Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 7, NUMBER 3 (2004) ISSN 1311-0454

AN EXPANSION FORMULA FOR FRACTIONAL DERIVATIVES AND ITS APPLICATION

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Dedicated to Professor Ivan H. Dimovski
on the occasion of his 70th birthday

Abstract

An expansion formula for fractional derivatives given as in form of a series involving function and moments of its k-th derivative is derived. The convergence of the series is proved and an estimate of the reminder is given. The form of the fractional derivative given here is especially suitable in deriving restrictions, in a form of internal variable theory, following from the second law of thermodynamics, when applied to linear viscoelasticity of fractional derivative type.

2000 Mathematics Subject Classification: 26A33
Key Words and Phrases: fractional derivatives, moments of the k-th derivative

1. Motivation

In the one dimensional isothermal theory of a viscoelastic body of generalized Kelvin-Voigt type, the stress σ is given in terms of strain ε by an equation of the form

$$\sigma(t) = E_{\infty}\varepsilon(t) + E_{\alpha}\varepsilon^{(\alpha)}(t), \tag{1}$$

where $0<\alpha<1, E_{\infty}$ and E_{α} are constants and

$$u^{(\alpha)}(t)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_{0}^{t}\frac{u(\tau)}{(t-\tau)^{a}}d\tau,\quad 0<\alpha<1 \tag{2}$$

is the Riemann-Liouville fractional derivative of real order α. The important problem concerning the constitutive equation of the type (1) is to find the restrictions on coefficients α, E_{∞} and E_{α} so that the Second Law of Thermodynamics, in the form of Clausius-Duhem inequality, is satisfied. There are several approaches to this problem (see for example [2], [5] and [1]). One of these approaches uses the method of internal variables. In it, in order to obtain the restrictions, one has to postulate the rate equations for internal variables. The Classical Thermodynamics deals with rate equations of involving integer (first) order derivatives only. This is a consequence of the assumption that the state variables (such as ε) are chosen properly, so that all relevant quantities can be expressed in terms of local values of state variables. The fractional derivative is a nonlocal operator, and therefore the formalism of Thermodynamics cannot be applied directly to the rate equations involving fractional derivatives.

One way to avoid such a difficulty is to use expression for fractional derivative in terms of integer order derivatives. It is well known that for analytic function $f(t)$ (see [8], p.278) the fractional derivative can be expanded in a power series of involving integer order derivatives as

$$f^{(\alpha)}(t)=\sum_{n=0}^{\infty}\binom{\alpha}{n}\frac{t^{n-\alpha}}{\Gamma(n+1-\alpha)}f^{(n)}(t),\ \text{ where }\binom{\alpha}{n}=\frac{(-1)^{n-1}\Gamma(n-\alpha)}{\Gamma(1-\alpha)\Gamma(n+1)}. \tag{3}$$

The expansion (3) is not useful for our purposes for two reasons. First, it involves derivatives of the function f of all (integer) orders, and second, it could be used for analytic functions only.

Our intention in this note is to derive an expansion formula for the fractional derivative of a function that will involve a function, a finite number of its integer order derivatives and time moments of a single integer order derivative. This expansion formula could be used for functions that are not analytic, but only finitely many times differentiable. Also, it offers a possibility to use internal variables method to derive the restrictions on the constitutive equations involving fractional order derivatives.

2. Preliminaries

The reader not familiar with spaces of generalized functions can pass over this part. Let $\mathcal{D}'(\mathbb{R})$ denote the space of Schwartz's distributions (cf. [9]) and $\mathcal{S}'(\mathbb{R})$ the space of tempered distributions. Then $\mathcal{D}'_+(\mathbb{R}) = \{T \in \mathcal{D}'(\mathbb{R});\ \text{supp}T \subset [0,\infty)\}$ and $\mathcal{S}'_+(\mathbb{R}) = \{T \in \mathcal{S}'(\mathbb{R})\ \text{supp}\,T \subset [0,\infty)\}$. The spaces $\mathcal{D}'_+$ and $\mathcal{S}'_+$ are commutative algebras with addition and convolution as operators. If $f \in L^1_{loc}(\mathbb{R})$, then it defines the distribution denoted by $[f]$. In the sequel,

$$\Delta^\alpha = \begin{cases} \delta^{(\alpha)}, & \alpha = 0,1,... \\ \frac{x_+^{-\alpha-1}}{\Gamma(-\alpha)}, & \alpha \in \mathbb{R}\setminus\{0,1,...\} \end{cases} \tag{4}$$

stands for the family belonging to $\mathcal{S}'_+$. If $\alpha < 0, \Delta^\alpha$ is defined by the function $f(x) = 0, x < 0; f(x) = x^{-\alpha-1}/\Gamma(-\alpha), x > 0$. $\{\Delta^\alpha; \alpha \in \mathbb{R}\}$ is a semi-group: $\Delta^{\alpha_1} * \Delta^{\alpha_2} = \Delta^{\alpha_1+\alpha_2}$ ($*$ is the sign of convolution).

