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ON THE RANGE OF THE FOURIER TRANSFORM ASSOCIATED WITH THE SPHERICAL MEAN OPERATOR

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Abstract

We characterize the range of some spaces of functions by the Fourier transform associated with the spherical mean operator $\mathcal{R}$ and we give a new description of the Schwartz spaces. Next, we prove a Paley-Wiener and a Paley-Wiener-Schawrtz theorems.

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1. Introduction

The spherical mean operator $\mathcal{R}$ is defined, for a function f on $\mathbb{R}^{n+1}$, even with respect to the first variable, by

$$\mathcal{R}(f)(r,x) = \int_{S^n} f(r\eta, x + r\xi) d\sigma_n(\eta,\xi), \quad (r,x) \in \mathbb{R} \times \mathbb{R}^n,$$

where S^n is the unit sphere $\{(\eta,\xi) \in \mathbb{R} \times \mathbb{R}^n : \eta^2 + \|\xi\|^2 = 1\}$ in $\mathbb{R}^{n+1}$ and σ_n is the surface measure on S^n normalized to have total measure one.

This operator plays an important role and has many applications, for example, in image processing of so-called synthetic aperture radar (SAR) data (see[5, [6]), or in the linearized inverse scattering problem in acoustics [4]. In [9] the second author with M.M. Nessibi and K. Trimèche have defined a generalized Fourier transform and a generalized convolution product associated with $\mathcal{R}$, and they have established some results in the theory of harmonic analysis (inversion formula, Paley-Wiener and Plancherel theorems, etc). Also, in [10], the second author with K. Trimèche have studied the Weyl transforms associated with the spherical mean operator $\mathcal{R}$. Vu Kim Tuan has studied in [13] the range of the Hankel and extended Hankel transforms on some spaces of functions.

Using the same idea as in [13] and the properties of the Fourier transforms associated with the spherical mean operator $\mathcal{R}$, we characterize in this paper the range of some subspaces of $L^2(\mathbb{R}_+ \times \mathbb{R}^n, r^n dr dx)$ (the space of square integrable functions on $\mathbb{R}_+ \times \mathbb{R}^n$ with respect to the measure $r^n dr dx$) by this transform. We give a new description of the spaces $S_*(\mathbb{R} \times \mathbb{R}^n)$ (the space of infinitely differentiable functions on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable, rapidly decreasing together with all their derivatives) and $S_*(\Gamma)$ (the space of infinitely differentiable functions, even with respect to the first variable, rapidly decreasing together with all their derivatives on the set $\Gamma = \mathbb{R} \times \mathbb{R}^n \cup \{(it, x); (t, x) \in \mathbb{R} \times \mathbb{R}^n, |t| \leq \|x\|\}$).

This paper is arranged as follows. In the first section, we recall some properties of the Fourier transform associated with the spherical mean operator $\mathcal{R}$. In the second section, we describe the range of rapidly decreasing functions by the Fourier transform associated with the spherical mean operator. In the third section, we will give an other characterization of the space $S_*(\mathbb{R} \times \mathbb{R}^n)$ and using the result of the precedent section, we obtain a description of the space $S_*(\Gamma)$. In the last section, a Paley-Wiener and a Paley-Wiener-Schawrtz theorems are established.

2. Fourier transform associated with the spherical mean operator

In this section, we recall some properties of the Fourier transform associated with the spherical mean operator. For more details see ([1],[4],[9],[10]).

Notation. We denote by:

- $\mathcal{E}_*(\mathbb{R} \times \mathbb{R}^n)$ the space of infinitely differentiable functions on $\mathbb{R} \times \mathbb{R}^n$, even with respect to the first variable.

- S^n the unit sphere in $\mathbb{R} \times \mathbb{R}^n$,

$$S^n = \{(\eta, \xi) \in \mathbb{R} \times \mathbb{R}^n; \eta^2 + \|\xi\|^2 = 1\},$$

where for $\xi = (\xi_1, ..., \xi_n)$, we have $\|\xi\|^2 = \xi_1^2 + ... + \xi_n^2$.

- $d\sigma_n$ the normalized surface measure on S^n.

DEFINITION 2.1. The spherical mean operator on $\mathcal{E}_*(\mathbb{R} \times \mathbb{R}^n)$ is defined by

$$\forall (r,x) \in [0,+\infty[\times \mathbb{R}^n, \;\; \mathcal{R}f(r,x) = \int_{S^n} f(r\eta, x + r\xi) d\sigma_n(\eta,\xi).$$

For $(\mu,\lambda) \in \mathbb{C} \times \mathbb{C}^n$, let us put

$$\forall (r,x) \in [0,+\infty[\times \mathbb{R}^n, \varphi_{\mu,\lambda}(r,x) = \mathcal{R}(\cos(\mu.)e^{-i<\lambda/.>})(r,x).$$

We have

$$\varphi_{\mu,\lambda}(r,x) = j_{\frac{n-1}{2}}(r\sqrt{\mu^2 + \|\lambda\|^2})e^{-i<\lambda/x>},$$

where $j_{\frac{n-1}{2}}$ is the normalized Bessel function defined by

$$j_{(n-1)/2}(x) = 2^{(n-1)/2}\Gamma((n+1)/2)\frac{J_{(n-1)/2}(z)}{z^{(n-1)/2}}. \tag{2.1}$$

Here $J_{(n-1)/2}$ is the Bessel function of first kind and index $(n-1)/2$ ([8],[14]), and if $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{C}^n$ and $x = (x_1, ..., x_n) \in \mathbb{R}^n$, we put $\lambda^2 = \lambda_1^2 + ... + \lambda_n^2$ and $< \lambda/x >= \lambda_1 x_1 + ... + \lambda_n x_n$.

The normalized Bessel function $j_{(n-1)/2}$ satisfies the following property

$$\forall \; k \in \mathbb{N}, \forall \; r \in \mathbb{R}; \;\; |j^{(k)}_{(n-1)/2}(r)| \leq 1. \tag{2.2}$$

Moreover for all $\lambda \in \mathbb{C}$; the function $r \mapsto j_{(n-1)/2}(\lambda r)$ is the unique solution of the differential equation

$$\begin{cases} L_n u(r) = -\lambda^2 u(r), \\ u(0) = 1, u'(0) = 0 \;, \end{cases} \tag{2.3}$$

where L_n is the Bessel operator defined on $\mathbb{R}_+^*$ by

$$L_n = (\frac{d}{dr})^2 + \frac{n}{r}\frac{d}{dr}.$$

We have, also, the following recurrence relation

$$\forall \; r \in \mathbb{R}, \; \forall \; \mu \in \mathbb{R}; \;\; \frac{\partial}{\partial \mu}(j_{(n-1)/2}(\mu r)) = \frac{-\mu r^2}{n+1} j_{(n+1)/2}(\mu r). \tag{2.4}$$

In the following we shall define the Fourier transform associated with the spherical mean operator and we give some properties.

