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GENERALIZED VARIATIONAL INEQUALITIES FOR UPPER HEMICONTINUOUS AND DEMI OPERATORS WITH APPLICATIONS TO FIXED POINT THEOREMS IN HILBERT SPACES

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Abstract. Existence theorems of generalized variational inequalities and generalized complementarity problems are obtained in topological vector spaces for demi operators which are upper hemicontinuous along line segments in a convex set X. Fixed point theorems are also given in Hilbert spaces for set-valued operators T which are upper hemicontinuous along line segments in X such that $I - T$ are demi operators.

1. Introduction. Throughout this paper $\mathbb{R}$ denotes the set of all real numbers. If A is a set, we shall denote by $2^A \setminus \{\emptyset\}$ the family of all non-empty subsets of A and by $\mathcal{F}(A)$ the family of all non-empty finite subsets of A. If A is

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a subset of a topological space X, we shall denote by $int_X(A)$ the interior of A in X and by $cl_X(A)$ the closure of A in X. If A is a subset of a vector space, we shall denote by $co(A)$ the convex hull of A.

Let X and Y be topological spaces and $T : X \to 2^Y \setminus \{\emptyset\}$. Then T is said to be (i) *upper* (respectively, *lower*) *semicontinuous* at $x_0 \in X$ if for each open set G in Y with $T(x_0) \subset G$ (respectively, with $T(x_0) \cap G \neq \emptyset$), there exists an open neighbourhood U of x_0 in X such that $T(x) \subset G$ (respectively, $T(x) \cap G \neq \emptyset$) for all $x \in U$; (ii) *upper* (respectively, *lower*) *semicontinuous* on X if T is upper (respectively, lower) semicontinuous at each point of X.

The following result which is Theorem 1 in [6] is a generalization of the celebrated 1972 Ky Fan's minimax inequality [9, Theorem 1].

Theorem A. *Let E be a topological vector space, X be a non-empty convex subset of E, $h : X \to \mathbb{R}$ be lower semicontinuous on $co(A)$ for each $A \in \mathcal{F}(X)$ and $f : X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be such that*

(a) *for each $A \in \mathcal{F}(X)$ and each fixed $x \in co(A)$, $y \longmapsto f(x, y)$ is lower semicontinuous on $co(A)$;*

(b) *for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$,* $\min_{x\in A}[f(x, y)+h(y)-h(x)] \leq 0$;

(c) *for each $A \in \mathcal{F}(X)$ and each $x, y \in co(A)$ and every net $\{y_\alpha\}_{\alpha\in\Gamma}$ in X converging to y with*

$$f(tx + (1-t)y, y_\alpha) + h(y_\alpha) - h(tx + (1-t)y) \leq 0 \text{ for all } \alpha \in \Gamma \text{ and all } t \in [0, 1],$$

we have $f(x, y) + h(y) - h(x) \leq 0$;

(d) *there exist a non-empty closed and compact subset K of X and $x_0 \in K$ such that $f(x_0, y) + h(y) - h(x_0) > 0$ for all $y \in X\setminus K$.*

Then there exists $\hat{y} \in K$ such that $f(x, \hat{y}) \leq h(x) - h(\hat{y})$ for all $x \in X$.

We shall use the following Kneser's minimax theorem [11, pp. 2418-2420] (see also Aubin [1, pp. 40-41]:

Theorem B. *Let X be a non-empty convex subset of a vector space and Y a non-empty compact convex subset of a Hausdorff topological vector space. Suppose that f is a real-valued function on $X \times Y$ such that for each fixed $x \in X$, $f(x, y)$ is lower semicontinuous and convex on Y and for each fixed $y \in Y$, $f(x, y)$ is concave on X. Then*

$$\min_{y\in Y} \sup_{x\in X} f(x, y) = \sup_{x\in X} \min_{y\in Y} f(x, y).$$

We shall first introduce the notions of h-demi and demi operators in Section 2 of this paper. As applications we shall present some existence theorems of generalized variational inequalities and existence theorems of generalized complementarity problems for upper hemicontinuous and demi operators in Section 3. Our results will extend the corresponding results in [2], [6], [7] and [13].

Finally, we shall investigate some fixed point theorems in Hilbert spaces which will extend some corresponding fixed point theorems in the literature, e.g., see [2], [6], [7] and [13].

2. Preliminaries. Let E be a topological vector space. We shall denote by E^* the continuous dual of E, by $\langle w, x\rangle$ the pairing between E^* and E for $w \in E^*$ and $x \in E$ and by $Re\langle w, x\rangle$ the real part of $\langle w, x\rangle$.

Let X be a non-empty subset of E. Then X is a cone in E if X is convex and $\lambda X \subset X$ for all $\lambda \geq 0$. If X is a cone in E, then $\widehat{X} = \{w \in E^* : Re\langle w, x\rangle \geq 0 \text{ for all } x \in X\}$ is also a cone in E^*, called the dual cone of X.

Let $y \in E$. Then the *inward* set of y with respect to X is the set $I_X(y) = \{x \in E : x = y + r(u - y) \text{ for some } u \in X \text{ and } r > 0\}$.