For $T \in \mathcal{D}'_+$ and $0 \le m-1 < \alpha < m, m \in \mathbb{N}$, the fractional distributional derivative $D^\alpha T = \Delta^\alpha * T$. If T is defined by a function $f \in \mathcal{C}[0,\infty), T = [f]$, then for $\alpha = m-\nu, m \in \mathbb{N}, 0< \nu <1$:

$$\begin{aligned} \Delta^\alpha * T &= \Delta^\alpha * [f] = \Delta^m * \Delta^{-\nu} * [f] = \delta^{(m)} * \left(\Delta^{-\nu} * [f]\right) \\ &= D^m \left(\frac{x_+^{\nu-1}}{\Gamma(\nu)} * [f]\right) = \left[\frac{d^m}{dx^m}\frac{1}{\Gamma(m-\alpha)}\int_0^x \frac{f(\tau)\,d\tau}{(x-\tau)^{-m+\alpha+1}}\right], \end{aligned}$$

and this is the definition of the α-th fractional derivative of the function f for $x > 0$ (cf. [4]).

In some cases we can express the relation between the distributional derivative $D^k[f]$ of a distribution $[f]$, given by the function f, and the classical derivative $f^{(k)}$ of the function f. Such a case is the following.

Suppose that $f(x)$ is a function defined for $x \in \mathbb{R}, f(x) = 0, x \in (-\infty,0)$. Let $f(x)$ have all derivatives in all points except at $\{x_n\}_{n\in\mathbb{N}}$, $\lim_{n\to\infty} x_n = \infty$ in which $f^{(p)}, p \in \mathbb{N}_0$ can have jumps $\left\{f_n^{(p)}\right\}_{n\in\mathbb{N}}, \left|f_n^{(p)}\right| < \infty, n \in \mathbb{N}$. Let $f_0^{(p)}(x), 0 \le p \le k$ be a function defined as $f_0^{(p)}(x) = f^{(p)}(x), x \in [0,\infty] \setminus \{x_1,...,x_r\}, \{x_1,...,x_r\}$ are jump points of $f^{(k)}$; $f_0^{(p)}(x_n)$ is not defined for $n = 1,...,r, 0 \le p \le k$. Then (cf.[9], p.37),

$$D^k[f] = \left[f_0^{(k)}\right] + \sum_{n=1}^r f_n^{(k-1)}\delta(x_n) + \cdots + \sum_{n=1}^r f_n \delta^{(k-1)}(x_n). \tag{5}$$

We write $V_n\left(f_0^{(p)}\right), n\in\mathbb{N}_0$, for the n-th moment of the function $f_0^{(p)}$,

$$V_n\left(f_0^{(p)}\right)(t)=\int_0^t f_0^{(p)}(\tau)\,\tau^n d\tau,\quad n\in\mathbb{N}_0,\ \ t\geq 0. \tag{6}$$

It is easily seen that $V_n\left(f_0^{(p)}\right)=V_n\left(f^{(p)}\right)$.

3. Main result

THEOREM 1. *Let $m,k\in\mathbb{N}$ and α,ν,q be real numbers with the properties: $0<\nu<1,\alpha=m-\nu,q=k-\alpha-1>0$. Suppose that:*

a) the function $f\in\mathcal{C}^k[0,\infty)$ and

$$f(0)=\cdots=f^{(k-1)}(0)=0. \tag{7}$$

Then for $t>0$ we have

$$\begin{aligned} f^{(\alpha)}(t)&=\frac{t^q}{\Gamma(k-\alpha)}\left(V_0\left(f^{(k)}\right)(t)+\sum_{p=1}^{\infty}\frac{\Gamma(p-q)}{\Gamma(-q)\,p!}\frac{1}{t^p}V_p\left(f^{(k)}\right)(t)\right)\\ &\equiv K_0\left(-k+\alpha,f^{(k)}\right)(t). \end{aligned} \tag{8}$$

