense in $(X, \tau_1 \vee \tau_2)$).

A quasi-uniform space $(X, \mathcal{U})$ is said to be *bicomplete* [10] provided that the uniformity $\mathcal{U}^*$ is complete.

For undefined concepts from the theory of quasi-uniformities we refer the reader to [10].

Following [9] we shall introduce three interesting quasi-uniformities on a paratopological group (G, τ). Let $\eta(e)$ denote the neighborhood filter at e. For each $U \in \eta(e)$ put $U_L = \{(x, y) : x^{-1}y \in U\}$. It follows that $\{U_L : U \in \eta(e)\}$ is a base for a quasi-uniformity $\mathcal{U}_L$ on G. Moreover put for each $U \in \eta(e)$, $U_R = \{(x, y) : yx^{-1} \in U\}$. Then $\{U_R : U \in \eta(e)\}$ is also a base for a quasi-uniformity $\mathcal{U}_R$ on G.

We have that $\tau(\mathcal{U}_L) = \tau$ and $\tau(\mathcal{U}_L^{-1}) = \tau^{-1}$; similarly, $\tau(\mathcal{U}_R) = \tau$ and $\tau(\mathcal{U}_R^{-1}) = \tau^{-1}$.

According to [9] the quasi-uniformities $\mathcal{U}_L$ and $\mathcal{U}_R$ are called the *left quasi-uniformity* and the *right quasi-uniformity* for (G, τ, τ^{-1}). The quasi-uniformity $\mathcal{U}_B = \mathcal{U}_L \vee \mathcal{U}_R$ is called the *two-sided quasi-uniformity* for (G, τ, τ^{-1}). Note that $\theta : (G, \mathcal{U}_B) \to (G, \mathcal{U}_B^{-1})$ is a quasi-uniform isomorphism.

Every parabitopological group (G, τ, τ^{-1}) generates a topological group (G, τ^*) with $\tau^* = \tau \vee \tau^{-1}$. If $\mathcal{L}^\vee$, $\mathcal{R}^\vee$ and $\mathcal{B}^\vee$ denote the left uniformity, the right uniformity and the two-sided uniformity for (G, τ^*), respectively, then $\mathcal{U}_L^* = \mathcal{L}^\vee, \mathcal{U}_R^* = \mathcal{R}^\vee$, and $\mathcal{U}_B^* = \mathcal{B}^\vee$[16].

It is known [16] that for any 2-Hausdorff parabitopological group (G, τ, τ^{-1}), there is a 2-Hausdorff parabitopological group which is bicomplete in its two-sided quasi-uniformity and has a 2-dense parabitopological subgroup isomorphic to G.

Although there does not exist a theory of completeness for arbitrary quasi-uniform spaces that is as well-behaved as the theory of completeness for uniform spaces, Doitchinov has introduced a conjugate-invariant class of quasi-uniform spaces, called quiet quasi-uniform T_0-spaces, for which a satisfactory theory of completeness exists. We next recall some basic facts of this theory.

A filter $\mathcal{G}$ on a quasi-uniform space $(X,\mathcal{U})$ is said to be *D-Cauchy* [5, 6, 8] provided that there exists a so-called *co-filter* $\mathcal{F}$ on X so that for each $U \in \mathcal{U}$ there are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \times G \subseteq U$. In this case, $(\mathcal{F},\mathcal{G})$ is called a *Cauchy filter pair* and we write $(\mathcal{F},\mathcal{G}) \to 0$. A quasi-uniform space $(X,\mathcal{U})$ is said to be *D-complete* if each D-Cauchy filter converges. A quasi-uniform space $(X,\mathcal{U})$ is called *uniformly regular* [1] provided that for each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $\mathrm{cl}_{\tau(\mathcal{U})}V(x) \subseteq U(x)$ for all $x \in X$. A quasi-uniform space $(X,\mathcal{U})$ is called *quiet* [5, 6] provided that for each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ so that for any Cauchy filter pair $(\mathcal{F},\mathcal{G})$ on X and all points $x, y \in X$, $V^{-1}(y) \in \mathcal{F}$ and $V(x) \in \mathcal{G}$ imply that $(x, y) \in U$. It is known that every quiet quasi-uniform space is uniformly regular.

Let $(X,\mathcal{U})$ be a quiet T_0-space. Two D-Cauchy filters $\mathcal{F}_1$ and $\mathcal{F}_2$ are called *equivalent* provided that they have a common co-filter (equivalently, see [5, Proposition 2], have the same co-filters). Denote the equivalence class of a D-Cauchy filter $\mathcal{F}$ by $\widehat{\mathcal{F}}$. On the set of equivalence classes $\widehat{X}$ of D-Cauchy filters on $(X,\mathcal{U})$ we define a base $\{\widehat{U} : U \in \mathcal{U}\}$ of a quasi-uniformity $\widehat{\mathcal{U}}$ by setting

$\widehat{U} := \{(\widehat{\mathcal{F}},\widehat{\mathcal{G}}) \in \widehat{X} \times \widehat{X}$: For some co-filter $\mathcal{F}_1$ of $\mathcal{F}$ and some $\mathcal{G}_2 \in \widehat{\mathcal{G}}$ on X there are $F_1 \in \mathcal{F}_1, G_2 \in \mathcal{G}_2$ such that $F_1 \times G_2 \subseteq U\}$.

The $\tau(\mathcal{U})$-neighborhood filter at $x \in X$ will be denoted by $\eta(x)$. Then $x \mapsto \widehat{\eta}(x)$ is a quasi-uniform embedding. Furthermore $(\widehat{X},\widehat{\mathcal{U}})$ is a quiet D-complete T_0-space. It is characterized as the unique quiet D-complete T_0-extension that contains a copy of $(X,\mathcal{U})$ as a doubly dense subspace [3]. The space $(\widehat{X},\widehat{\mathcal{U}})$ is called the *Doitchinov completion* of $(X,\mathcal{U})$. Instead of equivalence classes of D-Cauchy filters we could also work with minimal D-Cauchy filters (see [3]), because each D-Cauchy filter $\mathcal{G}$ in a quiet quasi-uniform space contains a unique minimal D-Cauchy filter generated by the base $\{M(U) : U \in \mathcal{U}\}$ on X where $M(U) = \bigcup\{G \in \mathcal{G}$: there is $F \in \mathcal{F}$ such that $F \times G \subseteq U\}$ whenever $U \in \mathcal{U}$. Here $\mathcal{F}$ is any fixed co-filter of $\mathcal{G}$. Note that neighborhood filters of points are minimal D-Cauchy filters.