NOTATION.(see [10]) We denote by:
- $d\nu(r,x)$ the measure defined on $[0,+\infty[\times I\!R^n$ by

$$d\nu(r,x) = k_n r^n dr \otimes dx,$$

where

$$k_n = \frac{1}{2^{(n-1)/2}\Gamma((n+1)/2)(2\pi)^{n/2}}.$$

- $L^p(d\nu), 1 \le p \le +\infty$, the space of measurable functions on $[0,+\infty[\times I\!R^n$, satisfying

$$\|f\|_{p,\nu} = \Big(\int_{I\!R^n}\int_0^\infty |f(r,x)|^p d\nu(r,x)\Big)^{1/p} < +\infty, \quad 1 \le p < +\infty,$$

$$\|f\|_{\infty,\nu} = \underset{(r,x)\in[0,+\infty[\times I\!R^n}{ess\ sup} |f(r,x)| < \infty, \quad p = +\infty.$$

-$d\gamma(\mu,\lambda)$ the measure on the set Γ defined by

$$\int_\Gamma f(\mu,\lambda)d\gamma(\mu,\lambda) = k_n\{\int_{I\!R^n}\int_0^\infty f(\mu,\lambda)(\mu^2+\|\lambda\|^2)^{(n-1)/2}\mu d\mu d\lambda$$

$$+\int_{I\!R^n}\int_0^{\|\lambda\|} f(i\mu,\lambda)(\|\lambda\|^2-\mu^2)^{(n-1)/2}\mu d\mu d\lambda\}$$

- $L^p(d\gamma), 1 \le p \le +\infty$, the space of measurable functions on Γ, satisfying

$$\|f\|_{p,\gamma} = \Big(\int_\Gamma |f(\mu,\lambda)|^p d\gamma(\mu,\lambda)\Big)^{1/p} < +\infty, \quad 1 \le p < +\infty,$$

$$\|f\|_{\infty,\gamma} = \underset{(\mu,\lambda)\in\Gamma}{ess\ sup} |f(\mu,\lambda)| < \infty, \quad p = +\infty.$$

DEFINITION 2.2. The Fourier transform associated with the spherical mean operator on $L^1(d\nu)$ is defined by

$$\forall(\mu,\lambda)\in\Gamma, \ \mathcal{F}f(\mu,\lambda) = \int_{I\!R^n}\int_0^\infty f(r,x)\varphi_{\mu,\lambda}(r,x)d\nu(r,x).$$

We have the following properties:

- $$\forall(\mu,\lambda)\in\Gamma, \ (\mathcal{F}f)(\mu,\lambda) = ((B\circ\widetilde{\mathcal{F}})(f))(\mu,\lambda) \tag{2.5}$$

where, $\forall(\mu, \lambda) \in I\!\!R \times I\!\!R^n$,

$$\widetilde{\mathcal{F}}f(\mu, \lambda) = \int_{I\!\!R^n}\int_0^\infty f(r, x) j_{(n-1)/2}(r\mu) e^{-i<\lambda/x>} d\nu(r, x), \tag{2.6}$$

and

$$\forall(\mu, \lambda) \in \Gamma, \;\; Bf(\mu, \lambda) = f(\sqrt{\mu^2 + \lambda^2}, \lambda).$$

- For $f \in L^1(d\nu)$ the function $\mathcal{F}f$ is continuous on Γ and

$$\lim_{|\mu|^2+\|\lambda\|^2 \to +\infty} \mathcal{F}f(\mu, \lambda) = 0 \tag{2.7}$$

- For $f \in L^1(d\nu)$ such that $\mathcal{F}f \in L^1(d\gamma)$, we have the inversion formula for $\mathcal{F}$: for almost everywhere $(r, x) \in [0, +\infty[\times I\!\!R^n$,

$$f(r, x) = \int_\Gamma \mathcal{F}f(\mu, \lambda)\overline{\varphi_{\mu,\lambda}(r, x)} d\gamma(\mu, \lambda).$$

- For all $p \in [1, +\infty]$ and $f \in L^p(d\nu)$,

$$Bf \in L^p(d\gamma) \;\; and \;\; \|Bf\|_{p,\gamma} = \|f\|_{p,\nu}. \tag{2.8}$$

In particular, the mapping B is an isometric isomorphism from $L^2(d\nu)$ onto $L^2(d\gamma)$.

- The mapping $\widetilde{\mathcal{F}}$ is an isometric isomorphism from $L^2(d\nu)$ onto itself. Consequently, the Fourier transform $\mathcal{F}$ is an isometric isomorphism from $L^2(d\nu)$ onto $L^2(d\gamma)$.

Thus,

$$\forall f \in L^2(d\nu); \;\; \mathcal{F}f \in L^2(d\gamma), \;\; and \;\|\mathcal{F}f\|_{2,\gamma} = \|f\|_{2,\nu}. \tag{2.9}$$

Notation. We denote by:

- $S_*(I\!\!R \times I\!\!R^n)$ the space of infinitely differentiable functions on $I\!\!R \times I\!\!R^n$, even with respect to the first variable, rapidly decreasing together with all their derivatives.

- $S_*(\Gamma)$ the space of infinitely differentiable functions on Γ, even with respect to the first variable, rapidly decreasing together with all their derivatives, which means

$\forall\; k_1, k_2 \in I\!\!N, \forall\; \alpha \in I\!\!N^n$,

$$sup\{(1 + |\mu|^2 + \|\lambda\|^2)^{k_1} |(\frac{\partial}{\partial\mu})^{k_2} D_\lambda^\alpha f(\mu, \lambda)|; (\mu, \lambda) \in \Gamma\} < +\infty,$$

where

$$\frac{\partial f}{\partial \mu}(\mu,\lambda) = \begin{cases} \frac{\partial}{\partial r}(f(r,\lambda)), & \text{if } \mu = r \in I\!R, \\ \frac{1}{i}\frac{\partial}{\partial t}(f(it,\lambda)), & \text{if } \mu = it, |t| \leq \|\lambda\|. \end{cases}$$

and

$$D^{\alpha}_{\lambda} = (\frac{\partial}{\partial \lambda_1})^{\alpha_1}(\frac{\partial}{\partial \lambda_2})^{\alpha_2}...(\frac{\partial}{\partial \lambda_n})^{\alpha_n}.$$

(see [9]).

Remark 1.1. From [9], the Fourier transform $\mathcal{F}$ is a topological isomorphism from $S_*(I\!R \times I\!R^n)$ onto $S_*(\Gamma)$. The inverse mapping is given by

$$\mathcal{F}^{-1}f(r,x) = \int_{\Gamma} f(\mu,\lambda)\overline{\varphi_{\mu,\lambda}(r,x)}d\gamma(\mu,\lambda).$$

3. Fourier transform of rapidly decreasing functions

This section consists to characterize, by the Fourier transform associated with the spherical mean operator, a space of functions having only some integral conditions at infinity. This permits, in the last section, to give an other description of the space $S_*(\Gamma)$. To prove the main result of this section, we need some lemmas.

Let f be a measurable function on $I\!R \times I\!R^n$. For every $k \in \{0,...,n\}$ and $(i_0,...,i_k) \in I\!N^{k+1}$ such that $0 \leq i_0 < ... < i_k \leq n$, we put:
$f_{i_0,...,i_k}(x;y)$

$$= f(y_0,...,y_{i_0-1},x_{i_0},y_{i_0+1},...,y_{i_p-1},x_{i_p},y_{i_p+1},...,y_{i_k-1},x_{i_k},y_{i_k+1},...,y_n) \tag{3.1}$$

with $x = (x_0,...,x_n)$ and $y = (y_0,...,y_n) \in I\!R \times I\!R^n$.

Lemma 3.1. *Let $I_n = \prod_{j=0}^{n}[a_j,b_j]$ where $a_0,...,a_n,b_0,...,b_n$ are real numbers such that for every $j \in \{0,...,n\}$, $a_j < b_j$. Let f be an infinitely differentiable function on I_n and g a measurable bounded function on I_n. Then we have*

$$\int_{I_n} f(t)g(t)dt = f(b)g^n(b) + \sum_{k=0}^{n}(-1)^{k+1}(\sum_{0\leq i_0<...<i_k\leq n} \int_{\prod_{j=0}^{k}[a_{i_j},b_{i_j}]}$$

$$(\frac{\partial^{k+1}}{\partial t_{i_0}...\partial t_{i_k}} f)_{i_0,...,i_k}(t;b) g^n_{i_0,...,i_k}(t;b)\ dt_{i_0}...dt_{i_k}),$$

where

$$\forall\ k \in \{0, ...n\};\ g^k(t) = \int_{\prod\limits_{j=0}^{k}[a_j,t_j]} g(u_0, ..., u_k, t_{k+1}, ..., t_n) du_0...du_k.$$

P r o o f. By integration by parts, the result follows by induction on n. ■

In the sequel, we denote by $S(I\!\!R^k)$ the usual Schwartz's space and $L^p(I\!\!R^k, dx), 1 \leq p \leq +\infty$, the Lebesgue space on $I\!\!R^k$.

