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FRAGMENTABILITY OF THE DUAL OF A BANACH SPACE WITH SMOOTH BUMP

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Abstract. We prove that if a Banach space X admits a Lipschitz β-smooth bump function, then $(X^*, weak^*)$ is fragmented by a metric, generating a topology, which is stronger than the τ_β-topology. We also use this to prove that if X^* admits a Lipschitz Gâteaux-smooth bump function, then X is sigma-fragmentable.

In [12] the authors proved that if a real Banach space admits an equivalent β-smooth norm, then every continuous convex function f defined on an open subset U of X is generically β-differentiable, that is, f is β-differentiable at the points of some dense G_δ subset of U. In particular, X is weak Asplund when we speak about the Gâteaux bornology. In [2] it was described how to weaken the hypothesis in this case, namely that the existence of Lipschitz Gâteaux-smooth bump is sufficient to guarantee that X is weak Asplund. Later, Li Yongxin and Shi Shuzhong [10] strenghtened the result of [12] in the general case (for generical β-differentiability) by proving that the conclusion in [12] is true even if

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the Banach space only admits a Lipschitz β-smooth bump function. This result is generalised there in the terms of minimal *weak** *usco* mappings ([10, Theorem 2], see Corollary 2 here). Meanwhile, Ribarska [14] has shown that if a Banach space X admits an equivalent β-smooth norm, then $(X^*, weak^*)$ is fragmented by a metric, generating a topology, which is stronger than the τ_β-topology (see the definition), which is formally stronger than the results in [12]. Here we shall see that the existence of a Lipschitz β-smooth bump is sufficient for the same conclusion (Theorem 3). This result is stronger in view of the example of a space with a Lipschitz Fréchet-smooth bump and no equivalent Gâteaux-smooth norm constructed in [4]. Thus we obtain a common strenghtening of the result in [14] and the mentioned results from [10].

We learned by the referee that M. Fosgerau has proved in his Ph.D. Thesis [3] that if a Banach space admits a Lipschitz Gâteaux-smooth bump function, then $(X^*, weak^*)$ is fragmentable. Theorem 3 here contains this result as a special case. The result of Fosgerau has not been published.

As a consequence we can also strengthen a result from [9], namely Corollary 0.5. there, saying that if X is a Banach space, such that its dual X^* has an equivalent (not necessarily dual) Gâteaux-smooth norm, then $(X, weak)$ is sigma-fragmentable by the norm. Here we prove this assertion under (possibly) weaker assumption of X^* having Lipschitz Gâteaux-smooth bump instead of equivalent Gâteaux-smooth norm.

We use a game introduced in [7] and a method used in [10] for proving our main theorem.

Definition 1. ([6]). *The topological space X is called fragmentable by a metric ρ if for every $\varepsilon > 0$, every subset of X has a nonempty relatively open subset of $\rho-$diameter less than ε*

Definition 2 ([5]). *The Banach space X is called sigma-fragmentable if for every $\varepsilon > 0$, X can be expressed as $X = \bigcup_{n\geq 1} X_n$ such that for every n, every subset of X_n has a nonempty relatively weakly open subset of norm-diameter less than ε*

In [7] the fragmentability of a space X was characterized by the existence of a winning strategy for the player Ω in the following ("fragmenting") game G. Two players (Σ and Ω) alternatively take *non-empty* subsets of X. Σ starts the game by choosing any subset A_1 of X and Ω answers by taking a *relatively open* subset $B_1 \subset A_1$. After that, on the n-th move Σ takes any subset A_n of the last move B_{n-1} of Ω and the latter answers again by taking a *relatively open* subset B_n of the set A_n just chosen by Σ. Using this way of selection, the players get a sequence of non-empty sets $A_1 \supset B_1 \supset A_2 \supset \cdots A_n \supset B_n \supset \cdots$, which is called a play. The player Ω is said to have won the play if the set $\bigcap_{n\geq 1} A_n$ contains at most one point.