The series in (8) converges in $\mathcal{C}(0,\infty)$ (converges uniformly on every compact set in $(0,\infty)$).

b) If $f\in\mathcal{C}^k[0,\infty)$, but has not the property (7), then

$$\begin{aligned} D^\alpha[f]&=\left[K_0\left(-k+\alpha,f^{(k)}\right)\right]\\ &\quad+f^{(k-1)}(0)\frac{t_+^{k-\alpha-1}}{\Gamma(k-\alpha)}+\cdots+f(0)\frac{t_+^{-\alpha}}{\Gamma(1-\alpha)}\\ &\equiv\left[K_1\left(-k+\alpha,f^{(k)}\right)\right]. \end{aligned} \tag{9}$$

c) If f has the k-th derivative in all points except in the points $\{x_1,...,x_r\}\subset[0,\infty)$ in which $f^{(p)},0\leq p\leq k-1$, has bounded jumps $f_i^{(p)},i=1,\cdots r$, then we have

$$\begin{aligned} D^\alpha[f]&=\left[K_1\left(-k+\alpha,f^{(k)}\right)\right]+\sum_{i=1}^{r}f_i\frac{(t-x_i)_+^{-\alpha}}{\Gamma(1-\alpha)}\\ &\quad+\cdots+\sum_{i=1}^{r}f_i^{(\kappa-1)}\frac{(t-x_i)_+^{k-\alpha-1}}{\Gamma(k-\alpha)}\equiv\left[K_2\left(-k+\alpha,f^{(k)}\right)\right]. \end{aligned} \tag{10}$$

d) If in b) and c) we have $0 < \alpha < 1$ *and* $k \geq 2$*, then* $D^{\alpha}[f] = f^{(\alpha)}(t), t \in (0,\infty)$ *and* $\frac{t_+^{\beta}}{\Gamma(1-\beta)}, \frac{(t-x_i)_+^{\beta}}{\Gamma(1-\beta)}, -\alpha \leq \beta \leq k-\alpha-1$*, are functions. The proof can be given without using spaces of generalized functions.*

e) Let $k > k_1 = k - w > 0$*. If* f *has only the* k_1*-th derivative in all points* $t \in [0,T]$ *except at the points* $\{x_1, ..., x_r\} \subset (0,T)$ *in which* $f^{(p)}, 0 \leq p \leq k_1 - 1$ *has bounded jumps* $f_i^{(p)}, i = 1, ..., r$*, then for* $t \in [0,T)$ *we have*

$$D^{\alpha}[f] = D^{k-k_1}\left[K_2\left(-k+\alpha, f^{(k_1)}\right)\right]. \tag{11}$$

P r o o f.

a) Let $t \in [\varepsilon, T], \varepsilon > 0, T < \infty$. The function $f^{(\alpha)}(t)$ defines the distribution $\left[f^{(\alpha)}\right] \subset \mathcal{D}'_+ \subset \mathcal{D}'(\mathbb{R})$. Then

$$\begin{aligned} D^{\alpha}[f] &= \Delta^{\alpha} * [f] = \Delta^{-k+\alpha} * \Delta^k * [f] \\ &= \frac{t^{k-\alpha-1}}{\Gamma(k-\alpha)} * D^k[f] = \frac{t^{k-\alpha-1}}{\Gamma(k-\alpha)} * \left[f^{(k)}\right]. \end{aligned}$$

Since $k - \alpha - 1 = q > 0$, and $f^{(k)} \in \mathcal{C}[0,\infty)$,

$$\begin{aligned} f^{(\alpha)}(t) &= \frac{1}{\Gamma(k-\alpha)} \int_0^t f^{(k)}(\tau)(t-\tau)^q \, d\tau \\ &= \frac{t^q}{\Gamma(k-\alpha)} \int_0^t f^{(k)}(\tau)\left(1 - \frac{\tau}{t}\right)^q d\tau, \quad t > 0. \end{aligned} \tag{12}$$

We will use the binomial formula (cf. [6], p.217)

$$\begin{aligned} (1+z)^q &= 1 + \sum_{p=1}^{N} \frac{\Gamma(q+1)}{\Gamma(q+1-p)\,p!} z^p + R_{N+1}, \\ R_{N+1} &= \frac{\Gamma(q+1)}{(N+1)!\Gamma(q-N)} z^{N+1} \\ &\times \int_0^1 (N+1)(1-\tau)^N (1+z\tau)^{q-N-1} \, d\tau, \end{aligned} \tag{13}$$

if $1 + z\tau \neq 0, \tau \in [0,1]$. We change the form of the coefficients in (13) using the equality

$$\frac{\Gamma(q+1)}{\Gamma(q+1-p)\,p!} = \frac{\Gamma(p-q)(-1)^p}{\Gamma(-q)\,p!}, \quad 0 < q \neq N.$$