For each $x_0 \in E$, each non-empty subset A of E and each $\epsilon > 0$, let $W(x_0; \epsilon) := \{y \in E^* : |\langle y, x_0\rangle| < \epsilon\}$ and $U(A; \epsilon) := \{y \in E^* : \sup_{x\in A} |\langle y, x\rangle| < \epsilon\}$. Let $\sigma\langle E^*, E\rangle$ be the (weak*) topology on E^* generated by the family $\{W(x; \epsilon) : x \in E \text{ and } \epsilon > 0\}$ as a subbase for the neighborhood system at 0 and $\delta\langle E^*, E\rangle$ be the (strong) topology on E^* generated by the family $\{U(A; \epsilon) : A \text{ is a non-empty bounded subset of } E \text{ and } \epsilon > 0\}$ as a base for the neighborhood system at 0. We note that E^*, when equipped with the (weak*) topology $\sigma\langle E^*, E\rangle$ or the (strong) topology $\delta\langle E^*, E\rangle$, becomes a locally convex Hausdorff topological vector space. Furthermore, for a net $\{y_\alpha\}_{\alpha\in\Gamma}$ in E^* and for $y \in E^*$, (i) $y_\alpha \to y$ in $\sigma\langle E^*, E\rangle$ if and only if $\langle y_\alpha, x\rangle \to \langle y, x\rangle$ for each $x \in E$ and (ii) $y_\alpha \to y$ in $\delta\langle E^*, E\rangle$ if and only if $\langle y_\alpha, x\rangle \to \langle y, x\rangle$ uniformly for $x \in A$ for each non-empty bounded subset A of E.

The following Definition is Definition 2.1(b) in [7]:

Definition 1. *Let E be a topological vector space and X be a non-empty subset of E. Let $T : X \to 2^{E^*} \setminus \{\emptyset\}$ be a map. Then T is said to be upper hemicontinuous on X if and only if for each $p \in E$, the function $f_p : X \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$f_p(z) = \sup_{u\in T(z)} Re\langle u, p\rangle \ \text{ for each } z \in X,$$

is upper semicontinuous on X (if and only if for each $p \in E$, the function g_p :

$X \to \mathbb{R} \cup \{-\infty\}$ *defined by*

$$g_p(z) = \inf_{u \in T(z)} Re\langle u, p\rangle \text{ for each } z \in X,$$

is lower semicontinuous on X).

Note that if X is convex, then the notion of upper hemicontinuity along line segments in X is independent of the vector topology τ on E as long as τ is Hausdorff and the continuous dual E^* remains unchanged. Note also that if T, $S : X \to 2^{E^*} \setminus \{\emptyset\}$ are upper hemicontinuous and $\alpha \in \mathbb{R}$, then $T + S$ and αT are also upper hemicontinuous.

The following is Proposition 2.4 in [7]:

Proposition 1. *Let E be a topological vector space and X be a non-empty subset of E. Let $T : X \to 2^{E^*} \setminus \{\emptyset\}$ be upper semicontinuous from relative topology on X to the weak* topology $\sigma\langle E^*, E\rangle$ on E^*. Then T is upper hemicontinuous on X.*

Note that there is a typo in Proposition 2.4 in [7]. The convexity of X is not needed.

The converse of Proposition 1 is not true in general as can be seen in Example 2.5 in [7] which is Example 2.3 in [15, p. 392]:

We shall now introduce the following definition:

Definition 2. *Let E be a topological vector space, X be a non-empty subset of E and $T : X \to 2^{E^*} \setminus \{\emptyset\}$. If $h : X \to \mathbb{R}$, then T is said to be an h-demi (respectively, a strong h-demi) operator if for each $y \in X$ and every net $\{y_\alpha\}_{\alpha\in\Gamma}$ in X converging to y (respectively, weakly to y) with*

$$\limsup_{\alpha\in\Gamma}[\inf_{u\in T(y)} Re\langle u, y_\alpha - y\rangle + h(y_\alpha) - h(y)] \leq 0$$

we have

$$\limsup_{\alpha\in\Gamma}[\inf_{u\in T(x)} Re\langle u, y_\alpha - x\rangle + h(y_\alpha) - h(x)] \geq \inf_{u\in T(x)} Re\langle u, y - x\rangle + h(y) - h(x)$$

for all $x \in X$.

T is said to be a demi (respectively, strong demi) operator if T is an h-demi (respectively, a strong h-demi) operator with $h \equiv 0$.

Clearly, a strong h-demi operator is also an h-demi operator.

As application of Definition 2, we shall obtain fixed point theorems in Hilbert spaces in Section 4.

For further applications of Definition 2, we refer the readers to [8].

The following is essentially a result of S. C. Fang (e.g. see [5] and [14, p. 59]) (see also [16, Lemma 2.4.2]):

Lemma 1. *Let X be a cone in a Hausdorff topological vector space E and $T : X \to 2^{E^*} \setminus \{\emptyset\}$ be a map. Then the following statements are equvalent:*

(a) There exist $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w}, \hat{y} - x\rangle \leq 0$ for all $x \in X$.

(b) There exist $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w}, \hat{y}\rangle = 0$ and $\hat{w} \in \widehat{X}$.