Lemma 3.2. *Let $\varphi : I\!\!R^{n+1} \times I_n \to \mathbb{C}$ be a measurable bounded function such that*

$$\lim_{\|\lambda\| \to +\infty} \int_{[\alpha_0,\beta_0]\times...\times[\alpha_n,\beta_n]} \varphi(\lambda, t) dt = 0, \tag{3.2}$$

uniformly in α_i and β_i for $a_i \leq \alpha_i \leq \beta_i \leq b_i;\ i \in \{0, ..., n\}$. Then for every integrable function f on I_n with respect to lebesgue measure, we have

$$\lim_{\|\lambda\| \to +\infty} \int_{I_n} f(t)\varphi(\lambda, t) dt = 0, \tag{3.3}$$

where I_n is defined in Lemma 3.1.

P r o o f. By using Lemma 3.1 with $g(t) = \varphi(\lambda, t)$ and according to the relation (3.2), we obtain the result for $f \in \mathcal{S}(I\!\!R^{n+1})$.

Since the function φ is bounded on $I\!\!R^{n+1} \times I_n$, we complete the proof by using the density of $\mathcal{S}(I\!\!R^{n+1})$ in $L^1(I\!\!R^{n+1}, dx)$. ■

Remark 3.1. In [12] and [7], the result of Lemma 3.2 is proved for $n = 0$.

Example 3.1. Let N be a real number such that $N \geq 1$ and

$$I_N = [0, N] \times [-N, N]^n.$$

Let φ be the function on $(I\!\!R \times I\!\!R^n) \times I_N$ defined by

$$\varphi(\mu, \lambda, r, x) = (r\mu)^{n/2} j_{(n-1)/2}(r\mu) e^{-i<\lambda,x>} 1_{[0,+\infty[}(\mu)$$

where $j_{(n-1)/2}$ is the normalized Bessel function defined by the relation (2.1).

It is well known ([8], [14]) that there exist two positive constants c_n and d_n such that
$\forall\ x \in [0, +\infty[$
$$|x^{n/2} j_{(n-1)/2}(x)| \leq c_n \tag{3.4}$$
and
$$|\int_0^x t^{n/2} j_{(n-1)/2}(t)dt| \leq d_n \tag{3.5}$$

According to the inequalities (3.4), (3.5) and by using Lemma 3.2, we deduce that for every integrable function

$$\lim_{\mu^2+\|\lambda\|^2 \to +\infty} \int_{I_N} f(r,x)\varphi(\mu,\lambda,r,x)drdx = 0. \tag{3.6}$$

In the following, we need the partial differential operators

$$\frac{\partial}{\partial \mu^2} = \frac{1}{2\mu}\frac{\partial}{\partial \mu}; \qquad K = 4(\mu^2 + \|\lambda\|^2)(\frac{\partial}{\partial \mu^2})^2 + 2(n+1)\frac{\partial}{\partial \mu^2},$$

$$L = L_n + \Delta; \quad A = K + \Sigma_{i=1}^n C_i^2,$$

where
$$C_i = \frac{\partial}{\partial \lambda_i} - 2\lambda_i \frac{\partial}{\partial \mu^2}, \quad 1 \leq i \leq n;$$

For all $f \in \mathcal{E}_*(I\!R \times I\!R^n)$, we have

- $$B(\frac{\partial}{\partial \mu^2} f) = \frac{\partial}{\partial \mu^2} Bf. \tag{3.7}$$

- $\forall\ k \in I\!N, \forall\ \alpha = (\alpha_1, ..., \alpha_n) \in I\!N^n$
$$B(L_n^k D_\lambda^\alpha f) = K^k C^\alpha Bf, \tag{3.8}$$

where $C^\alpha = C_1^{\alpha_1} ... C_n^{\alpha_n}$. and $D_\lambda^\alpha = (\frac{\partial}{\partial \lambda_1})^{\alpha_1} ... (\frac{\partial}{\partial \lambda_n})^{\alpha_n}$

- $\forall\ k \in I\!N$,
$$B(L^k f) = A^k(Bf), \tag{3.9}$$

We can now prove the main result of this section.

THEOREM 3.1. *Let f be a function in $L^2(d\nu)$. Then the following two assertions are mutually equivalent:*

1) For all $\alpha \in I\!N^n$ and $m \in I\!N$, the function

$$(r,x) \to r^m x^\alpha f(r,x)$$

belongs to $L^2(d\nu)$.

2) The Fourier transform $\mathcal{F}(f)$ of the function f satisfies the following properties:

i) The function $\mathcal{F}(f)$ is infinitely differentiable on Γ even with respect to the first variable.

ii) For all $\alpha \in I\!N^n$ and $m \in I\!N$ the function $K^m C^\alpha \mathcal{F}(f)$ belongs to $L^2(d\gamma)$.

iii) For all $\alpha \in I\!N^n$ and $m \in I\!N$

$$\lim_{|\mu|^2+\|\lambda\|^2 \to +\infty} (1 + (\mu^2 + \|\lambda\|^2)^{n/4}) K^m C^\alpha \mathcal{F}(f)(\mu,\lambda) = 0. \tag{3.10}$$

iv) For all $\alpha \in I\!N^n$ and $m \in I\!N$

$$\lim_{|\mu|^2+\|\lambda\|^2 \to +\infty} (\mu^2 + \|\lambda\|^2)^{(n+2)/4} \frac{\partial}{\partial \mu^2} K^m C^\alpha \mathcal{F}(f)(\mu,\lambda) = 0. \tag{3.11}$$

P r o o f. *Necessity.* Let f be a function in $L^2(d\nu)$ satisfying the assertion 1) of Theorem 3.1. Then, it is clear that for all $\alpha \in I\!N^n$ and $m \in I\!N$, the function

$$(r,x) \to r^m x^\alpha f(r,x)$$

belongs to $L^1(d\nu)$.

i) Using the relations (2.2) and (2.6), we deduce that the function $\widetilde{\mathcal{F}}(f)$ belongs to $\mathcal{E}_*(I\!R \times I\!R^n)$. On the other hand, it is known ([1], [9]) that if $g \in \mathcal{E}_*(I\!R \times I\!R^n)$ then, the function Bg is infinitely differentiable on Γ even with respect to the first variable. Thus from the relation (2.5), the function $\mathcal{F}(f)$ is infinitely differentiable on Γ, even with respect to the first variable.

ii) From the relations (2.3) and (2.6) we deduce that for all $\alpha \in I\!N^n$ and $m \in I\!N$ we have
$\forall (\mu,\lambda) \in I\!R \times I\!R^n$,

$$L_n^m D^\alpha \widetilde{\mathcal{F}}(f)(\mu,\lambda) = \widetilde{\mathcal{F}}((-r^2)^m (-i)^{|\alpha|} x^\alpha f(r,x))(\mu,\lambda) \tag{3.12}$$

Then, by the relations (2.5) and (3.8), we obtain

$$\forall (\mu,\lambda) \in \Gamma; \quad K^m C^\alpha \mathcal{F}(f)(\mu,\lambda) = \mathcal{F}((-r^2)^m (-i)^{|\alpha|} x^\alpha f(r,x))(\mu,\lambda). \tag{3.13}$$

Therefore, the property ii) follows from the relation (3.13) and the fact that the Fourier transform $\mathcal{F}$ is an isometric isomorphism from $L^2(d\nu)$ onto $L^2(d\gamma)$.

iii) From the relations (2.7) and (3.13), we deduce that

$$\forall \alpha \in I\!N^n,\ \forall m \in I\!N; \quad \lim_{|\mu|^2+\|\lambda\|^2 \to +\infty} K^m C^\alpha \mathcal{F}(f)(\mu,\lambda) = 0. \tag{3.14}$$