Finally let us observe that in [5] the sets $\breve{U} := \{(\widehat{\mathcal{F}},\widehat{\mathcal{G}}) \in \widehat{X} \times \widehat{X}$: For any co-filter $\mathcal{F}_1$ of $\mathcal{F}$ and any $\mathcal{G}_2 \in \widehat{\mathcal{G}}$ on X there are $F_1 \in \mathcal{F}_1, G_2 \in \mathcal{G}_2$ such that $F_1 \times G_2 \subseteq U\}$ are used as basic entourages of $\widehat{\mathcal{U}}$. It is straightforward to see that this filterbase is equivalent to the one given above:

Let $\breve{W} \in \widehat{\mathcal{U}}$. Furthermore let $\breve{H} \in \widehat{\mathcal{U}}$ witness double uniform regularity of $\widehat{\mathcal{U}}$ with respect to $\breve{W}$. Then $(\breve{H} \cap (X \times X))\widehat{\ } \subseteq \breve{W}$: Indeed suppose that $\widehat{\mathcal{F}}, \widehat{\mathcal{G}} \in \widehat{X}$ are such that there are a co-filter $\mathcal{F}_1$ of $\mathcal{F}$, $\mathcal{G}_2 \in \widehat{\mathcal{G}}$, $F_1 \in \mathcal{F}_1$ and $G_2 \in \mathcal{G}_2$ with $F_1 \times G_2 \subseteq \breve{H}$. Consequently $\mathrm{cl}_{\tau(\widehat{\mathcal{U}}^{-1})} F_1 \times \mathrm{cl}_{\tau(\widehat{\mathcal{U}})} G_2 \subseteq \breve{W}$. Since $\mathcal{F}_1$ converges to $\widehat{\mathcal{F}}$ with respect to $\tau(\widehat{\mathcal{U}}^{-1})$ and $\mathcal{G}_2$ converges to $\widehat{\mathcal{G}}$ with respect to $\tau(\widehat{\mathcal{U}})$, it follows that $(\widehat{\mathcal{F}}, \widehat{\mathcal{G}}) \in \breve{W}$. We conclude that the two bases are equivalent, since clearly $\breve{U} \subseteq \widehat{U}$.

3. Cauchy filter pairs.

Lemma 1. *Let $(G, \cdot)$ be a paratopological group. If $(\mathcal{F}, \mathcal{G})$ is a Cauchy filter pair with respect to $\mathcal{U}_B$, then $(\mathcal{G}^{-1}, \mathcal{F}^{-1})$ is a Cauchy filter pair with respect to $\mathcal{U}_B$.*

Proof. Let $U \in \eta(e)$. There are $F \in \mathcal{F}$ and $G \in \mathcal{G}$ such that $F \times G \subseteq U_L \cap U_R$. Thus $F^{-1}G \subseteq U$ and $GF^{-1} \subseteq U$. Therefore $(\mathcal{G}^{-1}, \mathcal{F}^{-1}) \to 0$ on $(G,\mathcal{U}_B)$. □

By an example we shall show below that with respect to the two-sided quasi-uniformity the product of two Cauchy filter pairs in a paratopological group need not be a Cauchy filter pair. However let us first give conditions under which such a pathological behaviour cannot occur. Recall that the envelope of a filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ is the filter on X generated by $\mathrm{fil}\{U(F) : F \in \mathcal{F}, U \in \mathcal{U}\}$.

Theorem 1. *Let $(G, \cdot)$ be a paratopological group. Suppose that $(\mathcal{F}_1, \mathcal{G}_1)$, $(\mathcal{F}_2, \mathcal{G}_2)$ are Cauchy filter pairs on $(G,\mathcal{U}_B)$ and that the envelopes of $\mathcal{G}_2$ and ${\mathcal{F}_1}^{-1}$ with respect to $\mathcal{U}_L$ are coarser than with respect to $\mathcal{U}_R$. Then $(\mathcal{F}_1\mathcal{F}_2, \mathcal{G}_1\mathcal{G}_2)$ is a Cauchy filter pair on $(G,\mathcal{U}_B)$.*

Proof. Let $V \in \eta(e)$. Choose $U \in \eta(e)$ such that $UU \subseteq V$. There are $F_2 \in \mathcal{F}_2$ and $G_2 \in \mathcal{G}_2$ such that $F_2^{-1}G_2 \subseteq U$. By our assumption on $\mathcal{G}_2$, there are $H \in \eta(e)$ and $G_2' \in \mathcal{G}_2$ such that $HG_2' \subseteq G_2U$. Choose $F_1 \in \mathcal{F}_1$ and $G_1 \in \mathcal{G}_1$ such that $F_1^{-1}G_1 \subseteq H$. Then $F_2^{-1}F_1^{-1}G_1G_2' \subseteq F_2^{-1}HG_2' \subseteq F_2^{-1}G_2U \subseteq UU \subseteq V$. The second inequality is verified similarly. Therefore $(\mathcal{F}_1\mathcal{F}_2, \mathcal{G}_1\mathcal{G}_2)$ is a Cauchy filter pair on $(G,\mathcal{U}_B)$. □

In order to prove the next result we shall need the following lemma.

Lemma 2. *Let $(G, \cdot)$ be a paratopological group, let P be a precompact subset of $(G,\mathcal{U}_L)$ and $U \in \eta(e)$. Then there is $H \in \eta(e)$ such that $HP \subseteq PU$.*

Proof. Choose $M \in \eta(e)$ such that $MM \subseteq U$. For any $p \in P$ find $H_p \in \eta(e)$ such that $H_pp \subseteq pM$. Since P is precompact in $(G,\mathcal{U}_L)$, there is a finite subset F of P such that $P \subseteq FM$. Set $H = \cap_{p \in F} H_p$. Thus $H \in \eta(e)$. Consider any $z \in HP$. There are $h \in H$ and $p \in P$ such that $z = hp$. Furthermore there exists $f_p \in F$ such that $p \in f_p M$. Thus $f_p^{-1} p \in M$ and $z = h f_p f_p^{-1} p \in H_{f_p} f_p M \subseteq f_p MM \subseteq PU$. $\square$