Theorem 1 ([7, Theorem 1.1]). *The topological space X is fragmentable if and only if the player Ω has a winning strategy for the game G.*

Theorem 2 ([8, Theorem 1.2]). *Let t be some topology, possibly different from the original topology τ on X. The topological space (X, τ) is fragmentable by a metric which majorizes the topology t if and only if there exists a strategy for the player Ω such that $\bigcap_{n\geq 1} A_n = \emptyset$ or $\bigcap_{n\geq 1} A_n = \{x\}$ and for every t-neighborhood U of x, there exists a positive integer k with $B_k \subset U$.*

Let X be a real Banach space, and let β be a bornology on X. For the notions of β-superdifferentiable and β-subdifferentiable extended real-valued functions, β-smooth function, as well as β-(sub/super)derivative we refer to [10], [1] or [11]. The β-derivative of a function f at a point x will be denoted by $\nabla_\beta f(x)$. The Gâteaux and Fréchet bornologies are denoted by G and F, respectively.

Definition 3. *Let β be a bornology on the space X. The (locally convex) τ_β-topology on the dual space X^* is given by the zero-neighborhood base $\{D_{S,\varepsilon} : S \in \beta,\ \varepsilon > 0\}$, where $D_{S,\varepsilon} = \{x^* \in X^* : \forall x \in S, \langle x^*, x\rangle < \varepsilon\}$*

In particular, τ_G is the *weak** topology and τ_F is the norm topology (on X^*).

Proposition 1 ([10]). *Let the Banach space X satisfy (H_β), that is, let there exist a Lipschitz β-smooth bump function $\nu : X \to [0, +\infty)$. Then X satisfies also (H'_β), that is, there exists a Lipschitz β-superdifferentiable function $\mu : X \to [0, 1]$ such that $\mu(0) = 0$ and $\mu(x) = 1$ for $\|x\| \geq 1$.*

Definition 4. *The continuous function $\rho : X \to [1, +\infty]$ is called a β-well function, if it is β-superdifferentiable, $\rho(0) < +\infty$ and $\rho(x) = +\infty$ for $\|x\| \geq 1$.*

Proposition 2 ([10]). *Let the Banach space X satisfy (H'_β) Then there exists a β-well function on X.*

Proposition 3 ([10]). *Let* ρ_0 *be a* β*-well function on* X*,* μ *be the function from the definition of* (H'_β)*,* $\mu_n(x) = \mu(nx)/2^n, n = 1, 2, \dots$ *and* $\{e_n\}_{n=1}^{\infty} \subset X$*. Then*

$$\rho_n(x) = \rho_0(x) + \sum_{k=1}^{n} \mu_k(x - e_k), \quad n = 1, 2, \dots,$$

and

$$\rho_\infty(x) = \rho_0(x) + \sum_{k=1}^{\infty} \mu_k(x - e_k)$$

are all β*-well functions on* X*.*

Definition 5. *Let* ρ *be a* β*-well function on* X*. The gauge function* ρ^* *on* X^* *is defined for any* $x^* \in X^*$ *by*

$$\rho^*(x^*) = \sup_{e \in X} \frac{\langle x^*, e\rangle}{\rho(e)}$$

Proposition 4 ([10]). *Let* ρ^* *be the gauge function from the last definition. Then there is some* $\varepsilon_0 \in (0, 1)$ *such that*

$$\forall x^* \in X^*,\ (1 - \varepsilon_0)\|x^*\| \le \rho^*(x^*) \le \|x^*\|.$$

