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A GENERALIZED CONVOLUTION WITH A WEIGHT FUNCTION FOR THE FOURIER COSINE AND SINE TRANSFORMS

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Dedicated to Professor Ivan H. Dimovski
on the occasion of his 70th birthday

Abstract

A generalized convolution with a weight function for the Fourier cosine and sine transforms is introduced. Its properties and applications to solving a system of integral equations are considered.

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1. Introduction

Let F_c and F_s be the Fourier cosine and sine transforms, respectively:

$$F_c(f)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y) \cos xy \, dy, \quad x > 0, \tag{1}$$

$$F_s(f)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y) \sin xy \, dy, \quad x > 0. \tag{2}$$

A convolution of two functions f and g for the Fourier cosine transform was given in 1941 by Churchill [1]:

$$(f \underset{F_c}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y)\,(g(|x-y|) + g(x+y))\,dy, \quad x > 0 \tag{3}$$

with the factorization property

$$F_c(f \underset{F_c}{*} g)(y) = (F_c f)(y)\,(F_c g)(y), \;\; \forall y > 0. \tag{4}$$

A convolution with the weight function $\gamma(y) = \sin y$ of functions f and g for the Fourier sine transform F_s was studied in [2], [7]:

$$\begin{aligned}(f \underset{F_s}{*} g)(x) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty f(t)\,(g(x+t+1) + g(|x-t+1|)\mathrm{sign}(x-t+1) \\ + \; g(|x+t-1|)\mathrm{sign}(x+t-1) + g(|x-t-1|)\mathrm{sign}(x-t-1))\,dt, \;\; x > 0,\end{aligned} \tag{5}$$

for which the factorization property holds:

$$F_s(f \underset{F_s}{*} g)(y) = \sin y\,(F_s f)(y)\,(F_s g)(y), \quad \forall y > 0. \tag{6}$$

A generalized convolution for the Fourier sine and cosine transforms was first introduced by Churchill in [1]:

$$(f \underset{1}{*} g)(x) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(y)\,(g(|x-y|) - g(x+y))\,dy, \quad x > 0, \tag{7}$$

that satisfies the factorization property

$$F_s(f \underset{1}{*} g)(y) = (F_s f)(y)(F_c g)(y), \qquad y > 0. \tag{8}$$

Yakubovich et al. [11], [12], [13] studied some special cases of generalized convolutions for nonconvolution integral transforms. In 1998, Kakichev and Nguyen Xuan Thao [3] proposed a constructive method of defining a generalized convolution for three (possibly different) integral transforms K_1, K_2, K_3 with a weight function $\gamma(x)$ with a factorization property

$$K_1(f \overset{\gamma}{*} g)(x) = \gamma(x)(K_2 f)(x)(K_3 g)(x).$$

In recent years, generalized convolutions, for instance, for the transforms of Stieltjes [15], Hilbert [6], [14], Hankel [16], H-transforms [4], I-transforms

[9], the Fourier cosine and sine transforms [8] have been studied. For example, a generalized convolution for the Fourier cosine and sine transforms has been introduced in [8], as follows:

$$(f \underset{2}{*} g(x) = \frac{1}{\sqrt{2\pi}} \int\limits_0^\infty f(y)\,(\mathrm{sign}(y-x)g(|y-x|) + g(y+x))\,dy,\ x>0. \quad (9)$$

It satisfies a factorization property

$$F_c(f \underset{2}{*} g)(y) = (F_s f)(y)(F_s g)(y),\ \forall y>0. \quad (10)$$

In this paper we give a new generalized convolution with a weight function for the Fourier cosine and sine transforms. We prove some of its properties as well as we point out some of its relationships to several well-known convolutions and apply this notion to solve a system of integral equations.

2. A generalized convolution

Definition 1. A generalized convolution with the weight function $\gamma(y) = \sin y$ for the Fourier cosine and sine transforms of functions f and g is defined by

$$(f \overset{\gamma}{*} g)(x) = \frac{1}{2\sqrt{2\pi}} \int\limits_0^\infty f(y)\,[(g(|x+y-1|) + g(|x-y+1|)$$
$$- g(x+y+1) - g(|x-y-1|))\,dy,\ \ \forall x>0. \quad (11)$$

We denote by $L(R_+)$ the set of all functions f defined on $(0,\infty)$ such that $\int\limits_0^\infty |f(x)|dx < \infty$.