Next we prove that $f^{(\alpha)}(t)$ can be written as:

$$f^{(\alpha)}(t)=\frac{t^{q}}{\Gamma(k-\alpha)}\int_{0}^{t}f^{(k)}(\tau)\left(1+\sum_{p=1}^{\infty}\frac{\Gamma(p-q)(-1)^{p}}{\Gamma(-q)\,p!}\left(\frac{\tau}{t}\right)^{p}\right)d\tau. \quad (14)$$

Let $M=\sup\left|f^{(k)}(t)\right|, 0<\varepsilon\leq t\leq T$, then by the asymptotic behavior of the function $\Gamma(z)$:

$$\left|\frac{\Gamma(p-q)(-1)^{p}}{\Gamma(-q)\,p!}\left(\frac{\tau}{t}\right)^{p}\right|\leq C\frac{1}{(p+1)^{q+1}},\quad t\in[\varepsilon,T],\quad \varepsilon>0, \quad (15)$$

and $0\leq\tau\leq t$. From inequality (15) it follows that the series in (14) converges uniformly in $[\varepsilon,T]$ for every $\varepsilon>0$ and $T<\infty$. This proves a).

b) The function $\bar{f}:\bar{f}(t)=f(t), t\in[0,\infty); \bar{f}(t)=0, t<0$, is such that $\bar{f}^{(p)}(t)$ exists for $t\neq 0$, $0\leq p\leq k$, and $\bar{f}^{(p)}(t)$ has a jump at $t=0, f^{(p)}(0), 0\leq p\leq k-1$. Then by (5),

$$\begin{aligned}D^{\alpha}[f]&=\frac{t_{+}^{k-\alpha-1}}{\Gamma(k-\alpha)}*\left(\left[f_{0}^{(k)}\right]+f_{0}^{(k-1)}(0)\,\delta(0)+\cdots+f(0)\,\delta^{(k-1)}(0)\right)\\&=\frac{t_{+}^{k-\alpha-1}}{\Gamma(k-\alpha)}*\left[f_{0}^{(k)}\right]+f_{0}^{(k-1)}\frac{t_{+}^{k-\alpha-1}}{\Gamma(k-\alpha)}+\cdots+f(0)\frac{t_{+}^{-\alpha}}{\Gamma(k-\alpha)},\end{aligned}$$

which proves b).

c) The proof of c) is just the same as the proof of b).

e) To prove (11), we consider

$$\begin{aligned}D^{\alpha}[f]&=\Delta^{\alpha}*[f]=\Delta^{k-k_{1}}*\Delta^{-k+\alpha}*\Delta^{k_{1}}[f]\\&=\delta^{(k-k_{1})}*\frac{t_{+}^{k-\alpha-1}}{\Gamma(k-\alpha)}*D^{k_{1}}[f]\\&=D^{k-k_{1}}\left[K_{2}\left(-k+\alpha,f^{(k_{1})}\right)\right].\end{aligned}$$

Since,

$$\begin{aligned}
&\left[K_2\left(-k+\alpha, f^{(k_1)}\right)\right] \\
&= \left[K_0\left(-k+\alpha, f^{(k_1)}\right)\right] + f^{(k_1-1)}(0)\,\frac{t_+^{k-\alpha-1}}{\Gamma(k-\alpha)} + \cdots \\
&+ f(0)\,\frac{t_+^{k-k_1-\alpha}}{\Gamma(1+k-k_1-\alpha)} + \sum_{i=1}^{r} f_i \frac{(t-x_i)_+^{k-k_1-\alpha}}{\Gamma(1+k-k_1-\alpha)} + \cdots \\
&+ \sum_{i=1}^{r} f_i^{(k_1-1)} \frac{(t-x_i)_+^{k-\alpha-1}}{\Gamma(k-\alpha)}\,, \qquad (16)
\end{aligned}$$

we have only to apply D^{k-k_1} to every addend in (16). We proved in a) that the series which defines $K_0\left(-k+\alpha, f^{(k_1)}\right)$ converges in $\mathcal{C}(0,\infty)$, consequently the D^{k-k_1}, applied to it, can be realized by applying D^{k-k_1} on every term of the series (cf. [9], p.76). ■