The following simple result is Lemma 2.1.6 in [16]:

Lemma 2. *Let E be a topological vector space and A be a non-empty bounded subset of E. Let C be a non-empty strongly compact subset of E^*. Define $f : A \to \mathbb{R}$ by $f(x) = \min_{u\in C} Re\langle u, x\rangle$ for all $x \in A$. Then f is weakly continuous on A.*

The following proposition justifies the validity of a demi operator.

Proposition 2. *Let X be a non-empty bounded subset of a topological vector space E, $h : X \to \mathbb{R}$ be weakly lower semicontinuous and $T : X \to 2^{E^*} \setminus \{\emptyset\}$ be an operator such that each $T(x)$ is strongly compact. Then T is an h-demi and a strong h-demi operator.*

Proof. Suppose $y \in X$ and $\{y_\alpha\}_{\alpha\in\Gamma}$ is a net in X converging to y (respectively, weakly to y) with

$$\limsup_{\alpha}[\inf_{u\in T(y)} Re\langle u, y_\alpha - y\rangle + h(y_\alpha) - h(y)] \leq 0.$$

Then for each $x \in X$,

$$\begin{aligned}
&\limsup_{\alpha}[\inf_{u\in T(x)} Re\langle u, y_\alpha - x\rangle + h(y_\alpha) - h(x)] \\
&\geq \liminf_{\alpha}[\inf_{u\in T(x)} Re\langle u, y_\alpha - x\rangle + h(y_\alpha) - h(x)] \\
&\geq \liminf_{\alpha}[\inf_{u\in T(x)} Re\langle u, y_\alpha - y\rangle] + \liminf_{\alpha} h(y_\alpha) - h(x) + \inf_{u\in T(x)} Re\langle u, y - x\rangle \\
&\geq \inf_{u\in T(x)} Re\langle u, y - x\rangle + h(y) - h(x)
\end{aligned}$$

since $\liminf_\alpha \inf_{u\in T(x)} Re\langle u, y_\alpha - y\rangle = \inf_{u\in T(x)} Re\langle u, y - y\rangle = 0$ by Lemma 2. Hence T is an h-demi (respectively, a strong h-demi) operator. $\square$

From Proposition 2 we see that every operator with strongly compact image satisfying the hypotheses of Proposition 2, is an h-demi and a strong h-demi operator. Thus the set of all demi operators contains all demi-monotone [6, Definition 1] and semi-monotone [2, p. 238-240] operators satisfying the hypotheses of Proposition 2.

We shall now prove the following lemma:

Lemma 3. *Let E be a Hausdorff topological vector space, $A \in \mathcal{F}(E)$, $X = co(A)$ and C be a non-empty weak*-compact subset of E^*. Let $f : X \times X \to \mathbb{R}$ be defined by $f(x, y) = \inf_{w \in C} Re\langle w, y - x\rangle$ for all $x, y \in X$. Then for each fixed $x \in X$, $y \longmapsto f(x, y)$ is continuous on X.*

Proof. Clearly, f is upper semicontinuous on X. It remains to show that f is also lower semicontinuous on X. Let $\lambda \in \mathbb{R}$ be given and $x \in X$ be arbitrarily fixed. Let $C_\lambda = \{y \in X : f(x, y) \leq \lambda\}$. Suppose $\{y_\alpha\}_{\alpha \in \Gamma}$ is a net in C_λ and $y_0 \in X$ such that $y_\alpha \to y_0$. Then for each $\alpha \in \Gamma$, $\lambda \geq f(x, y_\alpha) = \inf_{w \in C} Re\langle w, y_\alpha - x\rangle$ so that by weak*-compactness of C, there exists $w_\alpha \in C$ such that $\lambda \geq \inf_{w \in C} Re\langle w, y_\alpha - x\rangle = Re\langle w_\alpha, y_\alpha - x\rangle$. Since C is weak*-compact, there is a subnet $\{w_{\alpha'}\}_{\alpha' \in \Gamma'}$ of $\{w_\alpha\}_{\alpha \in \Gamma}$ and $w_0 \in E^*$ with $w_{\alpha'} \to w_0$ in the weak*-topology. Again, as C is also weak*-closed, $w_0 \in C$.

Suppose $A = \{a_1, \cdots, a_n\}$ and let $t_1, \cdots, t_n \geq 0$ with $\sum_{i=1}^n t_i = 1$ such that $y_0 = \sum_{i=1}^n t_i a_i$. For each $\alpha' \in \Gamma$, let $t_1^{\alpha'}, \cdots, t_n^{\alpha'} \geq 0$ with $\sum_{i=1}^n t_i^{\alpha'} = 1$ such that $y_{\alpha'} = \sum_{i=1}^n t_i^{\alpha'} a_i$. Since E is Hausdorff and $y_{\alpha'} \to y_0$, we must have $t_i^{\alpha'} \to t_i$ for each $i = 1, \cdots, n$. Thus

$$\begin{aligned} \lambda &\geq Re\langle w_{\alpha'}, y_{\alpha'} - x\rangle \\ &= \sum_{i=1}^n t_i^{\alpha'} Re\langle w_{\alpha'}, a_i - x\rangle \\ &\longrightarrow \sum_{i=1}^n t_i Re\langle w_0, a_i - x\rangle \\ &= Re\langle w_0, \sum_{i=1}^n t_i(a_i - x)\rangle = Re\langle w_0, y_0 - x\rangle \end{aligned}$$

so that $\lambda \geq \inf_{w \in C} Re\langle w, y_0 - x\rangle = f(x, y_0)$ and hence $y_0 \in C_\lambda$. Thus C_λ is closed in X for each $\lambda \in \mathbb{R}$. Therefore $y \longmapsto f(x, y)$ is lower semicontinuous on X. □

We obtain the following proposition by slight modification of Proposition 2 and by applying Lemma 3.