Proposition 5 ([10]). *Let* ρ *be a* β*-well function on* X*,* $e_0 \in X$ *with* $\rho(e_0) < +\infty$ *and* $x_0^* \in X^*$ *be such that*

$$c := \rho^*(x_0^*) = \frac{\langle x_0^*, e_0\rangle}{\rho(e_0)} > 0$$

then

(*i*) ρ *is* β*-differentiable at* e_0 *and* $x_0^* = c\nabla_\beta \rho(e_0)$*;*
(*ii*) $\forall S \in \beta, \forall \varepsilon > 0, \exists \delta > 0$ *such that*

$$D_{\rho,e_0,x_0^*,\delta} := \{x^* \in X^* : c - \delta < \frac{\langle x^*, e_0\rangle}{\rho(e_0)} \le \rho^*(x^*) < c + \delta\}$$
$$\subset x_0^* + D_{S,\varepsilon}.$$

Lemma 1. *Let the unit ball B^* of the Banach space $(X^*, weak^*)$ admit a strategy ω_1 for Ω, such that $\bigcap_{n\geq 1} A_n = \emptyset$ or $\bigcap_{n\geq 1} A_n = \{x^*\}$ and for every τ_β-neighborhood U of x^*, there exists a positive integer k with $B_k \subset U$. Then the whole space X^* also admits such a strategy.*

Proof. This statement is analogous to Proposition 2.1. from [8], and the proof follows the same idea.

As the space B^* admits a strategy ω_1 with the mentioned property, the space nB^* also does. Denote the latter strategy ω_n. Now we construct a strategy ω for the whole space. Let $A_1 \neq \emptyset$ be the first choice of Σ. If $A_1 \setminus B^* \neq \emptyset$, put $\omega(A_1) = A_1 \setminus B^*$ (this is a relatively *weak** open subset of A_1). Otherwise, if $A_1 \subset B^*$, then further follow the strategy ω_1. In general, let A_n be the n-th move of Σ. If $A_n \setminus nB^* \neq \emptyset$, put $\omega(A_1, B_1, \ldots, A_n) = A_n \setminus nB^*$. Otherwise, if $A_n \subset nB^*$, then find the least k for which $A_k \subset kB^*$ and follow the strategy ω_k.

For every play according to the strategy ω we have one of the following two alternatives: either (a) $B_n = A_n \setminus nB^* \neq \emptyset$ for all $n \geq 1$ (in this case $\bigcap_{n\geq 1} B_n \subset \bigcap_{n\geq 1}(X^* \setminus nB^*) = \emptyset$), or (b) for some positive integer k we get $A_k \subset kB^*$ and after that follow the strategy ω_k. But then, by the initial remark, $\bigcap_{n\geq k} A_n = \emptyset$ or $\bigcap_{n\geq k} A_n = \{x^*\}$ and for every τ_β-neighborhood U of x^*, there exists an integer $m \geq k$ with $B_m \subset U$. Thus ω has the desired property. □

Theorem 3. *Let the Banach space X satisfy (H_β). Then $(X^*, weak^*)$ is fragmentable by a metric d, such that the topology it generates is stronger than the τ_β-topology on X^*.*

Proof. **Proof.** We s/hall find a winning strategy ω for the player Ω in the fragmenting game G with the additional property from Theorem 2, i.e. $\bigcap_{n\geq 1} A_n = \emptyset$ or $\bigcap_{n\geq 1} A_n = \{x^*\}$ and for every τ_β-neighborhood $x^* + D_{S,\varepsilon}$ of x^*, there exists a positive integer k with $B_k \subset x^* + D_{S,\varepsilon}$. According to the last Lemma, it suffices to find such a strategy in B^* rather than in X^*. The frame of the proof anyway follows the idea from Theorem 1 in [10].

Let $A_1 \subset B^*$ be the first move of the player Σ. Put $s_0 = \sup\{\rho_0^*(x^*) : x^* \in A_1\}$. According to Proposition 4, $\exists \varepsilon_0 \in (0,1)$ such that

$$\forall x^* \in X^*,\ (1-\varepsilon_0)\|x^*\| \leq \rho^*(x^*) \leq \|x^*\|.$$

Therefore $s_0 < +\infty$. If $s_0 = 0$ then A_1 contains only one point the strategy is trivial (both the players have no choice in their moves and Ω wins). Let $s_0 > 0$.