Theorem 1. *Let $f, g \in L(R_+)$. Then the generalized convolution with the weight-functions $\gamma(y) = \sin\, y$ for the Fourier cosine and sine transforms belongs to $L(R_+)$, and the following factorization property holds:*

$$F_c(f \overset{\gamma}{*} g)(y) = \sin y\,(F_s f)(y)(F_c g)(y),\ \ \forall y>0. \quad (12)$$

P r o o f. Based on (11) and the fact that f and $g \in L(R_+)$ we have

$$\int\limits_0^\infty |(f \overset{\gamma}{*} g)|dx \le \frac{1}{2\sqrt{2\pi}} \int\limits_0^\infty \int\limits_0^\infty |f(y)|\,.\,|[g(|x+y-1|) + g(|x-y+1|)$$

$$- g(x+y+1) - g(|x-y-1|)]|\, dydx \le \frac{1}{2\sqrt{2\pi}} \int\limits_0^\infty |f(y)| \Bigg[\int\limits_0^\infty |g(|x+y-1|)| dx$$

$$+ \int\limits_0^\infty |g(|x-y-1|)| dx + \int\limits_0^\infty |g(x+y+1)|\, dx + \int\limits_0^{+\infty} |g(x-y-1)|\, dx \Bigg] dy. \tag{13}$$

On the other hand,

$$\int\limits_0^\infty |(g(x+y+1)|\, dx + \int\limits_0^\infty |g(|x-y-1|)|\, dx$$
$$= \int\limits_{y+1}^\infty |g(t)|\, dt + \int\limits_{-y-1}^\infty |g(|t|)|\, dt = 2 \int\limits_0^\infty |g(t)|\, dt. \tag{14}$$

Similarly,

$$\int\limits_0^\infty |g(|x-y+1|)|\, dx + \int\limits_0^\infty |g(|x+y-1|)|\, dx$$
$$= \int\limits_{1-y}^\infty |g(|t|)|\, dt + \int\limits_{y-1}^\infty |g(|t|)|\, dt = 2 \int\limits_0^\infty |g(t)|\, dt. \tag{15}$$

¿From (13), (14) and (15) it follows that

$$\int\limits_0^\infty \left| (f \overset{\gamma}{*} g)(x) \right| dx \le \sqrt{\frac{2}{\pi}} \int\limits_0^\infty |f(t)| dt \int\limits_0^\infty |g(t)| dt < \infty.$$

So $\left(f \overset{\gamma}{*} g \right)(x) \in L(R_+)$.

Now we prove the factorization property (12). Since

$$\sin x\, (F_s f)(x)\, (F_c g)(x) = \frac{2}{\pi} \int\limits_0^\infty \int\limits_0^\infty \sin x\, \sin xu\, \cos xv\, f(u)\, g(v)\, dudv$$

and

$$\sin x\, \sin xu\, \cos xv = \frac{1}{4} [\cos x(u-v-1) + \cos x(u+v-1)$$

$$-\cos x(u+v+1)-\cos x(u-v+1)],$$

we obtain

$$\sin x\,(F_s f)\,(x)\,(F_c g)\,(x)=\frac{1}{2\pi}\int\limits_0^\infty\int\limits_0^\infty[\cos x(u-v-1)+\cos x(u+v-1)$$

$$-\ \cos x(u+v+1)-\cos x(u-v+1)]\,f(u)g(v)dudv. \tag{16}$$

Taking a substitution $y=u$, $t=u+v+1$, we get

$$\frac{1}{2\pi}\int\limits_0^\infty\int\limits_0^\infty\cos x(u+v+1)f(u)g(v)dudv=\frac{1}{2\pi}\int\limits_0^\infty\int\limits_{y+1}^\infty\cos xt\,f(y)g(t-y-1)dtdy$$

$$=\frac{1}{2\pi}\int\limits_0^\infty\int\limits_0^\infty\cos xt\,f(y)g(|t-y-1|)dtdy-\frac{1}{2\pi}\int\limits_0^\infty\int\limits_0^{y+1}\cos xt\,f(y)g(y-t+1)dtdy. \tag{17}$$

Similarly, with a substitution $y=u, t=v-u-1$, we have

$$\frac{1}{2\pi}\int\limits_0^\infty\int\limits_0^\infty\cos x(u-v+1)\,f(u)g(v)dudv=\frac{1}{2\pi}\int\limits_0^\infty\int\limits_{-y-1}^\infty\cos xt\,f(y)g(t+y+1)dtdy$$

$$=\frac{1}{2\pi}\int\limits_0^\infty\int\limits_0^\infty\cos xt\,f(y)g(t+y+1)dtdy+\frac{1}{2\pi}\int\limits_0^\infty\int\limits_{-y-1}^0\cos xt\,f(y)g(t+y+1)dtdy. \tag{18}$$