Proposition 1. *Let $(G, \cdot)$ be a paratopological group such that each D-Cauchy filter on $(G, \mathcal{U}_B)$ has a base consisting of sets that are precompact with respect to $\mathcal{U}_L$. Then in $(G, \mathcal{U}_B)$ the product of any two Cauchy filter pairs is a Cauchy filter pair.*

Proof. By Lemma 2 the envelope of each D-Cauchy filter on $(G, \mathcal{U}_B)$ with respect to $\mathcal{U}_L$ is coarser than the envelope with respect to $\mathcal{U}_R$. The result follows from Theorem 1 and Lemma 1. $\square$

Lemma 3. *Let $(G, \cdot)$ be a paratopological group such that there is $V \in \eta(e)$ that is hereditarily precompact in $(G, \mathcal{U}_L)$. Then any D-Cauchy filter $\mathcal{G}$ on $(G, \mathcal{U}_L)$ contains an element that is hereditarily precompact in $(G, \mathcal{U}_L)$.*

Proof. There is $p \in G$ such that $pV \in \mathcal{G}$. But pV is hereditarily precompact in $(X, \mathcal{U}_L)$, because it is the image of V under the quasi-uniformly continuous left translation by p. $\square$

Proposition 2. *Let $(G, \cdot)$ be a paratopological group possessing $V \in \eta(e)$ that is hereditarily precompact in $(G, \mathcal{U}_L)$. Then the product of any two Cauchy filter pairs on $(G, \mathcal{U}_B)$ is a Cauchy filter pair on $(G, \mathcal{U}_B)$.*

Proof. The assertion is a consequence of Proposition 1 and Lemma 3. $\square$

Question. Let $(G, \cdot)$ be a paratopological group possessing $V \in \eta(e)$ that is (hereditarily) precompact in $(G, \mathcal{U}_L)$. Under which conditions is $(G, \cdot)$ a topological group?

In this context let us note the following result:

Theorem 2. *If G is a regular paratopological group possessing $V \in \eta(e)$ that is hereditarily precompact with respect to $\mathcal{U}_L$, then G is a topological group.*

Proof. Let $U \in \eta(e)$ be arbitrary. Then $e \notin (V \setminus U^{-1})U$. Choose $W \in \eta(e)$ such that $\overline{W} \subseteq U$. By our assumption on V there is a finite set $F \subseteq (V \setminus U^{-1})$ such that $V \setminus U^{-1} \subseteq FW$. Then $\overline{FW} \subseteq FU \subseteq (V \setminus U^{-1})U$. Consequently $e \in V \setminus \overline{FW} \subseteq U^{-1}$. We conclude that G is a topological group. $\square$

Example 3. There exists a regular T_0-paratopological group that (with respect to its two-sided quasi-uniformity) has two Cauchy filter pairs the product of which is not a Cauchy filter pair.

Proof. Take a set $M = \{a_n, b_n, u_n : n \in \omega\}$ where the elements are supposed to be pairwise distinct. Put $G = F(M)$, that is, G is the free group over M. For any $g \in G \setminus \{e\}$ let $g = \Pi_{i=1}^{r} g_i^{\epsilon_i}$ (where $\epsilon_i \in \{-1, 1\}$, $g_i \in M$) be the irreducible representation of g. Set $\operatorname{supp} g = \{g_1, g_2, \dots, g_r\}$.

For each $x \in M$, set $i(x) = n$ iff $x \in \{a_n, b_n, u_n\}$; for any $g \in G \setminus \{e\}$, let $i_m(g) = \min\{i(x) : x \in \operatorname{supp} g\}$ and $i_M(g) = \max\{i(x) : x \in \operatorname{supp} g\}$. Furthermore let $i_M(e) = i_m(e) = -1$. For $k \in \omega$, set $S_k = \{a_i b_j, b_i a_j, u_l : i, j, l \geq k\}$. Let H_k be the set of all elements in G that are equal to e or can be written as $\Pi_{i=1}^{r}(g_i {s_i}^{\epsilon_i} {g_i}^{-1})$ where $r \in \omega \setminus \{0\}$ and for each $i \in \{1, \dots, r\}$ we have $g_i \in G, s_i \in S_k, \epsilon_i \in \{-1, 1\}, i_M(g_i) < i_m(s_i)$, and for any $j \in \{1, \dots, r\}$, $i_m(s_j) = \min\{i_m(s_i) : i = 1, \dots, r\}$ implies that $\epsilon_j = 1$.

Fix $n \in \omega$. We have:

1. H_n is a subsemigroup; in particular $H_n H_n \subseteq H_n$;

2. $g H_n g^{-1} \subseteq H_n$ if $g \in G$ such that $i_M(g) < n$;

Indeed, let $g \in G$ and $h \in H_n \setminus \{e\}$ where $i_M(g) < n$. Then $h = e_1 \dots e_r$ where $r \in \omega \setminus \{0\}, e_i = g_i s_i^{\epsilon_i} g_i^{-1}$, $g_i \in G$, $s_i \in S_n$, $\epsilon_i \in \{-1, 1\}, i_M(g_i) < i_m(s_i)$ for all $i \in \{1, \dots, r\}$ and for any $j \in \{1, \dots, r\}$ we have that $i_m(s_j) = \min\{i_m(s_i) : i = 1, \dots, r\}$ implies that $\epsilon_j = 1$.

Thus $ghg^{-1} = \Pi_{i=1}^{r}((gg_i) s_i^{\epsilon_i} (gg_i)^{-1})$ where for each $i = 1, \dots, r$ we have $i_M(gg_i) < i_m(s_i)$, because $i_m(s_i) \geq n$. We conclude that $ghg^{-1} \in H_n$.

It follows that for each $n \in \omega$ and $g \in G$ we have $g H_m g^{-1} \subseteq H_n$ whenever $m > \max\{i_M(g), n\}$.

By [16] there exists a unique topology τ on G such that (G, τ) is a paratopological group and $\{H_n : n \in \omega\}$ is a neighborhood base at e.

Fact 1: (G, τ) is a regular Hausdorff space.

Proof. For fixed $k \in \omega$, let G_k be the smallest subgroup of G containing H_k. Let $g \in G \setminus \{e\}$. Then $i_M(g) < n$ for some $n \in \omega$.

Define a map $\gamma : M \to G$ by $\gamma(x) = x$, if $i(x) < n$ and $\gamma(x) = e$ if $i(x) \geq n$, where $x \in M$. Let $\hat{\gamma} : G \to G$ be the extension of γ to a homomorphism. Thus we see that $\hat{\gamma}(g) = g$ and by the definition of the elements of H_n we obtain $\hat{\gamma}(G_n) = \{e\}$. Therefore, $g \notin G_n$. Thus τ is a T_1-topology.

