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ON THE MELLIN TRANSFORMS OF DIRAC'S DELTA FUNCTION, THE HAUSDORFF DIMENSION FUNCTION, AND THE THEOREM BY MELLIN

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Abstract

We prove that Dirac's (symmetrical) delta function and the Hausdorff dimension function build up a pair of reciprocal functions. Our reasoning is based on the theorem by Mellin. Applications of the reciprocity relation demonstrate the merit of this approach.

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1. Introduction

Instead of discussing *non–continuous distributions of real argument*, we show that handling of non–continuous functions as a matter of function theory having a complex argument is possible and leads to sensible results. For this we give the following definitions of *functions*:

DEFINITION 1.1. By *Dirac's δ–function* we understand a non–continuous analytical function of complex argument, which is defined due to Dirac ([3], p. 625-626, and [16], eq. (2.7), p. 33) by the following three assumptions:

$$\delta(x) = 0 \quad \text{for} \quad x \neq 0\,, \tag{1}$$

$$\delta(x) = \delta(|x|)\,, \tag{2}$$

$$\int_0^{\exp(\mathrm{i}\,\varphi)} \delta(x)\,\mathrm{d}x = \frac{\exp(\mathrm{i}\,\varphi)}{2}\,. \tag{3}$$

Remark 1.1. The following result by Dirac holds ([3], eq. (1), p. 626) also for complex argument b:

$$\int_{-\infty}^{\infty} f(x)\,\delta(b-x)\,\mathrm{d}x = f(b). \tag{4}$$

Definition 1.2. As a *Hausdorff dimension function* we define the following transcendent non–continuous analytical function of complex argument with δ being a real positive number:

$$0^z := \lim_{\delta \to 0} \delta^z. \tag{5}$$

This definition of a Hausdorff dimension follows the fractal dimension given in ([11], eq. (2.4), p. 37; [4], section 3.1, p. 26–27).

Remark 1.2. The following values of the Hausdorff dimension function 0^z are found:

$$0^z = \begin{cases} 0 & \Re(z) > 0, \\ 1 & z = 0, \\ \infty & \Re(z) < 0, \\ \exp(+\mathrm{i}\,\log(0)) = \exp(-\mathrm{i}\,\infty) & \Re(z) = 0, \Im(z) > 0, \\ \exp(-\mathrm{i}\,\log(0)) = \exp(+\mathrm{i}\,\infty) & \Re(z) = 0, \Im(z) < 0. \end{cases} \tag{6}$$

Since the Hausdorff dimension $z = d$ is related to 0^{z-d}, there is another Hausdorff dimension function 0^{d-z} being the reciprocal of 0^{z-d}.

Definition 1.3. By a *pair of reciprocal functions* we understand a pair of two analytical functions with complex argument, that are connected by the following formulae of Mellin transform and its inverse ([10], eq. (62), p. 323):

$$F(z) = \int_0^{\infty} \Phi(x)\,x^{z-1}\,\mathrm{d}x, \tag{7}$$

$$\Phi(x) = \frac{1}{2\,\pi\,\mathrm{i}} \int_{a-\mathrm{i}\,\infty}^{a+\mathrm{i}\,\infty} F(z)\,x^{-z}\,\mathrm{d}z. \tag{8}$$

Of course, Mellin gave some conditions for $\Re(z)$, $\arg(x)$ and a, but did not mention explicitly, that in Mellin space an *analytical continuation* of the result is necessary to get back the original function via the residua of the Mellin transform $F(z)$ of $\Phi(x)$. In the context of Mellin's elaboration $F(z)$ is a generalized gamma function, also having a functional equation to be found, while $\Phi(x)$ allows an analytical continuation of its series expansion.

REMARK 1.3. In the context of this paper we stress, that a non–convergent integral just means, that there is a singularity in the result. Using divergent integrals may seem to the reader to be a "forbidden" operation similar to the discussion of $\sqrt{(-1)}$ five hundred years ago. Thus we claim, that a consequent handling of "forbidden" operations also can give sensible results. *To avoid fallacies it is essential to get a new result by at least two independent ways of calculation*! Thus a pair of reciprocal functions is found, if there is a way to get back to the original function. This can be achieved by inverse Mellin transformation of a Mellin transform or vice versa. The freedom of inverse Mellin transform (8) indeed consists in a clever choice of the constant a. An example for a choice $a \neq 0$ is given by Mellin himself ([10], eq. (120), p. 335).

THEOREM 1.4. The theorem by Mellin ([10], p. 323) is:

Von zwei reziproken Funktionen kann die eine nur dann identisch verschwinden, wenn auch die andere identisch gleich Null ist.

In a translation this is:

One of two reciprocal functions can be identical to null only if also the other one is equal to zero.

REMARK 1.4. Since unity is not identical to null, this theorem can be used to be the *proof of existence* for a lot of analytical functions. On the other hand it also can be used to show that Dirac's delta distribution exists as a function of complex argument, that may be non–continuous.