Furthermore,

$$\int\limits_0^\infty\int\limits_{-y-1}^0\cos xt\,f(y)g(t+y+1)dtdy=-\int\limits_0^\infty\int_{y+1}^0\cos xt\,f(y)g(y-t+1)dtdy$$

$$=\int_0^\infty\int_0^{y+1}\cos xt\,f(y)g(y-t+1)dtdy. \tag{19}$$

¿From (17), (18) and (19) we obtain

$$-\frac{1}{2\pi}\int_0^\infty\int_0^\infty[\cos x(u+v+1)+\cos x(u-v+1)]\,f(u)g(v)dudv$$

$$= -\frac{1}{2\pi}\int_0^\infty\int_0^\infty \cos xt\ [g(|t-y-1|)+g(t+y+1)]\,f(y)dtdy. \tag{20}$$

Similarly,

$$\frac{1}{2\pi}\int_0^\infty\int_0^\infty \cos x(u+v-1)f(u)g(v)dudv = \frac{1}{2\pi}\int_0^\infty\int_{y-1}^\infty \cos xtf(y)$$
$$\times g(t-y+1)dtdy = \frac{1}{2\pi}\int_0^\infty\int_0^\infty \cos xtf(y)$$
$$\times g(|t-y+1|)dtdy - \frac{1}{2\pi}\int_0^\infty\int_0^{y-1} \cos xt\,f(y)\,g(|t-y+1|)dtdy. \tag{21}$$

With a substitution $y=u, t=v-u+1$ we get

$$\frac{1}{2\pi}\int_0^\infty\int_0^\infty \cos x(u-v-1)f(u)g(v)dudv = \frac{1}{2\pi}\int_0^\infty\int_{1-y}^\infty \cos xt\,f(y)$$
$$\times g(t+y-1)dtdy = \frac{1}{2\pi}\int_0^\infty\int_0^\infty \cos xt\,f(y)$$
$$\times g(|t+y-1|)dtdy - \frac{1}{2\pi}\int_0^\infty\int_0^{1-y} \cos xt\,f(y)\,g(|t+y-1|)dtdy. \tag{22}$$

On the other hand,

$$\int_0^\infty\int_0^{y-1} \cos xtf(y)g(|t-y+1|)dtdy = -\int_0^\infty\int_{1-y}^0 \cos xtf(y)g(|t+y-1|)dtdy. \tag{23}$$

¿From (21), (22), and (23) we have

$$\frac{1}{2\pi}\int_0^\infty\int_0^\infty [\cos x(u+v-1)+\cos x(u-v-1)]\,f(u)g(v)dudv$$
$$= \frac{1}{2\pi}\int_0^\infty\int_0^\infty \cos xt\,[g(|t-y+1|)+g(|t+y-1|)]\,f(y)dtdy. \tag{24}$$

Finally, from (16), (20) and (24),

$$\sin x(F_s f)(x)(F_c f)(x) = \frac{1}{2\pi}\int_0^\infty \cos xy\Big\{\int_0^\infty f(y) \tag{25}$$
$$\times\ [g(|x+y-1|)+g(|x-y+1|)-g(x+y+1)-g(|x-y-1|)]\,dy\Big\}dx.$$

The last equality and (11) yield

$$\sin x\,(F_s f)(x)(F_c g)(x) = F_c(f \overset{\gamma}{*} g)(x).$$

The proof is complete. ∎

THEOREM 2. *In the space $L(R_+)$ the generalized convolution (11) is noncommutative:*

$$(f \overset{\gamma}{*} g)(x) = -(g \overset{\gamma}{*} f)(x) + \frac{1}{2\pi}[\operatorname{sign}(1-x)(f \underset{L}{*} g)(|x-1|) + (f \underset{L}{*} g)(x+1)]. \quad (26)$$