R e m a r k. If $\beta \in \mathbb{R}_+$, then (cf. [4])

$$J^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t f(\tau)(t-\tau)^{\beta-1}\, d\tau, \quad t>0,$$

denotes the fractional integral of order β. Since $J^\beta f(t)$ can be written as

$$\begin{aligned}
J^\beta f(t) &= \left(\frac{\tau^{\beta-1}}{\Gamma(\beta)} * f(\tau)\right)(t) \\
&\cong \left(\Delta^{-\beta}\right) * [f] = \Delta^{-k-\beta} * \Delta^k * [f] \\
&\cong \left[\frac{t^{k+\beta-1}}{\Gamma(k+\beta)}\right] * D^k[f]\,, \qquad (17)
\end{aligned}$$

where $k \in \mathbb{N}_0$ and $g \cong [g]$ means that $[g]$ is the regular distribution which corresponds to the function g. The form of $J^\beta f(t)$ given in (17) allows that Theorem 1 is applied to the fractional integral.

Also it is easily seen that $K_1\left(-k-\beta, f^{(k)}\right)(t)$ and $K_2\left(-k-\beta, f^{(k)}\right)(t)$,

$\beta > 0$ are functions which define distributions

$$\left[K_1\left(k-\beta, f^{(k)}\right)\right] \quad \text{and} \quad \left[K_2\left(k-\beta, f^{(k)}\right)\right].$$

4. Approximation of $f^{(\alpha)}$ and estimate of the reminder

We give an approximation of $f^{(\alpha)}$ in cases when it is not a generalized function (cf. a) and d)) in Theorem 1. In all these cases we can do it by taking only N terms in the series given by (8). To estimate the reminder (cf. (13)), let M_t denote

$$M_t = \max_{0\le\tau\le t}\left|f^{(k)}(\tau)\right|.$$

Then

$$\begin{aligned}
&\left|\frac{t^q}{\Gamma(k-\alpha)}\int_0^t f^{(k)}(\tau)\,R_{N+1}\left(\frac{\tau}{t}\right)d\tau\right| \\
&\le \frac{t^q}{\Gamma(k-\alpha)}\int_0^t \left|f^{(k)}(\tau)\right|\frac{\Gamma(N+1-q)}{\Gamma(1-q)\,N!}\left(\frac{\tau}{t}\right)d\tau \\
&\le M_t\left|\frac{\Gamma(N+1-q)}{\Gamma(k-\alpha)\,\Gamma(1-q)\,N!}\right|\frac{t^q}{t^{N+1}}\int_0^t \tau^{N+1}d\tau \\
&\le M_t\left|\frac{\Gamma(N+1-q)}{\Gamma(k-\alpha)\,\Gamma(1-q)\,N!}\right|\frac{t^{q+1}}{(N+2)},\quad t\in[\varepsilon,T],\ \varepsilon>0,\ T<\infty.
\end{aligned}\tag{18}$$

In this way we obtain

$$f^{(\alpha)}(t)=\frac{t^q}{\Gamma(k-\alpha)}\left(V_0\left(f^{(k)}\right)(t)+\sum_{p=1}^{N}\frac{\Gamma(p-q)}{\Gamma(-q)\,p!}\frac{1}{t^p}V_p\left(f^{(k)}\right)(t)\right)+Q_{N+1}(t),\tag{19}$$

where

$$Q_{N+1}(t)=\frac{t^q}{\Gamma(k-\alpha)}\int_0^t f^{(k)}(\tau)\,R_{N+1}\left(\frac{\tau}{t}\right)d\tau.\tag{20}$$

An estimate of $Q_{N+1}(t)$ is given by (18).

5. Applications

In this section we use the expansion (8) in two particular examples.