We next show that each H_m is closed; hence τ is regular: Fix $m \in \omega$ and suppose that $g \in G \setminus H_m$. Let $t > \max\{i_M(g), m\}$. Assume that $g H_t \cap H_m \neq \emptyset$. Then there are $h_m \in H_m$ and $h_t \in H_t$ such that $g = h_m {h_t}^{-1}$. Note that $h_t \neq e$, since $g \notin H_m$.

We can suppose that we have a representation of ${h_t}^{-1} = \Pi_{i=q+1}^{p}(g_i {s_i}^{-\epsilon_i} {g_i}^{-1})$ according to the definition of H_t.

Similarly as above, consider the morphism $\alpha : G \to G$ determined on M by $\alpha(m) = m$ if $i(m) < t$ and $\alpha(m) = e$ if $i(m) \geq t$. Since $i_M(g) < t$ we have $\alpha(g) = g$. Furthermore, because $\{s_{q+1}, \dots, s_p\} \subseteq S_t$, $\alpha(h_t^{-1}) = e$. Thus $g = \alpha(h_m)$. Therefore $\alpha(h_m) \neq e$. We can suppose that we have a representation of $h_m = \Pi_{i=1}^q (g_i s_i{}^{\epsilon_i} g_i^{-1})$ according to the definition of H_m.

Let $k_1 = \min\{i_m(s_i) : i = 1, \dots, q\}$ and $k_2 = \min\{i_m(s_i) : i = q + 1, \dots, p\}$. Since $\alpha(h_m) \neq e$, we have $k_1 < t$. Because $t \leq k_2$, we conclude that $h_m h_t^{-1} \in H_m$ by the definition of the elements in H_m —a contradiction. We have shown that H_m is closed and that τ is regular. □

For each k, put $A_k = \{a_n : n \in \omega, n \geq k\}$ and $B_k = \{b_n : n \in \omega, n \geq k\}$. Let $\mathcal{A}$ and $\mathcal{B}$ be the filters on G generated by $\{A_k^{-1} : k \in \omega\}$ and $\{B_k : k \in \omega\}$, respectively. Similarly, put $U_k = \{u_n : n \in \omega, n \geq k\}$ whenever $k \in \omega$ and let $\mathcal{U}$ be the filter on G generated by $\{U_k : k \in \omega\}$. Obviously the following two conditions are satisfied:

1. for any $n \in \omega$, there exist $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $A^{-1}B \subseteq H_n$ and $BA^{-1} \subseteq H_n$;
2. for each $n \in \omega$, $U_n \subseteq H_n$.

Finally, we need the following fact.

Fact 2: $a_n u_n b_n \notin H_1$ for any $n \in \omega$.

Assuming the contrary, let $n \in \omega$ and $a_n u_n b_n = g_1 s_1^{\epsilon_1} g_1^{-1} \dots g_k s_k^{\epsilon_k} g_k{}^{-1}$, where $k \in \omega \setminus \{0\}, g_i \in G, s_i \in S_1$, $i_m(s_i) > i_M(g_i)$ for any $i = 1, \dots, k$, and for each $j \in \{1, \dots, k\}$ such that $i_m(s_j) = \min\{i_m(s_i) : i = 1, \dots, k\}$ we have $\epsilon_j = 1$.

In the following we imagine that for each $j \in \{1, \dots, k\}$, we write g_j and s_j in their irreducible representations and plug them into the representation of $a_n u_n b_n$ above. Then by the uniqueness of the irreducible representation we can cancel the right-hand side to $a_n u_n b_n$, recording carefully how we cancel the letters u_n.

Let $\psi : F(M) \to A(M)$ be the canonical morphism that is the identity on M, where $A(M)$ denotes the free Abelian group over M. We have $a_n + u_n + b_n = \epsilon_1 \psi(s_1) + \dots + \epsilon_k \psi(s_k)$. Clearly $n = \min\{i_m(s_j) : j = 1, \dots, k\}$ by our assumption on the ϵ_j. Furthermore $u_n = s_l$ for some $l \in \{1, \dots, k\}$ and $u_n \notin \mathrm{supp} s_p$ whenever $p \in \{1, \dots, k\} \setminus \{l\}$. According to the equation above we can write $a_n u_n b_n = A_1(u_n) s_l Z_1(u_n)$ where $A_1(u_n)$ denotes the initial segment of the product preceding the letter s_l and $Z_1(u_n)$ denotes the final segment of the product succeeding the letter s_l.

Let $\beta : F(M) \to \mathbf{Z}$ be the canonical morphism determined on M by $\beta(m) = 1$ if $i(m) \geq n$ and $m \notin \{u_s : s \in \omega\}$, and $\beta(m) = 0$ otherwise. Note that $\beta(g_j s_j^{\epsilon_j} g_j^{-1})$ is even for any $j = 1, \dots, k$, because $n = \min\{i_m(s_i) : i = 1, \dots, k\}$.

Since $i_M(g_l) < n$ we conclude that both $\beta(A_1(u_n))$ and $\beta(Z_1(u_n))$ are even. If s_l was not cancelled in our cancelling process, then $a_n = A_1(u_n)$ and $b_n = B_1(u_n)$. Since $\beta(a_n) = \beta(b_n) = 1$, this is impossible. We conclude that the letter s_l was cancelled by some well-defined letter u_n^{-1}. This letter u_n^{-1} is necessarily part of a word $g_j^{\delta_j}$ for some $j \in \{1, \ldots, k\} \setminus \{l\}$ where $\delta_j \in \{-1, 1\}$. Since the letters between s_l and u_n^{-1} cancel themselves completely, we have $\beta(b) = 0$ for the word b between the letters s_l and u_n^{-1}. Note that there is a well-defined companion $l_2 := u_n$ of u_n^{-1} belonging to $g_j^{-\delta_j}$ such that the word between u_n^{-1} and l_2 is of the form $c := hs_j^{\epsilon_j}h^{-1}$ for some well-defined final segment h of g_j. Observe that $\beta(c)$ is even. The word between s_l and l_2 is obtained by concatenating the segments b and c or by deleting one from the other (if we disregard the letters s_l, u_n^{-1} and l_2, which are mapped to 0 by β anyway). We have that $a_n u_n b_n = A_2(u_n) l_2 Z_2(u_n)$ where $A_2(u_n)$ denotes the initial segment of the product preceding the letter l_2 and $Z_2(u_n)$ denotes the final segment of the product succeeding the letter l_2. In any case we conclude that both $\beta(A_2(u_n))$ and $\beta(Z_2(u_n))$ are even, since, for example, $\beta(A_2(u_n))$ is obtained from $\beta(A_1(u_n))$ by adding resp. subtracting appropriately the expressions $\beta(b)$ and $\beta(c)$. Again this is only possible if l_2 is cancelled during the cancelling process by some u_n^{-1}. Continuing in this way, we obviously reach after finitely many steps some letter u_n that was not cancelled, since no pair of companions u_n^{-1} and u_n can occur twice during the procedure, because each letter was cancelled at most once. However, then $\beta(A(u_n))$ and $\beta(Z(u_n))$ are even for the initial and final segments $A(u_n)$ and $Z(u_n)$ determined by that copy of u_n —a contradiction. Therefore we deduce that $a_n u_n b_n$ does not belong to H_1.