Proposition 3. *Let E be a Hausdorff topological vector space, $A \in \mathcal{F}(E)$ and $X = co(A)$. Let $h : X \to \mathbb{R}$ be lower semicontinuous and $T : X \to 2^{E^*} \setminus \{\emptyset\}$ be an operator such that each $T(x)$ is weak*-compact. Then T is an h-demi and a strong h-demi operator.*

The following is an example of a demi operator:

Example 1. *Let $T : [-1, 1] \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ be defined by*

$$T(x) = \begin{cases} [0, 2x], & \text{if } x \geq 0; \\ [2x, 0], & \text{if } x < 0. \end{cases}$$

Now, $[-1, 1] = co(B)$, where $B = \{-1, 1\} \in \mathcal{F}(\mathbb{R})$ and each $T(x)$ is compact in the usual topology (and therefore in the weak topology) of $\mathbb{R}$. Hence by Proposition 3, T is a demi operator.

Moreover for each $A \in \mathcal{F}([-1, 1])$ and each $y \in co(A)$ there exist $\overline{x} \in A$ and $\overline{u} \in T(\overline{x})$ such that

$$Re\langle \overline{u}, y - \overline{x} \rangle = \langle \overline{u}, y - \overline{x} \rangle = \overline{u}(y - \overline{x}) \leq 0.$$

This fact will justify the validity of a hypothesis (based on this fact) in Theorem 1 of the following Section 3.

3. Variational inequalities. In this section we shall present some existence theorems of generalized variational inequalities and generalized complementarity problems for upper hemicontinuous and demi operators.

The following result is Lemma 4.3 in [7]:

Lemma 4. *Let E be a topological vector space, X be a non-empty subset of E and $h : E \to \mathbb{R}$ be convex. Suppose $\hat{y} \in X$ and $\hat{w} \in E^*$ are such that $Re\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in X$, then $Re\langle \hat{w}, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in I_X(\hat{y})$.*

The following result is Lemma 4.2 in [7]:

Lemma 5. *Let E be a topological vector space, X be a non-empty convex subset of E, $h : X \to \mathbb{R}$ be convex and $T : X \to 2^{E^*} \setminus \{\emptyset\}$ be upper hemicontinuous along line segments in X. Suppose $\hat{y} \in X$ is such that $\inf_{u \in T(x)} Re\langle u, \hat{y} - x \rangle \leq h(x) - h(\hat{y})$ for all $x \in X$. Then*

$$\inf_{w \in T(\hat{y})} Re\langle w, \hat{y} - x \rangle \leq h(x) - h(\hat{y}) \text{ for all } x \in X.$$

We shall now establish the following result:

Theorem 1. *Let X be a non-empty convex subset of a Hausdorff topological vector space E and $h : E \to \mathbb{R}$ be convex. Let $T : X \to 2^{E^*} \setminus \{\emptyset\}$ be an h-demi (respectively, a strong h-demi) operator and be upper hemicontinuous along line segments in X to the weak*-topology on E^* such that each $T(x)$ is weak*-compact convex. Suppose that for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$ there exist $\overline{x} \in A$ and $\overline{u} \in T(\overline{x})$ such that $Re\langle \overline{u}, y - \overline{x}\rangle + h(y) - h(\overline{x}) \leq 0$. Suppose further that there exist a non-empty compact (respectively, weakly closed and weakly compact) subset K of X and $x_0 \in K$ such that for each $y \in X \setminus K$, $\inf_{u\in T(x_0)} Re\langle u, y-x_0\rangle + h(y) - h(x_0) > 0$. Then there exists $\hat{y} \in K$ and $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w}, \hat{y} - x\rangle \leq h(x) - h(\hat{y})$ for all $x \in I_X(\hat{y})$.*

Proof. We first note that for each $A \in \mathcal{F}(X)$, h is continuous on $co(A)$ (see e.g. [12, Corollary 10.1.1, p. 83]). Define $\phi : X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ by $\phi(x, y) = \min_{u\in T(x)} Re\langle u, y - x\rangle$, for each $x, y \in X$. Then we have the following.

(a) Clearly, for each $A \in \mathcal{F}(X)$ and each fixed $x \in co(A)$, since E is Hausdorff and $co(A)$ is compact, and the relative weak topology on $co(A)$ coincide with its relative topology; it follows that $y \longmapsto \phi(x, y)$ is lower semicontinuous (respectively, weakly lower semicontinuous) on $co(A)$, by Lemma 3.