Then there exist $x^+ \in A_1$ and $e_1 \in X$, such that $\langle x^+, e_1\rangle > \rho_0(e_1)(1-\varepsilon_0)s_0$. We put $B_1 = \{x^* \in A_1 : \langle x^*, e_1\rangle > \rho_0(e_1)(1-\varepsilon_0)s_0\} \ni x^+$. Then $B_1 = \omega(A_1)$ is a relatively *weak** open subset of A_1.

Now let Σ play some $A_2 \subset B_1$. Put

$$D_1 = \{e \in X : \sup_{x^* \in A_2} \langle x^*, e\rangle \geq \rho_0(e)(1-\varepsilon_0)s_0\}.$$

We have $e_1 \in D_1$ because $A_2 \subset B_1$. As A_2 is bounded, $x \mapsto \sup_{x^* \in A_2}\langle x^*, x\rangle$ is continuous and therefore D_1 is closed. Put $\rho_1(x) = \rho_0(x) + \mu_1(x - e_1)$, where μ_1 is as in Proposition 3. Let $s_1 = \sup\{\rho_1^*(x^*) : x^* \in A_2\}$. Then $\forall x^* \in A_2 \subset A_1$, one has

$$(1-\varepsilon_0)s_0 < \frac{\langle x^*, e_1\rangle}{\rho_0(e_1)} = \frac{\langle x^*, e_1\rangle}{\rho_1(e_1)} \leq s_1 \leq s_0.$$

Let $\varepsilon_1 \in (0, (1-\varepsilon_0)^2/2^2)$ be such that $(1-\varepsilon_0)s_0 < (1-\varepsilon_1)s_1$. Then $\exists x^+ \in A_2, \exists e_2 \in X$, such that $\langle x^+, e_2\rangle > \rho_1(e_2)(1-\varepsilon_1)s_1$. Now let Ω play $B_2 = \{x^* \in A_2 : \langle x^*, e_2\rangle > \rho_1(e_2)(1-\varepsilon_1)s_1\} \ni x^+$. Then $B_2 = \omega(A_1, B_1, A_2)$ is a relatively *weak** open subset of A_2.

In general, after Σ plays some $A_{n+1} \subset B_n$, put

$$D_n = \{e \in X : \sup_{x^* \in A_{n+1}} \langle x^*, e\rangle \geq \rho_{n-1}(e)(1-\varepsilon_{n-1})s_{n-1}\} \subset D_{n-1}.$$

We have $e_n \in D_n$ because $A_{n+1} \subset B_n$. Like before, D_n is closed. Put $\rho_n(x) = \rho_{n-1}(x) + \mu_{n-1}(x - e_n)$, where μ_{n-1} is as in Proposition 3. Let $s_n = \sup\{\rho_n^*(x^*) : x^* \in A_{n+1}\}$. Then for every $x^* \in A_{n+1} \subset A_n$, one has

$$(1-\varepsilon_{n-1})s_{n-1} < \frac{\langle x^*, e_n\rangle}{\rho_{n-1}(e_n)} = \frac{\langle x^*, e_n\rangle}{\rho_n(e_n)} \leq s_n \leq s_{n-1}.$$

Let $\varepsilon_n \in (0, (1-\varepsilon_0)^2/2^{n+1})$ be such that $(1-\varepsilon_{n-1})s_{n-1} < (1-\varepsilon_n)s_n$. Then $\exists x^+ \in A_{n+1}, \exists e_{n+1} \in X$, such that $\langle x^+, e_{n+1}\rangle > \rho_n(e_{n+1})(1-\varepsilon_n)s_n$. Now let Ω play $B_{n+1} = \{x^* \in A_{n+1} : \langle x^*, e_{n+1}\rangle > \rho_n(e_{n+1})(1-\varepsilon_n)s_n\} \ni x^+$. Then $B_{n+1} = \omega(A_1, B_1, A_2, \ldots, A_{n+1})$ is a relatively *weak** open subset of A_{n+1}.