We conclude that $(\mathrm{fil}\{e\}, \mathcal{U})$, as well as $(\mathcal{A}, \mathcal{B})$ are Cauchy filter pairs with respect to the two-sided quasi-uniformity of G; however by Fact 2 their product $(\mathrm{fil}\{e\} \cdot \mathcal{A}, \mathcal{U} \cdot \mathcal{B})$ is not a Cauchy filter pair with respect to the two-sided quasi-uniformity of G. □

4. Main results.

Theorem 3. *The two-sided quasi-uniformity $\mathcal{U}_B$ of a regular paratopological group is quiet.*

Proof. For $U \in \eta(e)$, choose $L \in \eta(e)$ such that $\overline{L} \subseteq U$. Furthermore let $V \in \eta(e)$ be such that $VV \subseteq L$. Suppose that $V_L^{-1}(x) \cap V_R^{-1}(x) \in \mathcal{F}$, $(\mathcal{F}, \mathcal{G}) \to 0$ with respect to $\mathcal{U}_B$ and $V_L(y) \cap V_R(y) \in \mathcal{G}$. For each $H \in \eta(e)$ choose $F_H \in \mathcal{F}$ and $G_H \in \mathcal{G}$ such that $F_H \times G_H \subseteq H_L \cap H_R$. For each $H \in \eta(e)$ find $f_H \in F_H \cap V_L^{-1}(x) \cap V_R^{-1}(x)$ and $g_H \in G_H \cap V_L(y) \cap V_R(y)$. If follows that $f_H^{-1}x \in V$, $y^{-1}g_H \in V$, $xf_H^{-1} \in V$ and $g_H y^{-1} \in V$. Furthermore $f_H^{-1}g_H \to e$ and

$g_H f_H^{-1} \to e$. Hence $y^{-1} g_H f_H^{-1} x \in VV \subseteq L$ and $y^{-1}(g_H f_H^{-1})x \to y^{-1}x$. Therefore $y^{-1}x \in \overline{L} \subseteq U$ and thus $(y, x) \in U_L$. Similarly, $x f_H^{-1} g_H y^{-1} \in VV \subseteq L$ and $x f_H^{-1} g_H y^{-1} \to xy^{-1}$, and thus $xy^{-1} \in U$ and $(y, x) \in U_R$. We have shown that $\mathcal{U}_B$ is quiet. □

Remark 1. An appropriate modification of the proof of Theorem 3 shows that the left and right quasi-uniformities of a regular paratopological group are uniformly regular.

In [4] Doitchinov develops a quasi-metric variant of his completion theory for so-called balanced quasi-metrics. We next outline how this theory can be applied to paratopological groups.

The following definition can be found in [16]. Let G be a group. An *absolute quasi-valued function* for G is a nonnegative real-valued function ρ on G such that (i) $\rho(e) = 0$; (ii) $\rho(xy) \leq \rho(x) + \rho(y)$ for all $x, y \in G$ and (iii) $\rho(x_n) \to 0$ implies $\rho(a x_n a^{-1}) \to 0$ for all $a \in G$.

Suppose that ρ is an absolute quasi-valued function on a group G. Then the functions defined on $G \times G$ by $d_L(x, y) = \rho(x^{-1}y)$ (resp. $d_R(x, y) = \rho(yx^{-1})$) are quasi-pseudometrics on G that induce the left (resp. right) quasi-uniformities with respect to the corresponding induced topology. It is shown in [16] that each first-countable paratopological group admits an absolute quasi-valued function and thus a left invariant quasi-pseudometric inducing its topology.

Example 4. Let $X = \mathbf{R}$ and $\rho(x) = x$ if $x \geq 0$ and $\rho(x) = 1$ if $x < 0$. The left invariant quasi-pseudometric induced by the absolute quasi-valued function ρ is the usual Sorgenfrey quasi-metric on X.

A sequence $(y_n)_{n \in \omega}$ in a quasi-pseudometric space (X, d) is called *D-Cauchy* [4, 12] provided that there exists a so-called *co-sequence* $(x_k)_{k \in \omega}$ in X such that for each $\epsilon > 0$ there is $N \in \omega$ with the property that $d(x_k, y_n) < \epsilon$ whenever $k, n \in \omega$ and $k, n \geq N$. In this case (x_k, y_n) is called *a Cauchy pair* of sequences and we write $(x_k, y_n) \to 0$.

A quasi-pseudometric space (X, d) is called *balanced* [4, 11] provided that for each Cauchy pair (x_k, y_n) of sequences and all $x, y \in X$ we have that $d(x, y) \leq \sup_{n \in \omega} d(x, y_n) + \sup_{k \in \omega} d(x_k, y)$. It is known that a balanced quasi-pseudometric d induces a quiet quasi-uniformity $\mathcal{U}_d$.