Here $f \underset{L}{*} g$ denotes the convolution for the Laplace integral transform

$$(f \underset{L}{*} g)(x) = \int_0^x f(t)\, g(x-t)\, dt.$$

P r o o f. Indeed, with the substitutions $t = x+y-1,\ t = y-x-1,\ t = x+y+1,\ t = y-x+1$, respectively, we have

$$(f \overset{\gamma}{*} g)(x) = \frac{1}{2\sqrt{2\pi}}\Big[\int_{x-1}^{\infty} g(|t|)f(t-x+1)dt + \int_{-(x+1)}^{\infty} g(|t|)f(t+x+1)dt$$
$$- \int_{x+1}^{\infty} g(|t|)f(t-x-1)dt - \int_{1-x}^{\infty} g(|t|)f(t+x-1)dt\Big] \quad (27)$$
$$= \frac{1}{2\sqrt{2\pi}}\Big\{\int_0^{\infty} [f(|t-x+1|)+f(t+x+1)-f(|t-x-1|)-f(|t+x-1|)]$$
$$\times g(t)dt + \int_{x-1}^{0} g(|t|)f(t-x+1)dt + \int_{-(x+1)}^{0} g(|t|)f(t+x+1)dt$$
$$- \int_{x+1}^{0} g(|t|)f(t-x-1)dt - \int_{1-x}^{0} g(|t|)f(t+x-1)dt\Big\}$$
$$= \frac{1}{2\sqrt{2\pi}}\Big\{\int_0^{\infty} [f(|x-t-1|) + f(x+t+1) - f(|x-t+1|) - f(|x+t-1|)]$$
$$\times g(t)dt + \int_{x-1}^{0} g(|t|)f(t-x+1)dt + \int_{-(x+1)}^{0} g(|t|)f(t+x+1)dt$$
$$- \int_{x+1}^{0} g(|t|)f(t-x-1)dt - \int_{1-x}^{0} g(|t|)f(t+x-1)dt\Big\}$$
$$= -(g \overset{\gamma}{*} f)(x) + \frac{1}{2\sqrt{2\pi}}\Big\{\int_{x-1}^{0} g(|t|)f(t-x+1)dt$$
$$-\int_{1-x}^{0} g(|t|)f(t+x-1)dt - \int_{x+1}^{0} g(|t|)f(t-x-1)dt + \int_{-(x+1)}^{0} g(|t|)f(t+x+1)dt\Big\}.$$

With the substitution $u = -t$ we get

$$\int_0^{x-1} g(|t|)f(|t-x+1|)du = -\int_0^{1-x} g(|u|)f(|u+x-1|)dv$$
$$\int_0^{-(x+1)} g(|t|)f(t+x+1)du = -\int_0^{1+x} g(u)f(|u-x-1|)du,$$

therefore,

$$\int_0^{1-x} g(|u|)f(|u+x-1|)du - \int_0^{x-1} g(|u|)f(|u-x+1|)dv$$
$$= \begin{cases} 2\int_0^{1-x} g(u)f(1-x-u)du, & x<1 \\ -2\int\limits_0^{x-1} g(u)f(x-1-u)du, & x \geq 1 \end{cases}$$
$$= 2\,\mathrm{sign}(1-x)\int\limits_0^{|x-1|} g(u)f(|x-1|-u)du = 2\,\mathrm{sign}(1-x)(f \underset{L}{*} g)(|x-1|).$$

Hence and with the substitutions $u=t-x+1$, $u=t+x-1$, $u=t-x-1$, $u=t+x+1$ we obtain

$$\int_{x-1}^0 g(|t|)f(t-x+1)dt - \int_{1-x}^0 g(|t|)f(t+x-1)dt - \int_{x+1}^0 g(t)f(t-x-1)dt$$
$$+\int_{-(x+1)}^0 g(|t|)f(t+x+1)dt = \int_0^{1-x} f(|u|)g(|u+x-1|)du$$
$$-\int_0^{x-1} f(|u|)g(|u-x+1|)du - \int_0^{-(x+1)} f(|u|)g(|u+x+1|)du$$
$$+\int_0^{x+1} f(u)g(|u-x-1|)du = 2\,\mathrm{sign}(1-x)(f \underset{L}{*} g)(|x-1|)$$
$$+2\int\limits_0^{(x+1)} f(u)g(x+1-u)du = 2\,\mathrm{sign}(1-x)(f \underset{L}{*} g)(|x-1|) + 2(f \underset{L}{*} g)(x+1).$$

Therefore,

$$(f \overset{\gamma}{*} g)(x) = -(g \overset{\gamma}{*} f)(x) + \frac{1}{\sqrt{2\pi}}\Big[\mathrm{sign}(1-x)(f \underset{L}{*} g)(|x-1|) + (f \underset{L}{*} g)(x+1)\Big].$$