If $x_n \in D_{n+1}$, then

$$\frac{\sup_{x^* \in A_{n+2}} \langle x^*, x_n\rangle}{\rho_n(x_n)} \geq (1-\varepsilon_n)s_n > (1-\varepsilon_{n-1})s_{n-1},$$

so

$$\exists x_n^* \in A_{n+2} : \frac{\langle x_n^*, x_n \rangle}{\rho_n(x_n)} > (1 - \varepsilon_{n-1}) s_{n-1},$$

that is,

$$\frac{\langle x_n^*, x_n \rangle}{(1 - \varepsilon_{n-1}) s_{n-1}} > \rho_n(x_n) = \rho_{n-1}(x_n) + \mu_n(x_n - e_n). \tag{1}$$

But $x_n^* \in A_{n+2} \subset A_{n+1}$, so

$$\frac{\langle x_n^*, x_n \rangle}{\rho_{n-1}(x_n)} \leq s_{n-1},\ i.e. \frac{\langle x_n^*, x_n \rangle}{s_{n-1}} \leq \rho_{n-1}(x_n). \tag{2}$$

Of course, $\|x_n\| < 1$ (otherwise $\rho_{n-1}(x_n) = +\infty$, which would contradict (1)). Then

$$\langle x_n^*, x_n \rangle \leq \|x_n^*\| \leq \frac{\rho_0^*(x_n^*)}{1 - \varepsilon_0} \leq \frac{s_0}{1 - \varepsilon_0}. \tag{3}$$

By (1),(2) and (3) we get

$$\mu_n(x_n - e_n) \leq \frac{\langle x_n^*, x_n \rangle}{(1 - \varepsilon_{n-1}) s_{n-1}} - \frac{\langle x_n^*, x_n \rangle}{s_{n-1}} \tag{4}$$

$$= \frac{\varepsilon_{n-1} \langle x_n^*, x_n \rangle}{(1 - \varepsilon_{n-1}) s_{n-1}} \leq \frac{\varepsilon_{n-1} s_0}{(1 - \varepsilon_{n-1}) s_{n-1} (1 - \varepsilon_0)}$$

But $(1 - \varepsilon_0) s_0 < (1 - \varepsilon_{n-1}) s_{n-1}$, so

$$\frac{s_0}{(1 - \varepsilon_{n-1}) s_{n-1}} < (1 - \varepsilon_0)^{-1}$$

and from (4) we get

$$\mu_n(x_n - e_n) < \frac{\varepsilon_{n-1}}{(1 - \varepsilon_0)^2} < 2^{-n},$$

so $\|x_n - e_n\| < n^{-1}$ by the definition of μ_n. Thus the diameters of the (closed) sets in the nested sequence $\{D_n\}$ tend to 0, so let $\bigcap_{n=1}^{\infty} D_n = \{e_\infty\}$.

Now let $y_\infty^* \in \bigcap_{n \geq 1} B_n$. As $y_\infty^* \in B_{n+1}$, we have

$$\langle y_\infty^*, e_{n+1} \rangle \geq \rho_n(e_{n+1})(1 - \varepsilon_n) s_n. \tag{5}$$

The sequence $\{s_n\}$ of positive reals is monotonely non-increasing, so let s_∞ be its limit. By Proposition 3,

$$\rho_\infty(x) = \rho_0(x) + \sum_{k=1}^{\infty} \mu_k(x - e_k)$$

is a β-well function on X, and $\rho_n \to \rho_\infty$ uniformly on the unit ball of X. Passing to limit in (5), we get

$$\langle y^*_\infty, e_\infty \rangle \geq \rho_\infty(e_\infty) s_\infty. \tag{6}$$

But as for every integer $n \geq 1$ we have $\rho_\infty \geq \rho_n$,

$$\frac{\langle y^*_\infty, e_\infty \rangle}{\rho_\infty(e_\infty)} \leq \rho^*_\infty(y^*_\infty) \leq \rho^*_n(y^*_\infty) \leq s_n.$$