Lemma 4. *Let $(G, \cdot)$ be a paratopological group and let ρ on G be an absolute quasi-valued function for $(G, \cdot)$ such that the quasi-pseudometric d_L induces the topology τ of G. Furthermore suppose that the map ρ is* continuous *on (G, τ). Then $d(x, y) = \rho(x^{-1}y) + \rho(yx^{-1})$ is a balanced quasi-pseudometric on G inducing the two-sided quasi-uniformity of G.*

P r o o f. We have to show only that d satisfies

$$d(x,y) \leq \sup_{n\in\omega} d(x,y_n) + \sup_{k\in\omega} d(x_k,y)$$

whenever $x, y \in G$ and $(x_k, y_n) \to 0$. Choose a subsequence (y_{n_l}) of (y_n) such that $d(x, y_{n_l}) \to \sup_{n\in\omega} d(x, y_n)$. Furthermore find a subsequence (x_{k_s}) of (x_k) such that $d(x_{k_s}, y) \to \sup_{k\in\omega} d(x_k, y)$. Then by property (ii) of ρ, we have $\rho(x^{-1}y_{n_l}x_{k_s}^{-1}y) + \rho(yx_{k_s}^{-1}y_{n_l}x^{-1}) \leq d(x, y_{n_l}) + d(x_{k_s}, y)$. Observe that $y_{n_l}x_{k_s}^{-1} \to e$ and $x_{k_s}^{-1}y_{n_l} \to e$ if $s, l \to \infty$. By the continuity of ρ we conclude that $d(x,y) = \rho(x^{-1}ey) + \rho(yex^{-1}) \leq \sup_{n\in\omega} d(x,y_n) + \sup_{k\in\omega} d(x_k, y)$. Hence d is balanced. □

J. Deák [2] has given an example of a quiet quasi-metrizable space that does not admit any balanced quasi-metric. The existence of this example motivates the following question.

Question. Does each completely regular first-countable T_0-paratopological group admit a balanced quasi-metric?

We next wish to prove the result on the Doitchinov completion of a paratopological group mentioned in the abstract.

Remark 2. Let us note that the condition on D-Cauchy filters stated in the following result is equivalent to the property that with respect to $\mathcal{U}_B$ the product of any two D-Cauchy filters is a D-Cauchy filter: Observe that the envelope of a filter $\mathcal{F}$ with respect to $\mathcal{U}_R$ (resp. $\mathcal{U}_L$) is the filter $\eta(e) \cdot \mathcal{F}$ (resp. $\mathcal{F} \cdot \eta(e)$). Hence if the product of any two D-Cauchy filters is a D-Cauchy filter, the condition is satisfied. The converse follows from the proof given below, which shows that under the stated condition the product of any two (minimal) D-Cauchy filters is D-Cauchy.

Theorem 4. *Let $(G,\cdot)$ be a regular T_0-paratopological group such that for each minimal D-Cauchy filter $\mathcal{G}$ on $(G,\mathcal{U}_B)$ the envelope of $\mathcal{G}$ with respect to $\mathcal{U}_L$ is coarser than the envelope of $\mathcal{G}$ with respect to $\mathcal{U}_R$. Then the Doitchinov completion $(\widehat{G},\widehat{\mathcal{U}_B})$ of $(G,\mathcal{U}_B)$ can be considered a regular T_0-paratopological group containing G as a doubly dense subgroup. Furthermore $\widehat{\mathcal{U}_B}$ is the two-sided quasi-uniformity of $\widehat{G}$.*

P r o o f. First recall that it is known that $(G,\mathcal{U}_B)$ is a doubly dense subspace of the D-complete quiet T_0-space $(\widehat{G},\widehat{\mathcal{U}_B})$. Let $\mathcal{G}_1, \mathcal{G}_2$ be any two minimal D-Cauchy filters on $(G,\mathcal{U}_B)$. Then we define the product $\widehat{\mathcal{G}_1} \cdot \widehat{\mathcal{G}_2}$ in $\widehat{G}$ as the equivalence class of $\mathcal{G}_1 \cdot \mathcal{G}_2$. It follows from Theorem 1 that the latter filter is D-Cauchy. In particular we conclude that under our assumption the envelope of a minimal D-Cauchy filter with respect to $\mathcal{U}_L$ and $\mathcal{U}_R$ are equal (compare Remark

2). It is also readily checked that $\cdot$ extends the multiplication of G to $\widehat{G}$ (under the identification $x \mapsto \widehat{\eta}(x)$), because we have $\eta(xy) = \eta(x) \cdot \eta(y)$ whenever $x, y \in G$. The operation $\cdot$ defined on $\widehat{G}$ is clearly associative.

Let $\mathcal{G}$ be a minimal D-Cauchy filter on $(G, \mathcal{U}_B)$ and let $\mathcal{G}_1$ be a co-filter of $\mathcal{G}$. We show that $\mathcal{G}_1$ is also a co-filter of $\eta(e) \cdot \mathcal{G}$ and $\mathcal{G} \cdot \eta(e)$. Indeed, let $U \in \eta(e)$ and let $W \in \eta(e)$ be such that $WW \subseteq U$. There are $G_1 \in \mathcal{G}_1$ and $G_2 \in \mathcal{G}$ such that $G_1^{-1}G_2 \subseteq W$ and $G_2G_1^{-1} \subseteq W$. By our assumption on $\mathcal{G}$ there are $H \in \eta(e)$ with $H \subseteq W$ and $F \in \mathcal{G}$ with $F \subseteq G_2$ such that $HF \subseteq G_2W$. Thus $G_1^{-1}(HF) \subseteq G_1^{-1}G_2W \subseteq WW \subseteq U$. Furthermore $(HF)G_1^{-1} \subseteq HW \subseteq WW \subseteq U$. Hence $\widehat{\eta}(e) \cdot \widehat{\mathcal{G}} = \widehat{\mathcal{G}}$. An analogous argument shows that $\widehat{\mathcal{G}} \cdot \widehat{\eta}(e) = \widehat{\mathcal{G}}$. Hence e is the unit element for the operation $\cdot$ on $\widehat{G}$.

Similarly, it is straightforward to check that $\widehat{\mathcal{F}^{-1}}$ is the inverse of $\widehat{\mathcal{G}}$ with respect to the operation $\cdot$ where $\mathcal{F}$ is any co-filter of $\mathcal{G}$. We have shown that $(\widehat{G}, \cdot)$ is a group.