On the other hand, from the relation (3.4), it follows that for $N \geq 1$, we have
$\forall\ (\mu,\lambda) \in I\!R_+ \times I\!R^n$

$$|\int_{[0,+\infty[\times I\!R^n} \mu^{n/2} r^{2m} x^\alpha f(r,x).e^{-i<\lambda/x>} j_{(n-1)/2}(r\mu).r^n drdx|$$

$$\leq c_n \int_{I\!R_+\times I\!R^n \setminus I_N} r^{2m+n/2} |x^\alpha f(r,x)| drdx$$
$$+ |\int_{I_N} \varphi(\mu,\lambda,r,x) r^{2m+n/2} x^\alpha f(r,x) drdx|,$$

where φ is the function given in Example 3.1, and this implies by using the hypothesis, Example 3.1 and the relations (2.6) and (3.12) that

$$\lim_{\mu^2+\|\lambda\|^2 \to +\infty} \mu^{n/2} L_n^m D^\alpha \widetilde{\mathcal{F}}(f)(\mu,\lambda) = 0. \tag{3.15}$$

Thus from the relations (2.5) and (3.8) we obtain

$$\lim_{|\mu|^2+\|\lambda\|^2 \to +\infty} (\mu^2 + \|\lambda\|^2)^{n/4} K^m C^\alpha \mathcal{F}(f)(\mu,\lambda) = 0. \tag{3.16}$$

Therefore, iii) follows from the relations (3.14) and (3.16).

iv) For $(\mu,\lambda) \in I\!R_+ \times I\!R^n$, and from the relations (2.4), (2.6) and (3.12) we obtain

$$\mu^{n/2} \frac{\partial}{\partial \mu} L_n^m D^\alpha \widetilde{\mathcal{F}}(f)(\mu,\lambda)$$

$$= \frac{\mu^{(n+2)/2}}{n+1} \int_{I\!R_+\times I\!R^n} (-r^2)^{m+1} (-i)^{|\alpha|} x^\alpha f(r,x)\, e^{-i<\lambda/x>} j_{(n+1)/2}(r\mu)\, d\nu(r,x).$$

By the same way as in iii), we deduce that for all $\alpha \in I\!N^n$ and $m \in I\!N$

$$\lim_{\mu^2+\|\lambda\|^2 \to +\infty} \mu^{(n+2)/2} \frac{\partial}{\partial \mu^2} L_n^m D^\alpha \widetilde{\mathcal{F}}(f)(\mu,\lambda) = 0.$$

Thus, iv) follows from the fact that
$\forall\ (\mu,\lambda)\in\Gamma$,

$$B(\mu^{(n+2)/2}\frac{\partial}{\partial\mu^2}L_n^m D^\alpha\widetilde{\mathcal{F}}(f))(\mu,\lambda)=(\mu^2+\|\lambda\|^2)^{(n+2)/4}\frac{\partial}{\partial\mu^2}K^m C^\alpha\mathcal{F}(f)(\mu,\lambda).$$

Sufficiency. Suppose now that the function f satisfies the assertion 2) of Theorem 3.1. Then from the property ii), we deduce that for all $\alpha\in\mathbb{N}^n$ and $m\in\mathbb{N}$ the function $K^m C^\alpha\mathcal{F}(f)$ belongs to $L^2(d\gamma)$. And this implies, by using the relations (2.5), (2.8) and (3.8) that the function $L_n^m D^\alpha\widetilde{\mathcal{F}}(f)$ belongs to $L^2(d\nu)$.

Therefore, for all $j\in\{1,...,n\}$ and $m\in\mathbb{N}$ there exists a null set $N_{j,m}\subset\mathbb{R}_+\times\mathbb{R}^{n-1}$ such that for every $(\mu,z)\in N_{j,m}^c$, the function defined on $\mathbb{R}$ by

$$f_{j,m,\mu,z}(t)=(\frac{\partial}{\partial\lambda_j})^m\widetilde{\mathcal{F}}(f)(\mu,z_1,...,z_{j-1},t,z_j,...,z_{n-1})$$

belongs to $L^2(\mathbb{R},dt)$.

And for all $m\in\mathbb{N}$ there exists a null set $M_m\subset\mathbb{R}^n$ such that for every $\lambda\in M_m^c$, the function $L_n^m\widetilde{\mathcal{F}}(f)(.,\lambda)$ belongs to $L^2(\mathbb{R}_+,t^n dt)$(the space of square integrable functions on $\mathbb{R}_+$ with respect to the measure $t^n dt$).

We introduce now, for $m\in\mathbb{N}$ and $j\in\{1,...,n\}$, the following sequences of functions:

- For $(\mu,z)\in\mathbb{R}_+\times\mathbb{R}^{n-1}$,

$$g_{j,m,\mu,z}^N(y)=\frac{1}{\sqrt{2\pi}}\int_{-N}^N f_{j,m,\mu,z}(t)e^{ity}dt.$$

- For $\lambda\in\mathbb{R}^n$,

$$h_{m,\lambda}^N(y)=\frac{1}{2^{(n-1)/2}\Gamma((n+1)/2)}\int_0^N L_n^m\widetilde{\mathcal{F}}(f)(t,\lambda)j_{(n-1)/2}(ty)t^n dt.$$

Then, for all $(\mu,z)\in N_{j,m}^c$ the sequence $(g_{j,m,\mu,z}^N)_N$ converges in $L^2(\mathbb{R},dt)$ to

$$g_{j,m,\mu,z}=\wedge_1^{-1}(f_{j,m,\mu,z}) \tag{3.17}$$

and for all $\lambda\in M_m^c$, the sequence $(h_{m,\lambda}^N)_N$ converges in $L^2(\mathbb{R}_+,t^n dt)$ to

$$h_{m,\lambda}=\mathcal{F}_B(L_n^m\widetilde{\mathcal{F}}(f)(.,\lambda)), \tag{3.18}$$

where $\wedge_1^{-1}$ is the inverse of the usual Fourier transform $\wedge_1$ on $I\!R$ defined by

$$\wedge_1(f)(y) = \lim_{N\to+\infty} \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} f(t)e^{-ity}dt,$$

in $L^2(I\!R, dt)$, and $\mathcal{F}_B$ is the Fourier-Bessel transform defined by

$$\mathcal{F}_B(f)(y) = \lim_{N\to+\infty} \frac{1}{2^{(n-1)/2}\Gamma((n+1)/2)} \int_0^N f(t)j_{(n-1)/2}(ty)t^n dt,$$

in $L^2(I\!R_+, t^n dt)$.

Now by integration by parts we obtain, for $m \in I\!N^*$:

• $\forall(\mu, z) \in I\!R_+ \times I\!R^{n-1}$,

$$g^N_{j,m,\mu,z}(y) = \frac{1}{\sqrt{2\pi}}[e^{ity}f_{j,m-1,\mu,z}(t)]^N_{-N} - iyg^N_{j,m-1,\mu,z}(y) \tag{3.19}$$

• $\forall\, \lambda \in I\!R^n$,

$$h^N_{m,\lambda}(y) = \frac{1}{2^{(n-1)/2}\Gamma((n+1)/2)}\{[j_{(n-1)/2}(ty)t^n \frac{\partial}{\partial t}(L_n^{m-1}\widetilde{\mathcal{F}}(f))(t,\lambda)]_0^N$$

$$-[\frac{-t^{n+1}y^2}{n+1}j_{(n+1)/2}(ty)(L_n^{m-1}\widetilde{\mathcal{F}}(f))(t,\lambda)]_0^N\} - y^2h^N_{m-1,\lambda}(y). \tag{3.20}$$

From the relations (2.5), (3.8) and the hypothesis iii), we deduce that for all $m \in I\!N^*$ and $j \in \{1, ..., n\}$, we have
$\forall\ (\mu, z) \in I\!R_+ \times I\!R^{n-1}$

$$\lim_{N\to+\infty}[e^{ity}f_{j,m-1,\mu,z}(t)]^N_{-N} = 0. \tag{3.21}$$

Then, from the relation (3.17),(3.19) and (3.21), we have

$$\forall\ (\mu, z) \in \bigcap_{l=0}^{k} N^c_{j,l}, \quad g_{j,k,\mu,z}(.) = (-it)^k g_{j,0,\mu,z}(.), \tag{3.22}$$

in $L^2(I\!R, dt)$.