(b) By hypothesis, for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$, there exist $\overline{x} \in A$ and $\overline{u} \in T(\overline{x})$ such that $Re\langle \overline{u}, y - \overline{x}\rangle + h(y) - h(\overline{x}) \leq 0$. It follows that for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$, $\min_{x\in A}[\min_{u\in T(x)} Re\langle u, y - x\rangle + h(y) - h(x)] \leq \min_{u\in T(\overline{x})} Re\langle u, y-\overline{x}\rangle + h(y) - h(\overline{x}) \leq 0$. Thus we have $\min_{x\in A}[\phi(x, y) + h(y) - h(x)] \leq 0$, for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$.

(c) Suppose that $A \in \mathcal{F}(X)$, $x, y \in co(A)$ and $\{y_\alpha\}_{\alpha\in\Gamma}$ is a net in X with $y_\alpha \to y$ in the relative topology (respectively, relative weak topology) such that

$$\phi(tx+(1-t)y, y_\alpha)+h(y_\alpha)-h(tx+(1-t)y) \leq 0 \quad \text{for all} \quad \alpha \in \Gamma \quad \text{and all} \quad t \in [0, 1].$$

Then for $t = 0$ we have $\phi(y, y_\alpha) + h(y_\alpha) - h(y) \leq 0$ for all $\alpha \in \Gamma$ so that $\min_{u\in T(y)} Re\langle u, y_\alpha - y\rangle + h(y_\alpha) - h(y) \leq 0$ for all $\alpha \in \Gamma$. Hence

$$\limsup_{\alpha\in\Gamma}[\min_{u\in T(y)} Re\langle u, y_\alpha - y\rangle + h(y_\alpha) - h(y)] \leq 0.$$

Since T is an h-demi (respectively, a strong h-demi) operator on X, we have

$$\limsup_{\alpha\in\Gamma}[\min_{u\in T(x)} Re\langle u, y_\alpha-x\rangle+h(y_\alpha)-h(x)]\geq \min_{u\in T(x)} Re\langle u, y-x\rangle+h(y)-h(x). \tag{3.1}$$

For $t = 1$ we also have $\phi(x, y_\alpha) + h(y_\alpha) - h(x) \leq 0$ for all $\alpha \in \Gamma$. Thus $\min_{u\in T(x)} Re\langle u, y_\alpha - x\rangle + h(y_\alpha) - h(x) \leq 0$ for all $\alpha \in \Gamma$. It follows that

$$\limsup_{\alpha\in\Gamma}[\min_{u\in T(x)} Re\langle u, y_\alpha - x\rangle + h(y_\alpha) - h(x)] \leq 0. \tag{3.2}$$

Hence by (3.1) and (3.2),

$$\min_{u\in T(x)} Re\langle u, y-x\rangle + h(y) - h(x) \le \limsup_{\alpha\in\Gamma}[\min_{u\in T(x)} Re\langle u, y_\alpha - x\rangle + h(y_\alpha) - h(x)] \le 0.$$

Consequently, $\phi(x, y) + h(y) - h(x) \le 0$.

(d) By assumption, K is a compact and therefore closed (respectively, weakly closed and weakly compact) subset of X and $x_0 \in K$ such that for each $y \in X \setminus K$, $\inf_{u\in T(x_0)} Re\langle u, y - x_0\rangle + h(y) - h(x_0) > 0$; it follows that for each $y \in X \setminus K$,

$$\min_{u\in T(x_0)} Re\langle u, y - x_0\rangle + h(y) - h(x_0) > 0,$$

i.e.,

$$\phi(x_0, y) + h(y) - h(x_0) > 0.$$

(If T is a strong h-demi operator, we equip E with the weak topology.) Then ϕ satisfies all the hypotheses of Theorem A. Hence by Theorem A, there exists a point $\hat{y} \in K$ with

$$\phi(x, \hat{y}) \le h(x) - h(\hat{y}) \text{ for all } x \in X;$$

in other words,

$$\min_{u\in T(x)} Re\langle u, \hat{y} - x\rangle \le h(x) - h(\hat{y}) \text{ for all } x \in X.$$

Since h is convex and T is upper hemicontinuous along line segments in X, by Lemma 5 we have

$$\min_{w\in T(\hat{y})} Re\langle w, \hat{y} - x\rangle \le h(x) - h(\hat{y}) \quad \text{for all} \quad x \in X.$$

Define $\psi : X \times T(\hat{y}) \to \mathbb{R}$ by

$$\psi(x, w) = Re\langle w, \hat{y} - x\rangle + h(\hat{y}) - h(x), \text{ for all } x \in X \text{ and for all } w \in T(\hat{y}).$$

Note that $T(\hat{y})$ is weak*-compact convex, and for each fixed $x \in X$, $w \longmapsto \psi(x, w)$ is weak* continuous and convex and for each fixed $w \in T(\hat{y})$, $x \longmapsto \psi(x, w)$ is concave. Hence by Theorem B we have

$$\min_{w\in T(\hat{y})} \sup_{x\in X}(Re\langle w, \hat{y}-x\rangle + h(\hat{y}) - h(x)) = \sup_{x\in X} \min_{w\in T(\hat{y})}(Re\langle w, \hat{y}-x\rangle + h(\hat{y}) - h(x)) \le 0.$$