We next verify that the multiplication $\cdot : \widehat{G} \times \widehat{G} \to \widehat{G}$ is continuous. Suppose that $(\widehat{\mathcal{F}_\delta})_{\delta \in H} \to \widehat{\mathcal{F}}$ and $(\widehat{\mathcal{G}_\delta})_{\delta \in H} \to \widehat{\mathcal{G}}$. Let $\mathcal{F}_1$ be the minimal co-filter of $\mathcal{F}$ and let $\mathcal{G}_1$ be the minimal co-filter of $\mathcal{G}$. First note that since $\widehat{\mathcal{F}_\delta} \to \widehat{\mathcal{F}}$, for any $U \in \eta(e)$ there are $\delta_0 \in H$, $F_1 \in \mathcal{F}_1$ and $F \subseteq G$ such that $F_1 \times F \subseteq \widehat{U_B}$ and $F \in \mathcal{F}_\delta$ whenever $\delta \in H$ and $\delta \geq \delta_0$: Indeed, choose $\widehat{V_B} \in \widehat{\mathcal{U}_B}$ such that $\widehat{V_B}^3 \subseteq \widehat{U_B}$ and set $F_1 = \widehat{V}_B^{-1}(\widehat{\mathcal{F}}) \cap G$ and $F = \widehat{V}_B^2(\widehat{\mathcal{F}}) \cap G$. Since $\widehat{\mathcal{F}_\delta} \to \widehat{\mathcal{F}}$, there is $\delta_0 \in H$ such that $\widehat{\mathcal{F}_\delta} \in \widehat{V}_B(\widehat{\mathcal{F}})$ whenever $\delta \geq \delta_0$. Thus $\widehat{V}_B(\widehat{\mathcal{F}_\delta}) \subseteq \widehat{V}_B^2(\widehat{\mathcal{F}})$ and since $(\widehat{V_B}(\widehat{\mathcal{F}_\delta}) \cap G) \in \mathcal{F}_\delta$, we have $\widehat{V}_B^2(\widehat{\mathcal{F}}) \cap G \in \mathcal{F}_\delta$, whenever $\delta \geq \delta_0$. This completes the proof of our assertion.

Consider any $U \in \eta(e)$ and let $W \in \eta(e)$ be such that $WW \subseteq U$. By the condition just verified there are $\delta_0 \in H$, $F_1 \in \mathcal{F}_1$ and $F_2 \subseteq G$ such that $F_1^{-1}F_2 \subseteq W$ and $F_2 \in \mathcal{F}_\delta$ whenever $\delta \in H$ and $\delta \geq \delta_0$. Since $\mathcal{F}_1^{-1}$ is a minimal D-Cauchy filter, there are $P \in \eta(e)$ and $E_1 \in \mathcal{F}_1$ such that $E_1^{-1}P \subseteq WF_1^{-1}$. Moreover by the condition verified above there are $G_1 \in \mathcal{G}_1$, $\delta_1 \in H$ such that $\delta_1 \geq \delta_0$ and $G_2 \subseteq G$ such that $G_1^{-1}G_2 \subseteq P$ and $G_2 \in \mathcal{G}_\delta$ whenever $\delta \geq \delta_1$. Then $(G_1E_1)^{-1}(G_2F_2) \subseteq E_1^{-1}G_1^{-1}G_2F_2 \subseteq E_1^{-1}PF_2 \subseteq WF_1^{-1}F_2 \subseteq WW \subseteq U$. The second inequality is shown similarly. By the definition of $\widehat{U_B}$ we conclude that $\widehat{\mathcal{G}_\delta}\widehat{\mathcal{F}_\delta} \to \widehat{\mathcal{G}}\widehat{\mathcal{F}}$ and thus the multiplication $\cdot$ is continuous on $\widehat{G}$.

Finally, we are going to show that the two-sided quasi-uniformity $\mathcal{U}_{\widehat{B}}$ of the paratopological group $\widehat{G}$ is equal to the quasi-uniformity $\widehat{\mathcal{U}_B}$. Let U be an arbitrary neighborhood of e in G.

Consider the following typical entourage of $\mathcal{U}_{\widehat{B}}$:

$$\{(\widehat{\mathcal{F}}, \widehat{\mathcal{G}}) \in \widehat{G} \times \widehat{G} : \widehat{\mathcal{F}}^{-1} \cdot \widehat{\mathcal{G}} \in \breve{U}_B(e) \text{ and } \widehat{\mathcal{G}} \cdot \widehat{\mathcal{F}}^{-1} \in \breve{U}_B(e)\}.$$

Clearly it is contained in

$$\{(\widehat{\mathcal{F}},\widehat{\mathcal{G}}) \in \widehat{G} \times \widehat{G} : \text{ For some co-filter } \mathcal{F}_1 \text{ of } \mathcal{F} \text{ and some } \mathcal{G}_2 \in \widehat{\mathcal{G}} \text{ there are } H \in \eta(e), F_1 \in \mathcal{F}_1 \text{ and } G_2 \in \mathcal{G}_2 \text{ such that } HF_1^{-1}G_2 \subseteq U, F_1^{-1}G_2H \subseteq U, HG_2F_1^{-1} \subseteq U \text{ and } G_2F_1^{-1}H \subseteq U\}.$$

The latter relation is contained in the following basic entourage of $\widehat{\mathcal{U}_B}$:

$$\{(\widehat{\mathcal{F}},\widehat{\mathcal{G}}) \in \widehat{G} \times \widehat{G} : \text{ For some co-filter } \mathcal{F}_1 \text{ of } \mathcal{F} \text{ and some } \mathcal{G}_2 \in \widehat{\mathcal{G}} \text{ there are } F_1 \in \mathcal{F}_1 \text{ and } G_2 \in \mathcal{G}_2 \text{ such that } F_1^{-1}G_2 \subseteq U \text{ and } G_2F_1^{-1} \subseteq U\}.$$

We conclude that $\widehat{\mathcal{U}_B} \subseteq \mathcal{U}_{\widehat{B}}$.

Let $U, W \in \eta(e)$ be such that $WW \subseteq U$. Consider the following typical entourage of $\widehat{\mathcal{U}_B}$:

$$\{(\widehat{\mathcal{F}},\widehat{\mathcal{G}}) \in \widehat{G} \times \widehat{G} : \text{ For any co-filter } \mathcal{F}_1 \text{ of } \mathcal{F} \text{ and any } \mathcal{G}_2 \in \widehat{\mathcal{G}} \text{ there are } F_1 \in \mathcal{F}_1 \text{ and } G_2 \in \mathcal{G}_2 \text{ such that } F_1^{-1}G_2 \subseteq W \text{ and } G_2F_1^{-1} \subseteq W\}.$$