We let $n \to \infty$ to get $\dfrac{\langle y^*_\infty, e_\infty \rangle}{\rho_\infty(e_\infty)} \leq s_\infty$ and having in mind (6) we conclude that

$$\frac{\langle y^*_\infty, e_\infty \rangle}{\rho_\infty(e_\infty)} = s_\infty$$

and $\rho^*_\infty(y^*_\infty) = s_\infty$. By Proposition 5(i) we get

$$y^*_\infty = s_\infty . \nabla_\beta \rho_\infty(e_\infty), \text{ so } |\bigcap_{n \geq 1} B_n| = 1$$

Now let $\delta > 0$ be given. There exists an integer N such that for $n > N$ one has $s_n < s_\infty + \delta$. Then

$$\forall y^* \in B_{n+1} \subset A_{n+1},\ \rho^*_\infty(y^*) \leq \rho^*_n(y^*) \leq s_n \leq s_\infty + \delta. \tag{7}$$

By the definition of B_{n+1} we have

$$\forall y^* \in B_{n+1},\ \frac{\langle y^*, e_{n+1} \rangle}{\rho_n(e_{n+1})} > (1 - \varepsilon_n) s_n. \tag{8}$$

By $\rho_\infty(e_\infty) < \infty$ we have $\|e_\infty\| < 1$, so

$$\left| \frac{\langle y^*, e_\infty \rangle}{\rho_\infty(e_\infty)} - \frac{\langle y^*, e_{n+1} \rangle}{\rho_n(e_{n+1})} \right| \leq \left| \frac{\langle y^*, e_\infty \rangle}{\rho_\infty(e_\infty)} - \frac{\langle y^*, e_\infty \rangle}{\rho_n(e_{n+1})} \right| + \left| \frac{\langle y^*, e_\infty \rangle}{\rho_n(e_{n+1})} - \frac{\langle y^*, e_{n+1} \rangle}{\rho_n(e_{n+1})} \right|$$

$$\leq \left|\langle y^*, e_\infty\rangle \left(\frac{1}{\rho_\infty(e_\infty)} - \frac{1}{\rho_n(e_{n+1})}\right)\right| + \left|\frac{\langle y^*, e_\infty - e_{n+1}\rangle}{\rho_n(e_{n+1})}\right|$$

$$\leq \|y^*\|. \left(\left|\frac{1}{\rho_\infty(e_\infty)} - \frac{1}{\rho_n(e_{n+1})}\right| + \|e_\infty - e_{n+1}\|\right)$$

$$\leq \frac{s_0}{1-\varepsilon_0}. \left(\left|\frac{1}{\rho_\infty(e_\infty)} - \frac{1}{\rho_n(e_{n+1})}\right| + \|e_\infty - e_{n+1}\|\right) \overset{n\to\infty}{\to} 0.$$

And by (8) we get (after choosing n large enough) that

$$\frac{\langle y^*, e_\infty\rangle}{\rho_\infty(e_\infty)} \geq s_\infty - \delta. \tag{9}$$

By (7), (9) and Proposition 5 (ii) we conclude that for any $D_{S,\varepsilon}$ from the τ_β-base $B_{n+1} \subset y^*_\infty + D_{S,\varepsilon}$, for n sufficiently large, provided that δ is chosen in the manner required in Proposition 5(ii). This fact, Theorem 2 and Lemma 1 show that $(X^*, weak^*)$ is fragmentable by a metric d, such that the topology it generates is stronger than the τ_β-topology on X^*. This finishes the proof. □

In [9] it is shown that if X^* admits an equivalent (not necessarily dual) Gâteaux-smooth norm, then X is sigma-fragmentable. Here we get the following (possibly stronger) result:

Corollary 1. *If X^* has a Lipschitz Gâteaux-smooth bump, then X is sigma-fragmentable.*