Hence, by (weak*) compactness of $T(\hat{y})$, there exists a point $\hat{w} \in T(\hat{y})$ such that

$$Re\langle \hat{w}, \hat{y} - x\rangle \le h(x) - h(\hat{y}) \text{ for all } x \in X. \tag{3.3}$$

Since h is defined on all of E and is convex, by (3.3) and Lemma 4, we have $Re\langle \hat{w}, \hat{y} - x\rangle \leq h(x) - h(\hat{y})$ for all $x \in I_X(\hat{y})$. $\square$

Remark 1. Theorem 1 extends Theorem 5 of Bae-Kim-Tan in [2, p. 238-240] in the following ways:

(1) T is an h-demi (or a strong h-demi) operator instead of a semi-monotone [2, pp. 236-237] operator,

(2) T is upper hemicontinuous along line segments instead of upper semi-continuous along line segments in X.

Note however that the coercive conditions in our Theorem 1 here and in Theorem 5 of [2] are not comparable.

Theorem 1 also extends Application 3 in [3, p. 297] in the following ways:

(3) T is a set-valued h-demi (or a strong h-demi) operator and is upper hemicontinuous along line segments in X to the weak* topology on E^* instead of single-valued pseudo-monotone [3, p. 297] and continuous on any finite dimensional subspace,

(4) h need not be lower semicontinuous on X.

By taking $h \equiv 0$ in Theorem 1 and applying Lemma 1 we have the following existence theorem of a generalized complementarity problem:

Theorem 2. *Let X be a cone in a Hausdorff topological vector space E. Let $T : X \to 2^{E^*} \setminus \{\emptyset\}$ be a demi (respectively, a strong demi) operator and be upper hemicontinuous along line segments in X to the weak*-topology on E^* such that each $T(x)$ is weak*-compact convex. Suppose that for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$ there exist $\overline{x} \in A$ and $\overline{u} \in T(\overline{x})$ such that $Re\langle \overline{u}, y - \overline{x}\rangle \leq 0$. Suppose further that there exist a non-empty compact (respectively, weakly closed and weakly compact) subset K of X and $x_0 \in K$ such that for each $y \in X \backslash K$, $\inf_{u \in T(x_0)} Re\langle u, y - x_0\rangle > 0$. Then there exist $\hat{y} \in K$ and $\hat{w} \in T(\hat{y})$ such that*

$$Re\langle \hat{w}, \hat{y}\rangle = 0 \text{ and } \hat{w} \in \widehat{X}.$$

Thus we see that Theorem 2 follows from Theorem 1 with $h \equiv 0$ by applying Lemma 1. Since X needs to be a cone in Theorem 2, we see that Theorem 2 does not imply Theorem 1 in general. Hence Theorem 1 and Theorem 2 are not equivalent in general.

4. Fixed point theorems. In this section H denotes a Hilbert space with inner product $\langle\ ,\ \rangle$ and the corresponding induced norm $\|\cdot\|$. Let d denote the metric on H induced by this norm $\|\cdot\|$.

If X is a non-empty subset of H, we shall denote by $\partial_H(X)$ the boundary of X in H. We shall denote by $bc(H)$ the family of all non-empty bounded closed subsets of H. Then the Hausdorff metric D on $bc(H)$ induced by the metric d is defined by

$$\begin{aligned} D(A_1, A_2) &= \inf\{r > 0 : A_1 \subset B_r(A_2) \text{ and } A_2 \subset B_r(A_1)\} \\ &= \max\{\sup_{x \in A_1} d(x, A_2), \sup_{y \in A_2} d(y, A_1)\}, \end{aligned}$$

where $d(x, A) = \inf\{\|x - y\| : y \in A\}$ and $B_r(A) = \{x \in H : d(x, A) < r\}$ for any $A \in 2^H$ and $r > 0$. (If $A = \{y\}$, we shall write $B_r(A) = B_r(y)$.)

If X is a non-empty subset of H, a map $T : X \to 2^H \setminus \{\emptyset\}$ is said to be *pseudo-contractive* on X [2, p. 240] if for each $x, y \in X$, and each $w \in T(y)$, there exists $u \in T(x)$ such that $\|x - y\| \leq \|(1 + r)(x - y) - r(u - w)\|$ for all $r > 0$.

A map $T : X \to bc(H)$ is said to be *nonexpansive* on X if for each $x, y \in X$, $D(T(x), T(y)) \leq \|x - y\|$.

Let K be a non-empty closed convex subset of a Hilbert space H. For each $x \in H$, there is a unique point $\pi_K(x)$ in K such that

$$\|x - \pi_K(x)\| = \inf_{z \in K} \|x - z\|.$$

$\pi_K(x)$ is called the projection of x on K.