Using the relation (3.17) and the Plancherel formula for the Fourier transform $\wedge$, we obtain for all $j \in \{1, ..., n\}$ and $k \in I\!N$;

$$\forall\ (\mu, z) \in \bigcap_{l=0}^{k} N^c_{j,l};\quad \int_{I\!R} |g_{j,k,\mu,z}(t)|^2dt = \int_{I\!R} |f_{j,k,\mu,z}(t)|^2dt.$$

Integrating over $[0,+\infty[\times I\!R^{n-1}$, with respect to the measure $\mu^n d\mu dz$, and using Fubini-Tonelli theorem, we obtain by virtue of the relation (3.22) that

$$\int_{I\!R_+\times I\!R^n} |(\frac{\partial}{\partial\lambda_j})^k \widetilde{\mathcal{F}} f(\mu,\lambda)|^2 \mu^n d\mu d\lambda$$
$$= \int_{I\!R} t^{2k} (\int_{I\!R_+\times I\!R^{n-1}} |g_{j,0,\mu,z}(t)|^2 \mu^n d\mu dz) dt. \qquad (3.23)$$

Let us now define the Fourier-transform $\widetilde{\mathcal{F}}_{n,n-1}$, on $[0,+\infty[\times I\!R^{n-1}$ by

$$\widetilde{\mathcal{F}}_{n,n-1}(g)(\mu,z) = k_{n,n-1} \int_{I\!R^{n-1}} \int_0^\infty g(r,x) j_{(n-1)/2}(r\mu) e^{-i<z/x>} r^n dr dx,$$

where

$$k_{n,n-1} = \frac{1}{2^{(n-1)/2}\Gamma((n+1)/2)(2\pi)^{(n-1)/2}}.$$

Then, the transform $\widetilde{\mathcal{F}}_{n,n-1}$ can be extended to an isometric isomorphism from $L^2([0,+\infty[\times I\!R^{n-1}, k_{n,n-1} r^n dr dx)$ (the space of square integrable functions on $[0,+\infty[\times I\!R^{n-1}$, with respect to the measure $k_{n,n-1}\ r^n$ $dr dx$) onto itself, and we have, for almost everywhere t

$$g_{j,0,\mu,z}(t) = \widetilde{\mathcal{F}}_{n,n-1}(f(...,\underset{\substack{\downarrow \\ j^{th} place}}{t},...))(\mu,z)\ .$$

Consequently, for almost everywhere t,

$$\int_{[0,+\infty[\times I\!R^{n-1}} |g_{j,0,\mu,z}(t)|^2 \mu^n d\mu dz$$
$$= \int_{[0,+\infty[\times I\!R^{n-1}} |f(\mu, z_1, ..., z_{j-1}, t, z_j, ..., z_{n-1})|^2 \mu^n d\mu dz. \qquad (3.24)$$

From the relations (3.23) and (3.24), we obtain

$$\int_{I\!R_+\times I\!R^n} |(\frac{\partial}{\partial\lambda_j})^k \widetilde{\mathcal{F}} f(\mu,\lambda)|^2 d\nu(\mu,\lambda) = \int_{I\!R_+\times I\!R^n} |x_j^k f(r,x)|^2 d\nu(r,x)$$

and by using the relations (2.5), (2.8) and (3.8), we deduce from the hypothesis ii) that for all $k\in I\!N$ and $j\in\{1,...,n\}$, the integral

$$\int_{I\!R_+\times I\!R^n} |x_j^k f(r,x)|^2 d\nu(r,x)$$

is finite. So, for all $\alpha \in \mathbb{N}^n$,

$$\int_{\mathbb{R}_+\times\mathbb{R}^n} |x^\alpha f(r,x)|^2 d\nu(r,x) < +\infty. \tag{3.25}$$

In the following, we will prove that for all $k \in \mathbb{N}$, the integral

$$\int_{\mathbb{R}_+\times\mathbb{R}^n} |r^k f(r,x)|^2 d\nu(r,x)$$

is finite.

From the relations (2.5), (3.8) and the hypothesis iii), we deduce that for all $m \in \mathbb{N}^*$,

$$\lim_{\mu^2+\|\lambda\|^2\to+\infty} \mu^{n/2} L_n^{m-1}\widetilde{\mathcal{F}}f(\mu,\lambda) = 0. \tag{3.26}$$

On the other hand, for all $y \in \mathbb{R}_+^*$, the relation (3.4) implies that $\forall\, m \in \mathbb{N}^*,\ \forall\,(\mu,\lambda) \in \mathbb{R}_+ \times \mathbb{R}^n$;

$$|\mu^{n+1}y^2 j_{(n+1)/2}(\mu y) L_n^{m-1}\widetilde{\mathcal{F}}f(\mu,\lambda)| \leq \frac{c_n}{y^{(n-2)/2}}|\mu^{n/2}L_n^{m-1}\widetilde{\mathcal{F}}f(\mu,\lambda)|.$$

Then, by the relation (3.26), we obtain for $y \in \mathbb{R}_+^*$, $\forall\, m \in \mathbb{N}^*,\ \forall\,\lambda \in \mathbb{R}^n$;

$$\lim_{N\to+\infty}[\frac{-t^{n+1}y^2}{n+1} j_{(n+1)/2}(ty)(L_n^{m-1}\widetilde{\mathcal{F}}f)(t,\lambda)]_{t=0}^{t=N} = 0. \tag{3.27}$$

Moreover, from the relations (2.5), (3.7), (3.8) and the hypothesis iv), we deduce that for all $m \in \mathbb{N}^*$,

$$\lim_{\mu^2+\|\lambda\|^2\to+\infty} \mu^{n/2}\frac{\partial}{\partial\mu} L_n^{m-1}\widetilde{\mathcal{F}}f(\mu,\lambda) = 0. \tag{3.28}$$

Now, from the relation (3.4), we have for all $y \in \mathbb{R}_+^*$;

$$|\mu^n j_{(n-1)/2}(\mu y)\frac{\partial}{\partial\mu} L_n^{m-1}\widetilde{\mathcal{F}}f(\mu,\lambda)| \leq \frac{c_n}{y^{n/2}}|\mu^{n/2}\frac{\partial}{\partial\mu}L_n^{m-1}\widetilde{\mathcal{F}}f(\mu,\lambda)|.$$

Then, by the relation (3.28), we get for all $m \in \mathbb{N}^*$, $y \in \mathbb{R}_+^*$ and $\lambda \in \mathbb{R}^n$;

$$\lim_{N\to+\infty}[j_{(n-1)/2}(ty)t^n\frac{\partial}{\partial t} L_n^{m-1}\widetilde{\mathcal{F}}f(t,\lambda)]_{t=0}^{t=N} = 0. \tag{3.29}$$

By the same way as the proof of the relation (3.25) and using the relations (3.20), (3.27) and (3.29), we deduce that for all $k \in \mathbb{N}$,

$$\int_{\mathbb{R}_+ \times \mathbb{R}^n} |r^k f(r,x)|^2 d\nu(r,x) < +\infty. \tag{3.30}$$

Thus, by the relations (3.25), (3.30) and the Cauchy-Schwarz inequality, it follows that for all $k \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$, the function

$$(r,x) \to r^k x^\alpha f(r,x)$$

belongs to $L^2(d\nu)$. This completes the proof of Theorem 3.1. ∎

4. Other characterizations of the spaces $S_*(\mathbb{R}^{n+1})$ and $S_*(\Gamma)$

In this section, we will give an other characterization of the space $S_*(\mathbb{R}^{n+1})$ which together with Theorem 3.1, permit to obtain a new description of the space $S_*(\Gamma)$.