E x a m p l e 1. It is known that a viscoelastic body of generalized Kelvin-Voigt type in a linear, isothermal deformation process is described by a constitutive equation in the form of so called three parameter model (see

[7]) given by (1). Suppose further that $\varepsilon(t)$ is continuous and has continuous first derivative, except at the point $t = 0$ where the first derivative has a jump $\varepsilon_0^{(1)}$. When the expansion (8) with $k = 2, p = 1, r = 1, q = 1 - \alpha$ is used in (8), we obtain

$$\sigma(t) = E\varepsilon(t) + E_\alpha \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \left\{ \varepsilon^{(1)}(t) + \sum_{j=1}^{\infty} \frac{\Gamma(j+\alpha-1)}{\Gamma(\alpha-1)\, j!} \frac{1}{t^j} V_j\left(\varepsilon^{(2)}\right)(t) \right\}$$
$$+ \varepsilon(0) \frac{t^{-\alpha}}{\Gamma(1-\alpha)} + \varepsilon_0^{(1)} \frac{t^{1-\alpha}}{\Gamma(2-\alpha)}, \tag{21}$$

with

$$V_j(t) = \int_0^t \varepsilon^{(2)}(\tau)\, \tau^j d\tau, \quad j = 1, ... \tag{22}$$

Let us identify the functions $V_j(t)$ in the constitutive equation (21) as internal variables $\xi_j, j = 1, ...$ In the equilibrium, internal variables are governed by thermodynamic variables while they can be considered as independent in non-equilibrium evolutions. Then (21) may be written as

$$\sigma(t) = E_\infty \varepsilon(t) + E_\alpha \sum_{i=1}^{\infty} \varphi_n(t)\, \xi_n(t), \tag{23}$$

where

$$\varphi_0(t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)};$$
$$\varphi_n(t) = \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \frac{\Gamma(n+\alpha-1)}{\Gamma(\alpha-1)\, n!} \frac{1}{t^n}; \quad n = 1, 2, ... \tag{24}$$

and the internal variables evolution law is

$$\xi_n^{(1)}(t) = \varepsilon^{(2)}(t)\, t^n, \quad n = 0, 1, ... \tag{25}$$

The internal variables are not components of a vector (since this is ruled out by the principle of material frame indifference, see [3]), but as a collection of scalar parameters. To derive the thermodynamical restrictions on the constitutive equation (23),(25) we follow the procedure used in [1] (see also [5] where other procedures are discussed). Thus, we consider a rod loaded as shown in **Figure 1**. We assume that the rod is deformed homogeneously so that the only independent variable is the time t.

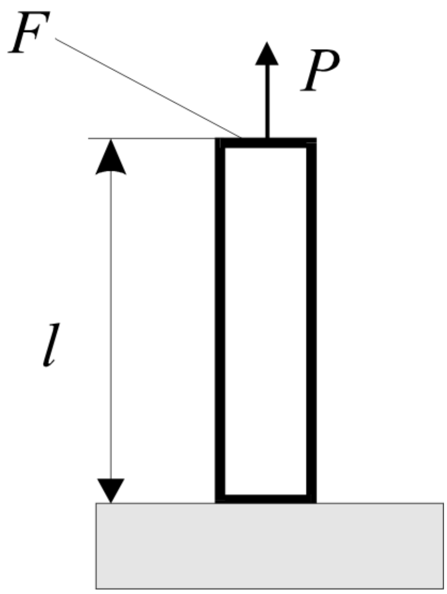

Figure 1: Loading configuration

The length of the rod is L in the undeformed state and $l(t)$ during the deformation. The rod is loaded by the force P and F is the cross-sectional area in the undeformed state. Thus stress, σ referred to the cross-sectional area of the unloaded rod and strain ε are given by

$$\sigma(t) = \frac{P}{F} \quad \text{and} \quad \varepsilon(t) = \frac{l}{L} - 1. \tag{26}$$

We describe a state of the body by two types of variables: the strain $\varepsilon(t)$ and a set of internal variables $\xi_n(t), n = 1, 2, ...$ The equilibrium state of the unloaded body corresponds to

$$\varepsilon = 0, \quad \xi = 0. \tag{27}$$

Thus the internal energy U, the entropy S and the free energy $U - TS$ are all functions of ε, (or l) and $\xi_n, n = 1, 2, ...$ The Gibbs equation for the free energy $U - TS$ reads

$$(U - TS)^{(1)} = \sigma V \varepsilon^{(1)} - \sum_{n=0}^{\infty} \Theta_n \xi_n^{(1)}. \tag{28}$$

T is the temperature, assumed to be constant. Θ_n is the "force" associated with the internal variable ξ_n so that $\Theta_n \frac{d\xi_n}{dt}$ is the power of the force Θ_n, and V is the volume of the body which is assumed to be constant. Since σ is linear in ε and ξ_n (see (23)), we assume that Θ_n are also linear in the same variables so that

$$\Theta_n = \gamma_n \varepsilon + \sum_{s=0}^{\infty} \Delta_{ns} \xi_s, \tag{29}$$

where $\gamma_n (t)$ and $\Delta_{sn} (t), s, n = 1, ...$ are specified constitutive functions. Note that with (23) the force P is given as

$$P(t) = F \left[E_{\infty} \varepsilon (t) + E_{\alpha} \sum_{i=1}^{\infty} \varphi_n (t) \xi_n (t) \right]. \tag{30}$$