The following result which is Theorem 1.2.3 in [10, p. 9] will characterize the projection $\pi_K(x)$ of x on K as illustrated:

Proposition 4. *Let K be a non-empty closed convex subset of H. Then for each $x \in H$ and $y \in K$, $y = \pi_K(x)$ if and only if*

$$Re\langle x - y, z - y\rangle \leq 0 \textit{ for all } z \in K.$$

As an application of Theorem 1, we have the following fixed point theorem:

Theorem 3. *Let X be a non-empty convex subset of H and $T : X \to 2^H \setminus \{\emptyset\}$ be an upper hemicontinuous map along line segments in X to the weak topology on H such that each $T(x)$ is weakly compact convex and $I - T$ is a demi (respectively, a strong demi) operator. Suppose that for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$ there exist $\overline{x} \in A$ and $\overline{u} \in T(\overline{x})$ such that $Re\langle \overline{x} - \overline{u}, y - \overline{x}\rangle \leq 0$. Suppose further that there exist a non-empty compact (respectively, weakly compact) subset*

K of X and $x_0 \in K$ such that for each $y \in X \backslash K$, $\inf_{u \in T(x_0)} Re\langle x_0 - u, y - x_0\rangle > 0$. Then there exists $\hat{y} \in K$ and $\hat{w} \in T(\hat{y})$ such that

$$Re\langle \hat{y} - \hat{w}, \hat{y} - x\rangle \leq 0 \quad \textit{for all} \quad x \in \overline{I_X(\hat{y})}.$$

Moreover, if either $\hat{y}$ is an interior point of X in H or $\pi_{T(\hat{y})}(\hat{y}) \in \overline{I_X(\hat{y})}$, then $\hat{y}$ is a fixed point of T, i.e., $\hat{y} \in T(\hat{y})$.

Proof. (If $I - T$ is a strong demi operator, we equip H with the weak topology.) Since T is upper hemicontinuous along line segments in X, $I - T : X \to 2^H \backslash \{\emptyset\}$ is also upper hemicontinuous along line segments in X and satisfies all the hypotheses of Theorem 1 with $h \equiv 0$, thus there exists $\hat{y} \in K$ and $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{y} - \hat{w}, \hat{y} - x\rangle \leq 0$ for all $x \in I_X(\hat{y})$. By continuity of $\hat{w}$,

$$Re\langle \hat{y} - \hat{w}, \hat{y} - x\rangle \leq 0 \text{ for all } x \in \overline{I_X(\hat{y})}. \tag{4.1}$$

Case 1. Suppose $\hat{y}$ is an interior point of X in H, *i.e.*, $\hat{y} \in int_H X$, then there exists $r > 0$ such that $B_r(\hat{y}) \subset X$. Then for each $z \in H$ with $z \neq \hat{y}$, let $u = \hat{y} + \frac{r}{2} \cdot \frac{\hat{y} - z}{\|\hat{y} - z\|}$, then $u \in B_r(\hat{y}) \subset X \subset I_X(\hat{y})$. Thus $Re\langle \hat{y} - \hat{w}, \frac{r}{2} \cdot \frac{z - \hat{y}}{\|\hat{y} - z\|}\rangle \leq 0$ so that $\frac{r}{2\|\hat{y} - z\|} Re\langle \hat{y} - \hat{w}, z - \hat{y}\rangle \leq 0$ and hence $Re\langle \hat{y} - \hat{w}, z - \hat{y}\rangle \leq 0$ for all $z \in H$.

It follows that $Re\langle \hat{y} - \hat{w}, z\rangle = 0$ for all $z \in H$ so that $\hat{y} = \hat{w} \in T(\hat{y})$.

Case 2. Suppose $p := \pi_{T(\hat{y})}(\hat{y}) \in \overline{I_X(\hat{y})}$. By Proposition 4, the projection p of $\hat{y}$ on $T(\hat{y})$ has the following property:

$$p \in T(\hat{y}) \text{ and } Re\langle \hat{y} - p, w - p\rangle \leq 0 \text{ for all } w \in T(\hat{y}). \tag{4.2}$$

Since $\hat{w} \in T(\hat{y})$, by (4.2) we have

$$\begin{aligned} 0 &\leq Re\langle p - \hat{y}, \hat{w} - p\rangle \\ &= Re\langle p - \hat{y}, \hat{w} - \hat{y} + \hat{y} - p\rangle \\ &= Re\langle p - \hat{y}, \hat{w} - \hat{y}\rangle - \|\hat{y} - p\|^2. \end{aligned}$$

Therefore

$$\|\hat{y} - p\|^2 \leq Re\langle \hat{y} - \hat{w}, \hat{y} - p\rangle \leq 0 \quad \text{by (4.1)}.$$

Thus $\hat{y} = p = \pi_{T(\hat{y})}(\hat{y}) \in T(\hat{y})$. $\square$

If we compare our Theorem 3 with Theorem 6 in [2], we see that the pseudo-contractivity (see definition in [2]) of T is not required here. But the coercive conditions of our Theorem 3 here and the Theorem 6 in [2] are not comparable.

By Theorem 3 and Proposition 3, we have the following corollary:

Corollary 1. *Let $X = co(B)$, for some $B \in \mathcal{F}(H)$ and $T : X \to 2^H \setminus \{\emptyset\}$ be upper hemicontinuous along line segments in X to the weak topology on H such that each $T(x)$ is weakly compact convex. Suppose that for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$ there exist $\overline{x} \in A$ and $\overline{u} \in T(\overline{x})$ such that $Re\langle \overline{x} - \overline{u}, y - \overline{x}\rangle \leq 0$. Then there exists $\hat{y} \in X$ and $\hat{w} \in T(\hat{y})$ such that*

$$Re\langle \hat{y} - \hat{w}, \hat{y} - x\rangle \leq 0 \quad \textit{for all} \quad x \in \overline{I_X(\hat{y})}.$$

Moreover, if either $\hat{y}$ is an interior point of X in H or $\pi_{T(\hat{y})}(\hat{y}) \in \overline{I_X(\hat{y})}$, then $\hat{y}$ is a fixed point of T, i.e., $\hat{y} \in T(\hat{y})$.