DEFINITION 1.5. By a *Fox's H–function* we understand the inverse Mellin transform of a generalized gamma function, represented by an integral of the following type

$$H_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix} (a_1,A_1),...,(a_n,A_n) & | & (a_{n+1},A_{n+1}),...,(a_p,A_p) \\ (b_1,B_1),...,(b_m,B_m) & | & (b_{m+1},B_{m+1}),...,(b_q,B_q) \end{matrix}\right]$$

$$:= \frac{1}{2\pi\mathrm{i}} \int\limits_{\mathcal{L}} \frac{\prod\limits_{j=1}^{m}\Gamma(b_j+B_j\,z)\,\prod\limits_{j=1}^{n}\Gamma(1-a_j-A_j\,z)}{\prod\limits_{j=m+1}^{q}\Gamma(1-b_j-B_j\,z)\,\prod\limits_{j=n+1}^{p}\Gamma(a_j+A_j\,z)}\,x^{-z}\,\mathrm{d}z \qquad (9)$$

and its analytical continuation ([5], eq. (1'), p. 81; [7], eq. (51), p. 408; [9], section 1.1, pp. 1-3; [14], eq. (1), p. 401).

Remark 1.5. The parameters $\{m, n, p, q\}$ are non–negative integer numbers in definition (9), while the $\{A_j, B_j\}$ are positive real numbers. The integration path $\mathcal{L}$ will be given in detail when needed.

In Section 2 we show that the Mellin transform of unity is connected to Dirac's delta function. We give some examples of pairs of reciprocal functions, where one of both functions is non–continuous. Then we show how to get the Hausdorff dimension function by Mellin transformation of Dirac's delta function and vice versa.

In Section 3 we discuss the application of the results in mathematics, physics, and computer algebra.

2. Non–continuous reciprocal functions: Basic results

The aim of this section is to show by examples, that handling divergent integrals of Mellin transform gives sensible results.

Example 2.1. The Mellin transform of unity does not converge for any z. Due to Remark 1.3 this elucidates, that the result allows at least one singularity. The Mellin integral distinguishes two cases $z = 0$ and $z \neq 0$:

$$\int_0^\infty x^{z-1}\,\mathrm{d}x \;=\; \begin{cases} \int_0^\infty \frac{1}{x}\,\mathrm{d}x = \infty & z = 0, \\ \int_0^1 x^{z-1}\,\mathrm{d}x + \int_1^\infty x^{z-1}\,\mathrm{d}x = \frac{1}{z} - \frac{1}{z} \;=\; 0 & z \neq 0. \end{cases} \tag{10}$$

The result shown for the case $z \neq 0$ in (10) is satisfied for $\Re(z) > 0$ *and* $\Re(z) < 0$, respectively. Due to Definition 1.3 an analytical continuation also of partial results takes place. The word *"and"* in this connection does not mean, that $\Re(z)$ is both positive and negative, but that one integral is won by the one and the other one by the other condition. The results are put together in Mellin space by analytical continuation.

Thus result (10) is proportional to Dirac's delta function (1)–(3). To get the proportionality multiplier we determined the inverse Mellin transform (8) of $2\,\pi\,\delta(z)$ according to definition (3) and equation (4):

$$\frac{2\,\pi}{2\,\pi\,\mathrm{i}} \int_{a-\mathrm{i}\,\infty}^{a+\mathrm{i}\,\infty} \delta(z) \left(\frac{1}{x}\right)^{z} \mathrm{d}z \;=\; 1 \quad \text{for} \quad a = 0. \tag{11}$$

Since the Mellin and Fourier transformations are closely connected for $a = 0$, the result $2\,\pi\,\delta(z)$ is already given in [16].

Thus unity and $2\pi\,\delta(z)$ are reciprocal functions due to Remark 1.3. On the other hand, the path $\mathcal{L}$ of the integral (11) goes straight through the singularity, we get due to definition (3) the result as zero for the residue of Dirac's delta function. The same is true as by a residual path around $z = 0$ and the use of definition (1). However if the integration path of the effective Mellin residue (8) is shifted over a delta singularity by the choice of a, the result will change! This elucidates the importance of Mellin's transform in function theory.

Example 2.2. The Mellin transform (7) of $\delta(x-1)$ yields unity due to definition (2) and equation (4):

$$\int_0^\infty \delta(1-x)\,x^{z-1}\,\mathrm{d}x \;=\; 1^{z-1} \;=\; 1. \tag{12}$$

Also the following integral results to unity:

$$\int_0^\infty \delta\left(1-\frac{1}{x}\right)\,x^{z-1}\,\mathrm{d}x \;=\; \int_0^\infty \delta(1-x)\,x^{-z-1}\,\mathrm{d}x \;=\; 1^{-z-1} \;=\; 1. \tag{13}$$