The theorem is proved. ■

Theorem 3. *In the space $L(R_+)$ the generalized convolution (11) is not associative and satisfies the following equalities:*

a) $f \overset{\gamma}{*} (g \overset{\gamma}{*} h) = g \overset{\gamma}{*} (f \overset{\gamma}{*} h)$;

b) $f \overset{\gamma}{*} (g \overset{\gamma}{*} h) = (f \underset{F_s}{*} g) \overset{\gamma}{*} h$, *where* $(f \underset{F_s}{*} g)$ *is the Fourier-sine convolution with a weight-function (5);*

c) $f \overset{\gamma}{*} (g \underset{F_s}{*} h) = (f \overset{\gamma}{*} g) \underset{F_c}{*} h$, *where* $(g \underset{F_c}{*} h)$ *is the convolution for the Fourier cosine transform (3);*

d) $f \overset{\gamma}{*} (g \underset{2}{*} h) = (f \underset{F_s}{*} g) \underset{2}{*} h$, *where* $(g \underset{2}{*} h)$ *is the generalized convolution with a weight-function for the Fourier cosine and sine transforms (9);*

e) $f \underset{1}{*} (g \overset{\gamma}{*} h) = (f \underset{1}{*} h) \overset{\gamma}{*} g$, *where* $(f \underset{1}{*} h)$ *is the generalized convolution for the Fourier sine and cosine transforms (7).*

P r o o f. **a)** From the factorization property, we have

$$F_c(f \overset{\gamma}{*} (g \overset{\gamma}{*} h))(x) = \sin x (F_s f)(x)\, F_c(g \overset{\gamma}{*} h)(x)$$

$$= \sin x\, (F_s f)(x)\, \sin x (F_s g)(x)\, (F_c h)(x) = \sin x\, (F_s g)(x)\, F_c(f \overset{\gamma}{*} h)(x)$$

$$= F_c \left\{ g \overset{\gamma}{*} (f \overset{\gamma}{*} h) \right\}(x), \quad \forall x > 0,$$

which implies that

$$(f \overset{\gamma}{*} g) \overset{\gamma}{*} h = g \overset{\gamma}{*} (f \overset{\gamma}{*} h).$$

b)

$$F_c(f \overset{\gamma}{*} (g \overset{\gamma}{*} h))(x) = \sin x (F_s f)(x)\, F_c(g \overset{\gamma}{*} h)(x)$$

$$= \sin x (F_s f)(x)\, \sin x (F_s g)(x)\, (F_c h)(x) = F_s(f \underset{F_s}{*} g)(x)\, \sin x\, (F_c h)(x)$$

$$= F_c \left\{ (f \underset{F_s}{*} g)(x) \overset{\gamma}{*} h \right\}(x), \quad \forall x > 0.$$

Hence $f \overset{\gamma}{*} (g \overset{\gamma}{*} h) = (f \underset{F_s}{*} g) \overset{\gamma}{*} h$.

The proofs for **c), d), e)** are similar to those of **a)** and **b)**. The theorem is proved. ∎

THEOREM 4. *The generalized convolution (11) is related to the convolution (7) as follows*

$$(f \overset{\gamma}{*} g)(x) = \frac{1}{2}[(f \underset{1}{*} g)(x+1) - (f \underset{1}{*} g)(x-1)\,\mathrm{sign}(x-1)]. \qquad (28)$$

P r o o f. Indeed,

i) For $x \geq 1$, from Definition 1 we obtain

$$\begin{aligned}(f \overset{\gamma}{*} g)(x) &= \frac{1}{2}\Big\{\frac{1}{\sqrt{2\pi}}\int_0^\infty [g(|x-y+1|) - g(x+y+1)]\, f(y)dy \\ &\quad -\frac{1}{2}\Big\{\frac{1}{\sqrt{2\pi}}\int_0^\infty [(|x-y-1|) - g(x+y-1)]\, f(y)dy\Big\} \\ &= \frac{1}{2}\Big[(f \underset{1}{*} g)(x+1) - (f \underset{1}{*} g)(x-1)\Big].\end{aligned}$$