LEMMA 4.1. *Let $m \in \mathbb{N}^*$. For all infinitely differentiable function f on $\mathbb{R}^m$, we have*

$$\int_{\prod_{j=1}^m [a_j,b_j]} \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_m} f(x)dx = f(b) + \sum_{k=1}^m (-1)^k [\sum_{1 \le i_1 < \ldots < i_k \le m} f_{i_1,\ldots,i_k}(a;b)] \tag{4.1}$$

where $a_1, \ldots, a_m, b_1, \ldots, b_m$ are real numbers such that, for all $j \in \{1, \ldots, m\}$; $a_j < b_j$.

P r o o f. The result follows by induction on m. ∎

PROPOSITION 4.1. *Let f be a continuous function on $\mathbb{R}^m$ and $f \in L^2(\mathbb{R}^m, dx)$. Then, the following assertions are equivalent:*

1) For all $\alpha \in \mathbb{N}^m$, the functions

$$x \to x^\alpha f(x) \quad \text{and} \quad x \to x^\alpha \wedge_m f(x)$$

belong to $L^2(\mathbb{R}^m, dx)$.

2) For all $\alpha \in \mathbb{N}^m$, the functions

$$x \to x^\alpha f(x) \quad \text{and} \quad x \to x^\alpha \wedge_m f(x)$$

are bounded on $\mathbb{R}^m$.

Here $\wedge_m$ is the Fourier-transform on $I\!R^m$ defined by

$$\wedge_m(f)(x) = \frac{1}{(2\pi)^{m/2}} \int_{I\!R^m} f(t) e^{-i<x,t>} dt.$$

P r o o f. *Necessity.* Let f be a continuous function on $I\!R^m$ such that $f \in L^2(I\!R^m, dx)$ and satisfying the assertion 1) of Proposition 4.1. Then, f is infinitely differentiable on $I\!R^m$ and all its derivatives are bounded on $I\!R^m$.

For all $\alpha \in I\!N^m$, we have

$$\frac{\partial}{\partial y_1} \cdots \frac{\partial}{\partial y_m} [(y^\alpha f(y))^{m+1}] = [\sum_{finite} c_{\gamma,\delta}\, y^\gamma \prod_{i=1}^{m} D^{\delta_i} f(y)] f(y), \tag{4.2}$$

where $\gamma \in I\!N^m$, $\delta = (\delta_1, ..., \delta_m) \in I\!N^m \times ... \times I\!N^m$ and $c_{\gamma,\delta}$ is a positive constant.

Let $p \in \{0, ..., m-1\}$ and $(i_1, ..., i_{m-p}) \in \{1, ..., m\}^{m-p}$, $1 \leq i_1 < ... < i_{m-p} \leq m$ such that for $\alpha = (\alpha_1, ..., \alpha_m) \in I\!N^m$, $\prod_{j=i_1,...,i_{m-p}} \alpha_j \neq 0$ and $\alpha_j = 0$ if $j \neq i_1, ..., i_{m-p}$.

By integrating the equality (4.2) over $(\prod_{j=i_1}^{i_{m-p}} [0, x_j]) \times (\prod_{j \neq i_1,...,i_{m-p}} [1, x_j])$ and according to Lemma 4.1 one can show by induction on p that

$$\sup_{x \in I\!R^m} |(\prod_{j=i_1,...,i_{m-p}} x_j^{\alpha_j}) f(x)| < +\infty,$$

Thus, for all $\alpha \in I\!N^m$, the function

$$x \to x^\alpha f(x)$$

is bounded on $I\!R^m$ and by the same argument, we prove that for all $\alpha \in I\!N^m$ the function

$$x \to x^\alpha \wedge_m f(x)$$

is bounded on $I\!R^m$. The implication 2) $\to$ 1) is clear. ∎

Proposition 4.2. *Let f be a continuous function on $I\!R \times I\!R^n$, even with respect to the first variable and $f \in L^2(d\nu)$. Then, the following assertions are mutually equivalent:*

1) For all $\alpha \in I\!N^n$ and $k \in I\!N$, the functions

$$(r, x) \to r^k x^\alpha f(r, x) \quad \text{and} \quad (\mu, \lambda) \to \mu^k \lambda^\alpha \widetilde{\mathcal{F}}(f)(\mu, \lambda)$$

are bounded on $I\!R_+ \times I\!R^n$.

2) For all $\alpha \in I\!N^n$ *and* $k \in I\!N$

$$\sup_{(r,x)\in I\!R_+\times I\!R^n} |r^k x^\alpha f(r,x)| < +\infty$$

the function f *is infinitely differentiable on* $I\!R^{n+1}$ *and all its partial derivatives are bounded.*

3) The function f *belongs to the space* $\mathcal{S}_*(I\!R^{n+1})$.

4) For all $\alpha \in I\!N^n$ *and* $k \in I\!N$ *the functions*

$$(r,x) \to r^k x^\alpha f(r,x) \qquad \text{and} \qquad (\mu,\lambda) \to \mu^k \lambda^\alpha \widetilde{\mathcal{F}}(f)(\mu,\lambda)$$

belong to $L^2(d\nu)$.

P r o o f.

1) $\Rightarrow$ 2) From the hypothesis 1), for all $k \in I\!N$ and $\alpha \in I\!N^n$; the function

$$(\mu,\lambda) \to \mu^k \lambda^\alpha \widetilde{\mathcal{F}}(f)(\mu,\lambda)$$

belongs to $L^1(d\nu)$. Then, by the inversion formula for the transform $\widetilde{\mathcal{F}}$ $\forall\,(r,x) \in I\!R \times I\!R^n$

$$f(r,x) = \int_{[0,+\infty[\times I\!R^n} \widetilde{\mathcal{F}}(f)(\mu,\lambda)\, e^{i<x/\lambda>} j_{(n-1)/2}(r\mu) d\nu(\mu,\lambda).$$

And using the relation (2.2), we deduce that the function f is infinitely differentiable on $I\!R \times I\!R^n$, even with respect the first variable and all its derivatives are bounded.

2) $\Rightarrow$ 3) For all $j \in \{0,...,n\}$ and for $k \in I\!N$ and $\alpha \in I\!N^n$ one can show that the function

$$y \to \int_0^{y_j} |u^k(y_0,...,y_{j-1},y_{j+1},...,y_n)^\alpha D_j f(y_0,...,y_{j-1},u,y_{j+1},...,y_n)|^2 du$$

is bounded on $I\!R^{n+1}$ and this leads us to see that the function

$$(r,x) \to r^k x^\alpha D_j f(r,x)$$

is bounded on $I\!R \times I\!R^n$.

Consequently, for all $j \in \{0,...,n\}$, the function $D_j f$ satisfies the same hypothesis as f. Hence, we deduce that for all $k \in I\!N$ and $\alpha \in I\!N^n$, the function $D_0^k D^\alpha f$ is rapidly decreasing.

3) $\Rightarrow$ 4) We know that the transform $\widetilde{\mathcal{F}}$ is an isomorphism from $S_*(I\!R \times I\!R^n)$ onto itself. Then, the assertion 4) follows from the fact that for all $k \in I\!N$ and $\alpha \in I\!N^n$, the mapping

$$f \rightarrow r^k x^\alpha f$$

is continuous from $S(I\!R \times I\!R^n)$ onto itself.

4) $\Rightarrow$ 1) Let $g : I\!R^{n+1} \times I\!R^n \rightarrow \mathbb{C}$ be the function defined by $g(t,x) = f(\|t\|, x)$. According to the relation

$$\widetilde{\mathcal{F}}f(\mu, \lambda) = \wedge_{2n+1} g(s, \lambda)$$

with $s \in I\!R^{n+1}$; $\|s\| = \mu$, then the result follows from Proposition 4.1. ∎

REMARK 4.1. By the same way, as in the proof of Proposition 4.2, we show that a continuous function f satisfies the assertion 2) of Proposition 4.1 if and only if the function f belongs to the space $S(I\!R^m)$.

In [3], it is proved that a continuous function f belongs to the space $S_*(I\!R)$ (the subspace of $S(I\!R)$ consisting of even functions), if and only if for all $k \in I\!N$, the functions

$$x \rightarrow x^k f(x) \quad \text{and} \quad x \rightarrow x^k \mathcal{F}_B f(x)$$

are bounded on $I\!R_+$.