The integrability for the free energy $(\partial\sigma/\partial\xi_n = -\partial\Theta_n/\partial\varepsilon)$ requires

$$E_{\alpha} \varphi_n (t) = -\gamma_n. \tag{31}$$

Therefore, with (31) the constitutive equations for σ and Θ_n become

$$\sigma (t) = E_{\infty} \varepsilon (t) + E_{\alpha} \sum_{i=1}^{\infty} \varphi_n (t) \xi_n (t),$$

$$\Theta_n(t) = -E_{\alpha} \varphi_n (t) \varepsilon(t) + \sum_{s=0}^{\infty} \Delta_{ns}(t) \xi_s(t), \quad n = 0, 1, ... \tag{32}$$

We return to the Gibbs equation (28) in which we replace $\frac{dU}{dt} = U^{(1)}$ and $\frac{d\xi_n}{dt} = \xi_n^{(1)}$ by the equations of balance of energy and of internal variable, viz.

$$U^{(1)} = Q^{(1)} + \sigma V \varepsilon^{(1)} (t),$$

$$\xi_n^{(1)} = \varepsilon^{(2)} (t) t^n, \quad n = 0, 1, ..., \tag{33}$$

$Q^{(1)}$ is the heating. Thus from (27) we obtain an equation of balance of entropy in the form

$$S^{(1)} - \frac{Q^{(1)}}{T} = \sum_{n=0}^{\infty} \frac{\Theta_n}{T} \xi_n^{(1)} = \sum_{n=0}^{\infty} \frac{\Theta_n}{T} \varepsilon^{(2)} (t) t^n \geq 0, \tag{34}$$

where we have indicated that the entropy production is non-negative. The condition (34) is equivalent to the equation (4.2) of [5]. The requirement that the equilibrium configuration $(\varepsilon_0 = (\xi_1)_0 = (\xi_2)_0 = ... = 0)$ is stable leads to the condition that Gibbs free energy $U - TS - Pl$ tends to a minimum at $(\varepsilon_0 = (\xi_1)_0 = (\xi_2)_0 = ... = 0)$. This condition is equivalent to

$$\begin{bmatrix} \frac{\partial\sigma}{\partial\varepsilon} & \frac{\partial\sigma}{\partial\xi_1} & . & \frac{\partial\sigma}{\partial\xi_n} & . \\ -\frac{\partial\Theta_1}{\partial\varepsilon} & -\frac{\partial\Theta_1}{\partial\xi_1} & . & -\frac{\partial\Theta_1}{\partial\xi_n} & . \\ . & . & . & . & . \\ -\frac{\partial\Theta_n}{\partial\varepsilon} & -\frac{\partial\Theta_n}{\partial\xi_1} & . & -\frac{\partial\Theta_n}{\partial\xi_n} & . \\ . & . & . & . & . \end{bmatrix}, \text{ positive definite,} \tag{35}$$

or

$$\begin{bmatrix} E_\infty & E_\alpha\varphi_1 & . & E_\alpha\varphi_n & . \\ E_\alpha\varphi_1(t) & -\Delta_{11} & . & -\Delta_{1n} & . \\ . & . & . & . & . \\ E_\alpha\varphi_n(t) & -\Delta_{n1} & . & -\Delta_{nn} & . \\ . & . & . & . & . \end{bmatrix}, \text{ positive definite.} \tag{36}$$

The condition (36) is satisfied if we take, for example, $E_\infty > 0, E_\alpha > 0$ and $\Delta_{mn} = -\delta_{nm}$, where δ_{mn} is the Kronecker symbol. We analyze next (34). By using (29) this condition leads to

$$\frac{1}{T}\sum_{n=0}^{\infty}\left\{-E_\alpha\varphi_n(t)\,\varepsilon(t) - \int_0^t \varepsilon^{(2)}(\tau)\,\tau^n d\tau\right\}\varepsilon^{(2)}(t)\,t^n \geq 0, \quad t \geq 0. \tag{37}$$

Note that (see (24),(25)), $\varphi_n(t) > 0, n = 1, 2, ...$ It is obvious that, the condition (37) cannot be satisfied for all $t \geq 0$ and arbitrary $\varepsilon(t)$. For example, let $\varepsilon = \varepsilon_0 \sin\omega t$ with ε_0 and ω being constants. For sufficiently small t the condition (37) is violated. Thus, the constitutive equation (1) does not satisfy the restrictions following from the second law of Thermodynamics. This result, obtained here by the method of internal variables, is in agreement with the conclusion reached in [2] by using different approach.