ii) Let $0 < x < 1$. We have

$$\begin{aligned}(f \overset{\gamma}{*} g)(x) &= \frac{1}{2}\Big\{\frac{1}{\sqrt{2\pi}}\int_0^\infty [g(|x-y+1|) - g(x+y+1)]\, f(y)dy \\ &\quad -\frac{1}{\sqrt{2\pi}}\int_0^\infty [g(|x-y-1|) - g(|x+y-1|)]\, f(y)dy\Big\} \\ &= \frac{1}{2}\Big\{\frac{1}{\sqrt{2\pi}}\int_0^\infty [g(|x-y+1|) - g(x+y+1)]\, f(y)dy \\ &\quad -\frac{1}{\sqrt{2\pi}}\int_0^\infty [g(|x-y-1|) - g(|x+y-1|)]\, f(y)dy\Big\} \\ &= \frac{1}{2}\Big[(f \underset{1}{*} g)(x+1) + (f \underset{1}{*} g)(1-x)\Big] \\ &= \frac{1}{2}\Big[(f \underset{1}{*} g)(x+1) - (f \underset{1}{*} g)(|x-1|)\mathrm{sign}(x-1)\Big].\end{aligned}$$

Hence

$$(f \overset{\gamma}{*} g)(x) = \frac{1}{2}\Big[(f \underset{1}{*} g)(x+1) - (f \underset{1}{*} g)(|x-1|)\mathrm{sign}(x-1)\Big].$$

The theorem is proved. ■

THEOREM 5. *In the space $L(R_+)$ the generalized convolution (11) does not have a unit element.*

P r o o f. Suppose that there exists a unit element of the generalized convolution (11) in $L(R_+) : e \overset{\gamma}{*} g = g \overset{\gamma}{*} e = g$, for any function $g \in L(R_+)$. It follows that $F_c(e \overset{\gamma}{*} g)(y) = (F_c g)(y), \ \forall y > 0$. Hence, $\sin y \, (F_s e)(y) \, (F_c g)(y) = (F_c g)(y), \ \forall y > 0$. The last equation is equivalent to the equality

$$(F_c g)(y)[\sin y (F_s e)(y) - 1] = 0, \ \ \forall y > 0$$

for any function $g(y) \in L(R_+)$. Choosing g such that $(F_c g)(y) \neq 0, \ \forall y > 0$, we see that $\sin y (F_s e)(y) - 1 = 0$ or $(F_s e)(y) = \frac{1}{\sin y}, \ \forall y > 0$.

Thus, $(F_s e)(y)$ does not approach 0 as y tends to infinity, as it must be if $e \in L(R_+)$, and so $e \notin L(R_+)$. This is a contradiction. The theorem is proved. ■

Set $L(R_+, e^x) = \{h(x), \text{ for all } e^x h(x) \in L(R_+)\}$

THEOREM 6. (Titchmarch-type theorem) *Let* $f, g \in L(R_+, e^x)$. *If*

$$(f \overset{\gamma}{*} g)(x) = 0, \ \ \forall x > 0,$$

then either $f(x) = 0$ *or* $g(x) = 0, \ \forall x > 0$.

P r o o f. Under the hypothesis $(f \overset{\gamma}{*} g)(x) = 0, \ \forall x > 0$, it follows that $F_c(f \overset{\gamma}{*} g)(y) = 0, \ \forall y > 0$. By virtue of Theorem 1,

$$\sin y (F_s f)(y)(F_c g)(y) = 0, \ \ \forall y > 0. \tag{29}$$

Consider the Fourier cosine integral transform

$$(F_c g)(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(x) \cos(yx) dx, \quad y \in R_+.$$

Since

$$\Big| \frac{d^n}{dy^n} \left(\cos(yx) g(x)\right) \Big| = \Big| g(x) x^n \cos\Big(yx + n\frac{\pi}{2}\Big) \Big| \leq |g(x) x^n|$$

$$= |e^{-x} x^n g_1(x)| = |e^{-x} x^n| \, |g_1(x)| \leq C |g_1(x)|, \quad g_1(x) = e^x \, g(x) \in L(R_+),$$

for x large enough, due to Weierstrass' criterion, the integral

$$\int_0^\infty \frac{d^n}{dy^n} \left[\cos(yx) g(x)\right] dx$$

uniformly converges on R_+. Therefore, based on the differentiability of integrals depending on parameter, we conclude that $(F_c g)(y)$ is analytic. Similarly, $(F_s f)(y)$ is analytic. So from (29) we have $(F_s f)(y) = 0, \ \forall y > 0$ or $(F_c g)(y) = 0, \ \forall y > 0$. It follows that either $f(x) = 0$ or $g(x) = 0, \ \forall x > 0$. The theorem is proved. ■