Here $\mathcal{F}_B$ is the Fourier-Bessel transform defined by

$$\mathcal{F}_B(f)(x) = \frac{1}{2^{(n-1)/2}\Gamma((n+1)/2)} \int_0^{+\infty} f(t) j_{(n-1)/2}(tx) t^n dt,$$

with $f \in L^1(I\!R_+, t^n dt)$.

COROLLARY 4.1. *Let f be a continuous function on Γ, and $f \in L^2(d\gamma)$, even with respect to the first variable. Then the following assertions are mutually equivalent:*

a) For all $\alpha \in I\!N^n$ and $k \in I\!N$

$$\sup_{(z,\lambda)\in\Gamma} |(z^2 + \|\lambda\|^2)^{m/2} \lambda^\alpha f(z,\lambda)| < +\infty \quad \textit{and}$$

$$\sup_{(r,x)\in I\!R_+ \times I\!R^n} |r^m x^\alpha \mathcal{F}^{-1}(f)(r,x)| < +\infty.$$

b) The function f belongs to $\mathcal{S}_*(\Gamma)$.

c) For all $\alpha \in \mathbb{N}^n$ and $m \in \mathbb{N}$ the function

$$(\mu, \lambda) \to (\mu^2 + \|\lambda\|^2)^{m/2} \lambda^\alpha f(\mu, \lambda)$$

belongs to $L^2(d\gamma)$.

- *The function f satisfies the properties i), ii), iii) and iv) of the assertion 2) of Theorem 3.1.*

P r o o f. Since the Fourier-transform is an isomorphism from $S_*(\mathbb{R} \times \mathbb{R}^n)$ onto $S_*(\Gamma)$, there exists a function $h \in S_*(\mathbb{R} \times \mathbb{R}^n)$ such that $g = \mathcal{F}^{-1}(f)$. Therefore by using the relations (2.5) and (2.8), the result follows from Proposition 4.2 and Theorem 3.1. ∎

5. Fourier transform of functions with bounded support

In this section, we prove some results characterizing some spaces of functions with bounded support. From these characterizations we deduce a Paley-Wiener and Paley-Wiener-Schwartz theorems for the Fourier transform associated with the spherical mean operator.

We recall that for a measurable function f on $\mathbb{R} \times \mathbb{R}^n$, supp$(f)$ is the smallest closed set, outside it the function vanishes almost everywhere [15].

Using similar techniques as in [13], we prove the following theorem describing the range of square integrable functions with bounded support.

THEOREM 5.1. *(Paley-Wiener) Let f be a function in $L^2(d\nu)$.*

1) If the function f has a bounded support, then f satisfies the assertion 2) of Theorem 3.1 and moreover, the sequence $(\|A^k \mathcal{F}(f)\|_{2,\gamma}^{1/2k})_k$ converges to $\sigma_{\mathcal{F}(f)}$.

2) Conversely, if the function f satisfies the assertion 2) of Theorem 3.1 and the sequence $(\|A^k \mathcal{F}(f)\|_{2,\gamma}^{1/2k})_k$ admits a finite limit σ, then the function f has a bounded support and $\sigma = \sigma_{\mathcal{F}(f)}$, where

$$\sigma_{\mathcal{F}(f)} = max\{\|y\| : y \in suppf\}.$$

Now, we will prove a second result characterizing the space of infinitely differentiable functions with bounded support.

NOTATION. Let $m \in \mathbb{N}^*$ and $\delta > 0$, we denote by

•$\mathcal{H}^\delta(\mathbb{C}^m)$ the space of entire functions $g : \mathbb{C}^m \to \mathbb{C}$, slowly increasing of exponential type, i.e.: there exists a positive integer k such that

$$\sup_{(\lambda_1,\ldots,\lambda_m) \in \mathbb{C}^m} \{(1 + |\lambda_1|^2 + \ldots + |\lambda_m|^2)^{-k}$$

$$\times|f(\lambda_1,...,\lambda_m)|e^{-\delta(|Im\lambda_1|+...+|Im\lambda_m|)}\} < \infty.$$

• $\mathcal{E}'_\delta(I\!R^m)$ the space of distributions T on $I\!R^m$ such that supp T$\subset \{x \in I\!R^m; \|x\| \le \delta\}$.

• $S'(I\!R^m)$ the space of tempered distributions T on $I\!R^m$.

• $\mathcal{E}(I\!R^m)$ the space of infinitely differentiable functions on $I\!R^m$.

• $\mathcal{D}_*(I\!R \times I\!R^n)$ the subspace of $\mathcal{E}(I\!R^{n+1})$ consisting of functions, even with respect to the first variable, with bounded support.

• For all $f \in \mathcal{H}^\delta(\mathbb{C}^{n+1})$ we put

- $\delta_f = sup\{\|y\|, y \in supp \wedge_{n+1}^{-1} f\}$.

- $\delta_{f,i} = sup\{|t|, t \in P_i(supp \wedge_{n+1}^{-1} f)\}; \quad i \in \{0,...,n\}$,

where for $m \in I\!N^*$, $\wedge_m$ is the usual Fourier transform on $I\!R^m$ defined in Proposition 4.1 and $P_i((y_0,...,y_{n+1})) = y_i$.

We design by $\widetilde{\gamma}$ the measure on Γ defined by

$$d\widetilde{\gamma}(\mu,\lambda) = (1/k_n)\frac{d\gamma(\mu,\lambda)}{(\mu^2+\|\lambda\|^2)^{n/2}}.$$

• $L^p(d\widetilde{\gamma})$, $1 \le p \le \infty$, the space of measurable functions on Γ satisfying

$$\|f\|_{p,\widetilde{\gamma}} = \left(\int_\Gamma |f(\mu,\lambda)|^p d\widetilde{\gamma}\right)^{1/p} < +\infty, \quad 1 \le p < +\infty,$$

$$\|f\|_{\infty,\widetilde{\gamma}} = \underset{(\mu,\lambda)\in\Gamma}{ess\ sup}\ |f(\mu,\lambda)| < \infty, \quad p = +\infty.$$

Remark 5.1.

• The Fourier transform $\wedge_m$ is a bijection from $\mathcal{E}'_\delta(I\!R^m)$ onto $\mathcal{H}^\delta(\mathbb{C}^m)$(Paley-Wiener theorem) and from $S'(I\!R^m)$ onto itself.

• For every $f \in S_*(I\!R \times I\!R^n)$ we have

$$\|E^k C^\alpha B f\|_{p,\widetilde{\gamma}} = \|(\frac{\partial}{\partial x_0})^k D^\alpha f\|_{p,I\!R^{n+1}}, \tag{5.1}$$

where $E = 2(\mu^2+\|\lambda\|^2)^{1/2}\dfrac{\partial}{\partial\mu^2}$ and C^α is defined in the relation (3.8).

Using the Bernstein's inequality and the theorem of Kolmogoroff (see [11]), we obtain the following results.

Proposition 5.1. *Let $p \in [1,\infty]$ and $i \in \{0,...,n\}$. Then for all $f \in \mathcal{H}^\delta(\mathbb{C}^{n+1}) \bigcap L^p(I\!R^{n+1}, dx)$ we have*

$$\|\frac{\partial}{\partial x_i} f\|_{p,I\!R^{n+1}} \le \delta_{f,i}\|f\|_{p,I\!R^{n+1}}.$$

PROPOSITION 5.2. *Let* $p \in [1, \infty]$ *and* $f \in \mathcal{E}(\mathbb{R}^{n+1})$ *such that for all* $\alpha \in \mathbb{N}^{n+1}$

$$D^\alpha f \in L^p(\mathbb{R}^{n+1}, dx).$$

Then for all $i \in \{0, ..., n\}$ *and* $r \in \mathbb{N}^*$ *we have*

$$\|(\frac{\partial}{\partial x_i})^k f\|^r_{p,\mathbb{R}^{n+1}} \leq (\pi/2)^r \|f\|^{r-k}_{p,\mathbb{R}^{n+1}} \|(\frac{\partial}{\partial x_i})^r f\|^k_{p,\mathbb{R}^{n+1}}$$

with $k \in \mathbb{N}$, $0 < k < r$.