E x a m p l e 2. Consider the problem of expanding the $\alpha-$th derivative of a function that is not analytic. Thus, we consider the function

$$f(t) = \begin{cases} 0, & 0 \leq t \leq 1 \\ (t-1)^2, & t \geq 1 \end{cases}. \tag{38}$$

Its derivative of the order $\alpha = 1/2$ is

$$f^{(1/2)}(t) = \begin{cases} 0, & 0 \leq t \leq 1 \\ \frac{1}{\sqrt{\pi}}\frac{8}{3}(t-1)^{3/2}, & t \geq 1 \end{cases}. \tag{39}$$

In this case $k = 2, r = 1, q = 1 - \alpha = 1/2$. From (19) we have

$$f^{(1/2)}(t) = \frac{t^{1/2}}{\Gamma(2-1/2)}\left(V_0\left(f^{(2)}\right)(t) + \sum_{p=1}^{\infty}\frac{\Gamma(p-1/2)}{\Gamma(-1/2)\,p!}\frac{1}{t^p}V_p\left(f^{(k)}\right)\right).$$

Since

$$f^{(2)}(t) = \begin{cases} 0, & 0 \leq t \leq 1 \\ 2, & t > 1 \end{cases},$$

it follows that

$$V_p\left(f^{(2)}\right)(t) = \begin{cases} 0, & 0 \le t \le 1 \\ 2\frac{t^{p+1}-1}{p+1}, & t > 1, \quad p = 0, 1, \end{cases}.$$

Therefore the series (8) becomes

$$f^{(1/2)}(t) = \begin{cases} 0\,, & 0 \le t \le 1 \\ \frac{4t^{1/2}}{\sqrt{\pi}}\left((t-1) + \sum_{p=1}^{\infty} \frac{\Gamma(p-1/2)}{-2\sqrt{\pi}p!}\frac{1}{t^p}\frac{t^{p+1}-1}{p+1}\right), & t \ge 1 \end{cases}. \tag{40}$$

By calculating the first ten terms ($N = 10$ in (19)), we obtain

$$\begin{aligned} f^{(1/2)}(t) = \frac{4t^{\frac{1}{2}}}{\sqrt{\pi}}\Bigg((t-1) &- \frac{1}{4}\frac{t^2-1}{t} - \frac{1}{24}\frac{t^3-1}{t^2} - \frac{1}{64}\frac{t^4-1}{t^3} \\ &- \frac{1}{128}\frac{t^5-1}{t^4} - \frac{7}{1536}\frac{t^6-1}{t^5} - \frac{3}{1024}\frac{t^7-1}{t^6} - \frac{33}{16384}\frac{t^8-1}{t^7} \\ &- \frac{143}{98304}\frac{t^9-1}{t^8} - \frac{143}{131072}\frac{t^{10}-1}{t^9} - \frac{221}{262144}\frac{t^{11}-1}{t^{10}} - ...\Bigg), \\ t \ge 1. \end{aligned} \tag{41}$$

In **Figure 2** we draw the exact expression (39) and the approximation given by (41) for $t \ge 1$.

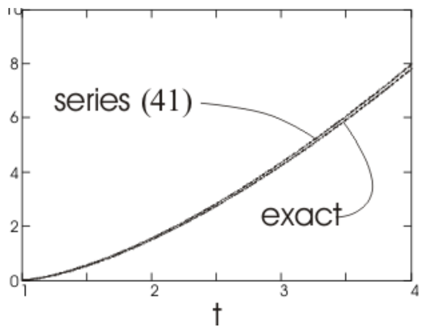

Figure 2: Exact and approximate value of the derivative

6. Conclusions

In this note we proved an expansion formula for fractional derivatives given by (8). It contains integer derivatives up to the finite order k and time moments of k-th derivative, given by (6). The mechanical interpretation of the moments, as internal variables, leads to the possibility of applying the Clausius-Duhem inequality for materials with internal variables in order to obtain the restrictions on the coefficients in the constitutive equations. Also it allows for determining series expansion of fractional derivatives for functions that are not analytic.

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Received: September 22, 2004

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