3. A system of integral equations

Consider a system of integral equations

$$\begin{cases} \lambda_{11} f(y) + \lambda_{12} \int\limits_0^\infty \varphi(t) g_1(y,t)dt &= k(y), \\ \lambda_{21} \int\limits_0^\infty f(t)\psi_1(y,t)dt + \lambda_{22} g(y) &= h(y), \qquad y > 0. \end{cases} \tag{30}$$

Here, $\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}$ are complex numbers and $\varphi, \psi, k, h \in L(R_+)$, f and g are unknown functions of $L(R_+)$ and

$$g_1(y,t) = \frac{1}{\sqrt{2\pi}} \left[g(|t-y|) + g(t+y) \right],$$

$$\psi_1(y,t) = \frac{1}{\sqrt{2\pi}} \left[\psi(|y+t-1|) + \psi(|y-t+1|) - \psi(y+t+1) - \psi(|y-t-1|) \right].$$

THEOREM 7. *Let* $1 + C \operatorname{sign} y\, (F_s\varphi)(y)(F_c\psi)(y) \neq 0$. *Then there exists a solution in* $L(R_+)$ *of* (30) *which is defined by*

$$f(y) = \frac{1}{\lambda} \left\{ \lambda_{22} k(y) - \lambda_{12} (\varphi \underset{1}{*} h)(y) - (k \underset{1}{*} l)(y) + \lambda_{12} \left((\varphi \underset{1}{*} h) \underset{1}{*} l \right)(y) \right\} \in L(R_+),$$

$$g(y) = \frac{1}{\lambda} \left\{ \lambda_{11} h(y) - \lambda_{21} (k \overset{\gamma}{*} \psi)(y) - \lambda_{11} (h \underset{F_c}{*} l)(y) + \lambda_{21} ((k \overset{\gamma}{*} \psi) \underset{F_c}{*} l)(y) \right\} \in L(R_+).$$

Here, $l \in L(R_+)$ *is defined by*

$$(F_c l)(y) = \frac{C\, F_c(\varphi \overset{\gamma}{*} \psi)(y)}{1 + C F_c(\varphi \overset{\gamma}{*} \psi)(y)}, \quad \lambda = \lambda_{11}\lambda_{22}, \quad C = -\frac{\lambda_{12}\lambda_{21}}{\lambda_{11}\lambda_{22}}.$$

P r o o f. System (30) can be written in the form

$$\begin{cases} \lambda_{11} f(y) + \lambda_{12} (\varphi \underset{1}{*} g)(y) &= k(y), \\ \lambda_{21} (f \overset{\gamma}{*} \psi)(y) + \lambda_{22} g(y) &= h(y), \qquad y > 0. \end{cases}$$

Using the factorization properties of the convolutions (7) and (11) we have

$$\begin{cases} \lambda_{11} (F_s f)(y) + \lambda_{12} (F_s \varphi)(y)(F_c g)(y) &= (F_s k)(y), \\ \lambda_{21} \sin y (F_s f)(y)(F_c \psi)(y) + \lambda_{22} (F_c g)(y) &= (F_c h)(y), \qquad y > 0. \end{cases}$$

Since,

$$\begin{aligned}
\Delta &= \begin{vmatrix} \lambda_{11} & \lambda_{12}(F_s\varphi)(y) \\ \lambda_{21}(F_c\psi)(y)\sin y & \lambda_{22} \end{vmatrix} \\
&= \lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21}\sin y(F_s\varphi)(y)(F_c\psi)(y) \\
&= \lambda_{11}\lambda_{12}\Big[1 - \frac{\lambda_{12}\lambda_{21}}{\lambda_{11}\lambda_{22}}\sin\, y(F_s\varphi)(y)(F_c\psi)(y)\Big] \\
&= \lambda[1 + C\,\mathrm{sign}\, y(F_s\varphi)(y)(F_c\psi)(y)(y)] \neq 0, \\
\Delta_1 &= \begin{vmatrix} (F_sk)(y) & \lambda_{12}(F_s\varphi)(y) \\ (F_ch)(y) & \lambda_{22} \end{vmatrix} = (F_sk)(y)\lambda_{22} - \lambda_{12}(F_s\varphi)(y)(F_ch)(y).
\end{aligned}$$