REMARK 5.2. From Proposition 5.2 it follows that for every $f \in \mathcal{E}(\mathbb{R}^{n+1})$ satisfying for all $\alpha \in \mathbb{N}^{n+1}$,

$$D^\alpha f \in L^p(\mathbb{R}^{n+1}, dx),$$

there always exists the limit of the sequences $\left(\|(\frac{\partial}{\partial x_i})^k f\|^{1/k}_{p,\mathbb{R}^{n+1}}\right)_k$, $i \in \{0, ..., n\}$.

THEOREM 5.2. *Let* $p \in [1, \infty]$ *and* f *be a function satisfying the hypothesis of Proposition 5.2.*

1) If $\delta_f < \infty$ *then for all* $i \in \{0, ..., n\}$ *the sequence* $\left(\|(\frac{\partial}{\partial x_i})^k f\|^{1/k}_{p,\mathbb{R}^{n+1}}\right)_k$ *converges to* $\delta_{f,i}$.

2) If there exists $M > 0$ *such that for all* $\alpha \in \mathbb{N}^{n+1}$

$$\|D^\alpha f\|_{p,\mathbb{R}^{n+1}} \leq M^{|\alpha|}$$

then $\delta_f < \infty$, *and for all* $i \in \{0, ..., n\}$ *the sequence* $\left(\|(\frac{\partial}{\partial x_i})^k f\|^{1/k}_{p,\mathbb{R}^{n+1}}\right)_k$ *converges to* $\delta_{f,i}$.

P r o o f. Without loss of generality we may assume that $i = 0$ and using the fact that the function f belongs to the space $L^p(\mathbb{R}^{n+1}, dx)$ it follows that $f \in S'(\mathbb{R}^{n+1})$.

1) Assume that $\delta_f < \infty$. Then, from the Paley-Wiener theorem for the Fourier transform $\wedge_{n+1}$ we deduce that $f \in \mathcal{H}^{\delta_f}(\mathbb{C}^{n+1})$. According to Proposition 5.1 and using the same reasoning as in [2], we obtain the result of the assertion 1).

2) Assume that there exists $M > 0$ such that for all $\alpha \in \mathbb{N}^{n+1}$

$$\|D^\alpha f\|_{p,\mathbb{R}^{n+1}} \leq M^{|\alpha|}.$$

If $p = \infty$, then it follows that $f \in \mathcal{H}^M(\mathbb{C}^{n+1})$ and from the Paley-Wiener theorem for the Fourier transform $\wedge_{n+1}$ we deduce that *supp* $\wedge^{-1}_{n+1}(f) \subset \{x \in \mathbb{R}^{n+1}, \|x\| \leq M\}$.

For $p \in [1, +\infty[$, let $\varphi \in S(I\!R^{n+1})$ such that

$$\begin{cases} 0 \leq \varphi \leq 1, \\ supp\,\varphi \subset \{x \in I\!R^{n+1},\ \|x\| \leq 1\}\ . \end{cases}$$

For all $m \in I\!N^*$, we put

$$\varphi_m(x) = m^{n+1}\varphi(mx) \ \text{ and } \ f_m(x) = \int_{I\!R^{n+1}} f(x+t)\varphi_m(t)dt,$$

and we have

$$\forall\, \alpha \in I\!N^{n+1}, \quad \|D^\alpha f_m\|_{p, I\!R^{n+1}} \leq M^{|\alpha|}.$$

Therefore $f_m \in \mathcal{H}^M(\mathbb{C}^{n+1})$ and this implies that $\delta_{f_m} < \infty$. Hence, by proceeding as in the proof of Theorem 1 in [2], it follows that for all $i \in \{0, ..., n\}$

$$\delta_{f,i} \leq M.$$

Thus, for all $p \in [1, +\infty]$, $\delta_f < \infty$ and from 1) we deduce that for all $i \in \{0, ..., n\}$ the sequence $\left(\|(\frac{\partial}{\partial x_i})^k f\|^{1/k}_{p, I\!R^{n+1}}\right)_k$ converges to $\delta_{f,i}$. ■

Lemma 5.1. *(see[9]) The mapping $W_{(n-1)/2}$ on $D_*(I\!R \times I\!R^n)$ defined by*

$$W_{(n-1)/2}g(r,x) = \frac{2\Gamma((n+1)/2)}{\sqrt{\pi}\Gamma(n/2)} \int_r^{+\infty} (t^2 - r^2)^{(n-2)/2} f(t,x)tdt$$

is a topological isomorphism from $D_*(I\!R \times I\!R^n)$ onto itself. Moreover, for all $g \in D_*(I\!R \times I\!R^n)$

$$sup\{|t|, t \in P_i(suppW_{(n-1)/2}(g))\} = sup\{|t|, t \in P_i(suppg)\}$$

$i \in \{0, ..., n\}$.

Corollary 5.1. *Let f be a function in $S_*(I\!R \times I\!R^n)$. Then the function $\widetilde{\mathcal{F}}^{-1} f$ belongs to the space $\mathcal{D}_*(I\!R \times I\!R^n)$ if and only if there exist $M > 0$ and $p \in [1, \infty]$ such that for all $\alpha \in I\!N^{n+1}$*

$$\|D^\alpha f\|_{p, I\!R^{n+1}} \leq M^{|\alpha|+1}$$

and moreover, for all $i \in \{0, ..., n\}$, the sequence $\left(\|(\frac{\partial}{\partial x_i})^k f\|^{1/k}_{p, I\!R^{n+1}}\right)_k$ converges to $\delta_{f,i}$.

P r o o f. Since the transform $\widetilde{\mathcal{F}}$ satisfies the relation (see [9])

$$\widetilde{\mathcal{F}} = \frac{\sqrt{\pi}}{2^{n/2}\Gamma(n+1)/2} \wedge_{n+1} oW_{(n-1)/2}, \tag{5.2}$$

Then, by virtue of Lemma 5.1, the result follows from Proposition 5.1 and Theorem 5.2). ■

THEOREM 5.3. *(Paley -Wiener-Schwartz) Let f be a function in $S_*(\Gamma)$. Then $\mathcal{F}^{-1}f$ belongs to the space $\mathcal{D}_*(I\!R \times I\!R^n)$ if and only if there exists $M > 0$ and $p \in [1, \infty]$ such that for all $(k, \alpha) \in I\!N \times I\!N^n$,*

$$\|E^k C^\alpha f\|_{p,\widetilde{\gamma}} \leq M^{k+|\alpha|+1},$$

and moreover the sequences $\left(\|E^k f\|_{p,\widetilde{\gamma}}^{1/k}\right)_k$ and $\left(\|C_i^k f\|_{p,\widetilde{\gamma}}^{1/k}\right)_k$, $i \in \{1, ..., n\}$, converge respectively to $\sigma_{f,0}$ and $\sigma_{f,i}$. Here,

$$\sigma_{f,i} = sup\{|t|, t \in P_i(supp\mathcal{F}^{-1}f)\}$$

with

$$P_i(y) = y_i \quad , \quad y = (y_0, ..., y_n) \in I\!R^{n+1}.$$

P r o o f. Since the mapping B is an isomorphism from $S_*(I\!R \times I\!R^n)$ onto $S_*(\Gamma)$, then from the relation (2.5) we deduce that

$$\mathcal{F}^{-1}f = \widetilde{\mathcal{F}}^{-1}(B^{-1}f).$$

According to the relations (2.5), (5.2) and Lemma 5.1, we get for all $i \in \{0, ..., n\}$

$$\delta_{B^{-1}f,i} = \sigma_{f,i}. \tag{5.3}$$

Hence, Theorem 5.3 follows from Corollary 5.1, the relations (5.1) and (5.3). ∎

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