P r o o f. The last theorem shows that under the given condition, $(X^{**}, weak^*)$ is fragmented by a metric, such that the topology it generates is stronger than the τ_G-topology, that is, than the $weak^*$ topology. Taking into account the canonical embedding of $(X, weak)$ into $(X^{**}, weak^*)$ we conclude that $(X, weak)$ is fragmented by a metric whose topology is stronger than the weak topology on X. By Theorem 1.4 from [8] this means that X is sigma-fragmentable. □

Remark. Of course, the existence of an equivalent Gâteaux-smooth norm implies the existence of a Lipschitz Gâteaux-smooth bump. In view of a known example from [4], the hypothesis in the corresponding result from [9] is stronger than ours in arbitrary Banach space setting, but we don't know whether it's different for dual Banach spaces.

We now show that indeed Theorem 1 from [10] and its generalisation Theorem 2 [10] are corollaries of the last theorem. We remind that a map $F : Z \to 2^Y$, where Z,Y are Hausdorff spaces, is called an *usco map* if it is nonempty compact valued and upper semicontinuous. Such a map is called a *minimal usco map*, if it is minimal with respect to the inclusion of the graphs among all usco maps with the same domain. When $Y = (X^*, w^*)$ for some Banach space X, we call F $w^* - usco$ (correspondingly, *minimal* $w^* - usco$). If F is also convex-valued, it is called *convex* $w^* - usco$, and such a map which is minimal w.r.t the inclusion is called a *minimal convex* $w^* - usco$.

We need the following lemma.

Lemma 2 ([13, Proposition 2.5.]). *Let $F : Z \to 2^Y$ be a minimal usco map on the Baire space Z. Let Y be a Hausdorff space, fragmented by a metric d. Then there exists a dense G_δ subset D of Z such that F is single-valued and d-upper semicontinuous at every $z \in D$.*

Lemma 3 ([11, Lemma 7.12]). *Let $T : Z \to 2^{X^*}$ be a w^*-usco map on the Hausdorff space Z. For $z \in Z$, define $\overline{co}T(z)$ to be the weak* closed convex hull of $T(z)$. Then the map $\overline{co}T$ is convex w^*-usco.*

Corollary 2 ([10, Theorem 2]). *If X satisfies (H_β), Z is a Baire space and $F : Z \to 2^{X^*}$ is a minimal convex w^*-usco map, then F is single-valued and τ_β-upper semicontinuous in all the points of some dense G_δ subset D of Z.*

P r o o f. Let T be a minimal w^*-usco map contained in F (for the existence of such T see [11, Proposition 7.3]). By Theorem 3, X^* is fragmentable by a metric d, which generates a topology stronger than the τ_β-topology on X^*. By the Lemma 2, T is single-valued and d-upper semicontinuous in all the points of some dense G_δ subset D of Z. But as the d-topology is stronger than the τ_β-topology, T is also τ_β-upper semicontinuous in the points of D. By Lemma 3, $\overline{co}T$ is convex w^*-usco, and the minimality of F implies $\overline{co}T = F$. Of course, F is single-valued in the points of D, and we now see that it is τ_β-upper semicontinuous there. Let W be some τ_β-open set containing $F(z_0)$ for some $z_0 \in D$. Take some $S \in \beta, \varepsilon > 0$, such that for the basic τ_β-open (convex) set

$$U = D_{S,\varepsilon} = \{x^* \in X^* : \forall x \in S, \langle x^*, x\rangle < \varepsilon\}$$

we have $F(z_0)+2U \subset W$. Now T is τ_β-upper semicontinuous in z_0, so let $V \ni z_0$ be open neighborhood with $T(V) \subset T(z_0)+U$. Then for every $z \in V$, we have

$$F(z) = \overline{co}T(z) \subset \overline{co}(T(z_0)+U) \subset \overline{\overline{co}T(z_0)+U} \subset \overline{co}T(z_0)+2U = F(z_0)+2U \subset W.$$

Thus F is τ_β-upper semicontinuous in the points of D. $\square$

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