Therefore,

$$\begin{aligned}
(F_sf)(y) &= \frac{\Delta_1}{\Delta} = \frac{\Delta_1}{\lambda} - \frac{\Delta_1}{\lambda}\frac{C\sin\, y(F_s\varphi)(y)(F_c\psi)(y)}{1 + C\sin\, y(F_s\varphi)(y)(F_c\psi)(y)}, \\
&= \frac{\Delta_1}{\lambda} - \frac{\Delta_1}{\lambda}\frac{CF_c(\varphi \overset{\gamma}{*} \psi)(y)}{1 + CF_c(\varphi \overset{\gamma}{*} \psi)(y)}, \quad y > 0.
\end{aligned}$$

Due to Wiener-Levi's theorem [7] there exists a function $l \in L(R_+)$ such that

$$(F_cl)(y) = \frac{CF_c(\varphi \overset{\gamma}{*} \psi)(y)}{1 + CF_c(\varphi \overset{\gamma}{*} \psi)(y)}\,.$$

It follows that

$$(F_sf)(y) = \frac{1}{\lambda}\left[\Delta_1 - \Delta_1(F_cl)(y)\right] = \frac{1}{\lambda}\left\{(F_sk)(y)\lambda_{22}\right.$$

$$\left. - \lambda_{12}(F_s\varphi)(y)(F_ch)(y) - \left[(F_sk)(y)\lambda_{22} - \lambda_{12}(F_s\varphi)(y)(F_ch)(y)\right](F_cl)(y)\right\}$$

$$= \frac{1}{\lambda}\Big\{(F_sk)(y)\lambda_{22} - \lambda_{12}F_s(\varphi \underset{1}{*} h)(y) - F_s(k \underset{1}{*} l)(y) + \lambda_{12}F_s(\varphi \underset{1}{*} h)(y)(F_cl)(y)\Big\}$$

$$= \frac{1}{\lambda}\Big\{(F_sk)(y)\lambda_{22} - \lambda_{12}F_s(\varphi \underset{1}{*} h)(y) - F_s(k \underset{1}{*} l)(y) + \lambda_{12}F_s\Big((\varphi \underset{1}{*} h) \underset{1}{*} l\Big)(y)\Big\}.$$

Hence,

$$f(y) = \frac{1}{\lambda}\Big\{k(y)\lambda_{22} - \lambda_{12}(\varphi \underset{1}{*} h)(y) - (k \underset{1}{*} l)(y) + \lambda_{12}\Big((\varphi \underset{1}{*} h) \underset{1}{*} l\Big)(y)\Big\}.$$

Similarly,

$$\Delta_2 = \begin{vmatrix} \lambda_{11} & (F_sk)(y) \\ \lambda_{21}\sin y(F_c\psi)(y) & (F_ch)(y) \end{vmatrix}$$

$$= \lambda_{11}(F_c h)(y) - \lambda_{21} \sin y\, (F_s k)(y)(F_c \psi)(y) = \lambda_{11}(F_c h)(y) - \lambda_{21} F_c(k \overset{\gamma}{*} \psi)(y).$$

It follows that

$$(F_c g)(y) = \frac{\Delta_2}{\Delta} = \frac{\Delta_2}{\lambda} - \frac{\Delta_2}{\lambda}(F_c l)(y) = \frac{1}{\lambda}\{\lambda_{11}(F_c h)(y)$$

$$- \lambda_{21} F_c(k \overset{\gamma}{*} \psi)(y) - \lambda_{11}(F_c h)(y)(F_c l)(y) + \lambda_{21} F_c(k \overset{\gamma}{*} \psi)(y)(F_c l)(y)\Big\}$$

$$= \frac{1}{\lambda}\Big\{\lambda_{11}(F_c h)(y) - \lambda_{21} F_c(k \overset{\gamma}{*} \psi)(y)$$

$$- \lambda_{11}\Big(F_c(h \underset{F_c}{*} l)\Big)(y) + \lambda_{21} F_c\Big((k \overset{\gamma}{*} \psi) \underset{F_c}{*} l\Big)(y)\Big\}.$$

Hence

$$g(y) = \frac{1}{\lambda}\Big\{\lambda_{11} h(y) - \lambda_{21}(k \overset{\gamma}{*} \psi)(y) - \lambda_{11}(h \underset{F_c}{*} l)(y) + \lambda_{21}\Big((k \overset{\gamma}{*} \psi) \underset{F_c}{*} l\Big)(y)\Big\}.$$

The proof is complete. ∎

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