Note that, by Proposition 3, both T and $I - T$ are demi operators in Corollary 1.

The following fixed point theorem is an immediate consequence of Theorem 3:

Theorem 4. *Let X be a non-empty convex subset of H and $T : X \to bc(H)$ be upper hemicontinuous along line segments in X to the weak topology on H such that each $T(x)$ is weakly compact and convex and $I - T$ is a demi (respectively, a strong demi) operator. Suppose that for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$ there exist $\overline{x} \in A$ and $\overline{u} \in T(\overline{x})$ such that $Re\langle \overline{x} - \overline{u}, y - \overline{x}\rangle \leq 0$. Suppose further that there exist a non-empty compact (respectively, weakly compact) subset K of X and $x_0 \in K$ such that (i) for each $y \in K \cap \partial_H(X)$, $\pi_{T(y)}(y) \in \overline{I_X(y)}$ and (ii) for each $y \in X \setminus K$, $\inf_{u \in T(x_0)} Re\langle x_0 - u, y - x_0\rangle > 0$. Then T has a fixed point in K.*

In view of Proposition 3, we have the following immediate consequence of Theorem 4:

Corollary 2. *Let $X = co(B)$, for some $B \in \mathcal{F}(H)$ and $T : X \to bc(H)$ be upper hemicontinuous along line segments in X to the weak topology on H such that each $T(x)$ is weakly compact convex. Suppose that for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$ there exist $\overline{x} \in A$ and $\overline{u} \in T(\overline{x})$ such that $Re\langle \overline{x} - \overline{u}, y - \overline{x}\rangle \leq 0$. Suppose further that for each $y \in \partial_H(X)$, $\pi_{T(y)}(y) \in \overline{I_X(y)}$. Then T has a fixed point in X.*

By Theorem 4 and Proposition 2, we have the following corollary:

Corollary 3. *Let X be a non-empty bounded convex subset of H and $T : X \to bc(H)$ be upper hemicontinuous along line segments in X to the weak topology on H such that each $T(x)$ is (norm) compact and convex. Suppose that* *for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$ there exist $\overline{x} \in A$ and $\overline{u} \in T(\overline{x})$ such that $Re\langle \overline{x} - \overline{u}, y - \overline{x}\rangle \leq 0$. Suppose further that there exist a non-empty weakly compact subset K of X and $x_0 \in K$ such that (i) for each $y \in K \cap \partial_H(X)$, $\pi_{T(y)}(y) \in \overline{I_X(y)}$ and (ii) for each $y \in X \setminus K$, $\inf_{u \in T(x_0)} Re\langle x_0 - u, y - x_0\rangle > 0$. Then T has a fixed point in K.*

It will be interesting to compare Corollary 3 with Theorem 6 of Bae-Kim-Tan in [2, pp. 242-243].

Corollary 4. *Let X be a non-empty compact (respectively, bounded closed) convex subset of H and $T : X \to bc(H)$ be upper hemicontinuous along line segments in X to the weak topology on H such that each $T(x)$ is (norm) compact convex. Suppose that for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$ there exist $\overline{x} \in A$ and $\overline{u} \in T(\overline{x})$ such that $Re\langle \overline{x} - \overline{u}, y - \overline{x}\rangle \leq 0$. Suppose further that for each $y \in \partial_H(X)$, $\pi_{T(y)}(y) \in \overline{I_X(y)}$. Then T has a fixed point in X.*

Corollary 5. *Let X be a non-empty compact (respectively, bounded closed) convex subset of H and $T : X \to bc(X)$ be upper hemicontinuous along line segments in X to the weak topology on H such that each $T(x)$ is (norm) compact convex. Suppose that for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$ there exist $\overline{x} \in A$ and $\overline{u} \in T(\overline{x})$ such that $Re\langle \overline{x} - \overline{u}, y - \overline{x}\rangle \leq 0$. Then T has a fixed point in X.*

The following result also follows from Corollary 4:

Corollary 6. *Let X be a non-empty bounded closed convex subset of H and $T : X \to bc(H)$ be upper semi-continuous along line segments in X to the weak topology on H such that each $T(x)$ is (norm) compact convex. Suppose that for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$ there exist $\overline{x} \in A$ and $\overline{u} \in T(\overline{x})$ such that $Re\langle \overline{x} - \overline{u}, y - \overline{x}\rangle \leq 0$. Suppose further that for each $y \in \partial_H(X)$, $\pi_{T(y)}(y) \in \overline{I_X(y)}$. Then T has a fixed point in X.*

It will be interesting to the readers to compare Corollary 6 with Browder's fixed point theorem [4, Theorem 1].

For further applications of upper hemicontinuous and demi operators in generalized quasi-variational inequalities on non-compact sets, we refer to [8].

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