That relation is contained in

$$\{(\widehat{\mathcal{F}},\widehat{\mathcal{G}}) \in \widehat{G} \times \widehat{G} : \text{ For some co-filter } \mathcal{F}_1 \text{ of } \mathcal{F} \text{ and some } \mathcal{G}_2 \in \widehat{\mathcal{G}} \text{ there are } H \in \eta(e), F_1 \in \mathcal{F}_1 \text{ and } G_2 \in \mathcal{G}_2 \text{ such that } HF_1^{-1}G_2 \subseteq U, F_1^{-1}G_2H \subseteq U, HG_2F_1^{-1} \subseteq U \text{ and } G_2F_1^{-1}H \subseteq U\}.$$

The latter relation is contained in the following basic entourage of $\mathcal{U}_{\widehat{B}}$:

$$\{(\widehat{\mathcal{F}},\widehat{\mathcal{G}}) \in \widehat{G} \times \widehat{G} : \widehat{\mathcal{F}}^{-1} \cdot \widehat{\mathcal{G}} \in \widehat{U_B}(e) \text{ and } \widehat{\mathcal{G}} \cdot \widehat{\mathcal{F}}^{-1} \in \widehat{U_B}(e)\}.$$

We conclude that the two quasi-uniformities $\mathcal{U}_{\widehat{B}}$ and $\widehat{\mathcal{U}_B}$ are equal. □

Let us observe that the quasi-uniformity $\widehat{\mathcal{U}_B}^{-1}$ yields the Doitchinov completion of the two-sided quasi-uniformity $\mathcal{U}_B^{-1}$ of the paratopological group (G, τ^{-1}) (see [6]).

Corollary 1. *Let G be a regular T_0-paratopological group such that $\mathcal{U}_L = \mathcal{U}_R$. Then the Doitchinov completion of its two-sided quasi-uniformity $(\widehat{G},\widehat{\mathcal{U}_B})$ can be made a paratopological group that contains G as a doubly dense subgroup.*

Example 5. The restriction of the Sorgenfrey line to the set **Q** of rationals provides an example of a metrizable paratopological group such that the two-sided quasi-uniformity is quasi-metrizable by a stable[1] quiet bicomplete non D-complete quasi-metric. Note that the D-completion of this paratopological group is the Sorgenfrey line.

5. Completeness properties. We finish this paper with some remarks on completeness properties of paratopological groups. Recall that a filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ is said to be a *left K-Cauchy filter* [11, 19] provided that for each $U \in \mathcal{U}$ there exists $F \in \mathcal{F}$ such that $U(x) \in \mathcal{F}$ whenever $x \in F$. A quasi-uniform space $(X,\mathcal{U})$ is called *left K-complete* provided that each left K-Cauchy filter converges. A quasi-metric space (X, d) is said to be *left K-complete* provided that its induced quasi-uniformity $\mathcal{U}_d$ is left K-complete.

Our investigations are motivated by the following open problem.

Question. Let $(G, \cdot)$ be a (regular) paratopological group such that $(G,\mathcal{U}_B)$ is left K-complete. Is G a topological group?

In the following we obtain some partial results related to this question.

Proposition 3. *Each regular paratopological group that admits a left K-complete quasi-metric is a topological group.*

P r o o f. Let us recall that a sequence $(x_n)_{n\in\omega}$ in a quasi-metric space (X, d) is called *left K-Cauchy* provided that for any $\epsilon > 0$ there is $N \in \omega$ such that $m, n \in \omega$ and $m \geq n \geq N$ imply that $d(x_n, x_m) < \epsilon$. A quasi-metric space (X, d) is known to be left K-complete if and only if each left K-Cauchy sequence is convergent [18]. The assertion follows now from [17, Proof of (B)]. □

Remark 3. It is known that each topological space admits a left K-complete quasi-uniformity [14]. Therefore it is not possible to formulate the preceding result for quasi-uniformities instead of quasi-metrics.

A quasi-uniform space $(X,\mathcal{U})$ is called *S-completable* if each left K-Cauchy filter is a Cauchy filter with respect to the uniformity $\mathcal{U}^*$ (see e.g. [11]).

Proposition 4. *Let $(G, \cdot)$ be a regular first-countable T_0-paratopological group such that the two-sided quasi-uniformity $\mathcal{U}_B$ is S-completable. Then G is a topological group.*

P r o o f. The bicompletion $(\widetilde{G},\widetilde{\mathcal{U}_B})$ is a first-countable quiet [13, Proposition 12] paratopological group that is S-complete (see e.g. [11]), i.e. in particular

[1] This concept is explained below.

left K-complete. Thus it is a topological group by the preceding result. Therefore G is a topological group. □

Question. Can "first-countable" be omitted in Proposition 4?

A filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ is called *stable* [1] provided that $\cap_{F\in\mathcal{F}}U(F)$ belongs to $\mathcal{F}$ whenever $U\in\mathcal{U}$. A quasi-uniform space is called *stable* [7, 12] provided that each D-Cauchy filter is stable.

Proposition 5. *A T_1-paratopological group $(G,\cdot)$ having the property that its two-sided quasi-uniformity $\mathcal{U}_B$ is stable and left K-complete is a topological group.*

P r o o f. Suppose that $\mathcal{G}\to e$ in (G,τ), but $\mathcal{G}\not\to e$ in (G,τ^{-1}) for some filter $\mathcal{G}$ on G. In particular for some τ-neighborhood U of e, we have $U^{-1}\notin\mathcal{G}$. Let $\mathcal{H}$ be an ultrafilter on G containing $\mathcal{G}\cup\{G\setminus U^{-1}\}$. Then $(\text{fil}\{e\},\mathcal{H})\to 0$ with respect to $\mathcal{U}_B$; thus $\mathcal{H}$ is stable in $(G,\mathcal{U}_B)$, i.e. $\mathcal{H}$ is left K-Cauchy in $(G,\mathcal{U}_B^{-1})$ [19]. Therefore $\mathcal{H}\to a$ in τ^{-1}, because $(G,\mathcal{U}_B^{-1})$ is left K-complete. Since $\tau(\mathcal{U}_B)$ is a T_1-topology, we have $e=a$. Thus $\mathcal{H}\to e$ with respect to τ^{-1} — a contradiction. We conclude that $\mathcal{G}\to e$ with respect to τ^{-1} and that G is a topological group. □

Corollary 2. *A regular T_1-paratopological group $(G,\cdot)$ with the property that its two-sided quasi-uniformity $\mathcal{U}_B$ is stable and S-completable is a topological group.*

P r o o f. Its bicompletion is a stable [13, Proposition 13], left K-complete (quiet) T_1-paratopological group. □

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