The inverse Mellin transform of unity no longer satisfies the conditions for equation (8). However, if we follow Mellin ([10], eqs. (73)+(74), p. 327) by choosing an integration path called $(-\infty)$ for $|x| > 1$, we find:

$$\frac{1}{2\pi\mathrm{i}}\left(\int_{-\infty-\mathrm{i}\infty}^{a-\mathrm{i}\infty} x^{-z}\,\mathrm{d}z + \int_{a-\mathrm{i}\infty}^{a+\mathrm{i}\infty} x^{-z}\,\mathrm{d}z + \int_{a+\mathrm{i}\infty}^{-\infty+\mathrm{i}\infty} x^{-z}\,\mathrm{d}z\right), \tag{14}$$

and $(+\infty)$ for $|x| < 1$:

$$\frac{1}{2\pi\mathrm{i}}\left(\int_{\infty+\mathrm{i}\infty}^{a+\mathrm{i}\infty} x^{-z}\,\mathrm{d}z + \int_{a+\mathrm{i}\infty}^{a-\mathrm{i}\infty} x^{-z}\,\mathrm{d}z + \int_{a-\mathrm{i}\infty}^{\infty-\mathrm{i}\infty} x^{-z}\,\mathrm{d}z\right). \tag{15}$$

This yields the following results due to the fact, that the residue of the regular function x^{-z} is zero:

$$\frac{1}{2\pi\mathrm{i}}\int_{-\infty-\mathrm{i}\infty}^{-\infty+\mathrm{i}\infty} x^{\mp z}\,\mathrm{d}z \;=\; {}_0y_0(-x) \;=\; \delta(1-x) \;=\; \begin{cases} \infty & x = 1, \\ 0 & x \neq 1. \end{cases} \tag{16}$$

In case of (14) $\Re(a) < \infty$ is assumed to get convergence, in case of (15) $\Re(a) > -\infty$ is necessary. The result (16) is delivered by the residues (null) of the two regular functions $x^{\mp z}$. The case x^{+z} follows from a substitution $z \to -z$ when evaluating (15). The divergence for $x = 1$ results in both cases of (16).

We note, that in the context by Mellin ([10], eqs. (73)+(74), p. 327) the integration path $(-\infty)$ is valid for $|x| < 1$ and $(+\infty)$ for $|x| > 1$, which elucidates, that the case $|x| = 1$ being discussed here is not included by default. Indeed $\delta(1-x)$ gives a singularity for $|x| = 1$. The notation of the hypergeometric function ${}_0y_0(-x)$ within the result (16) is due to the case $m = n = 0$ in the notation of Mellin. The corresponding hypergeometric differential equation ([10], eq. (82), p. 329) of the form

$$(1+x)\,x^0\,{}_0{y_0}^{(0)}(x) = 0 \tag{17}$$

is reduced to an algebraic equation. This kind of equation is also mentioned by Dirac ([3], p. 626) in the form $x\,\delta(x) = 0$. This equation due to Dirac determines the essential property of his delta function.

To show the full properties of a complex valued function, it is also necessary to discuss a complex value x. If $x = |x|\,\exp(\mathrm{i}\,\varphi)$ is not real, the following integration path, now called $(-\exp(\mathrm{i}\,\varphi)\,\infty)$, is needed to get the inverse Mellin transform of unity for $|x| > 1$ (and analogously by $\varphi \to \varphi + \pi$ the integration path $(+\exp(\mathrm{i}\,\varphi)\,\infty)$ for $|x| < 1$):

$$\frac{1}{2\,\pi\,\mathrm{i}}\left(\int_{\mathrm{e}^{\mathrm{i}\,\varphi}\,(-\infty-\mathrm{i}\,\infty)}^{\mathrm{e}^{\mathrm{i}\,\varphi}\,(a-\mathrm{i}\,\infty)} x^{-z}\,\mathrm{d}z + \int_{\mathrm{e}^{\mathrm{i}\,\varphi}\,(a-\mathrm{i}\,\infty)}^{\mathrm{e}^{\mathrm{i}\,\varphi}\,(a+\mathrm{i}\,\infty)} x^{-z}\,\mathrm{d}z + \int_{\mathrm{e}^{\mathrm{i}\,\varphi}\,(a+\mathrm{i}\,\infty)}^{\mathrm{e}^{\mathrm{i}\,\varphi}\,(-\infty+\mathrm{i}\,\infty)} x^{-z}\,\mathrm{d}z\right)$$

$$= 0 - \frac{1}{2\,\pi\,\mathrm{i}}\int_{\mathrm{e}^{\mathrm{i}\,\varphi}\,(-\infty+\mathrm{i}\,\infty)}^{\mathrm{e}^{\mathrm{i}\,\varphi}\,(-\infty-\mathrm{i}\,\infty)} x^{-z}\,\mathrm{d}z \tag{18}$$

$$= \frac{1}{2\,\pi\,\mathrm{i}}\int_{-\infty-\mathrm{i}\,\infty}^{-\infty+\mathrm{i}\,\infty} x^{-\exp(-\mathrm{i}\,\varphi)\,z}\,\mathrm{d}z = \delta(1-x) \equiv \delta\left(1-\frac{1}{x}\right). \tag{19}$$

Thus result (19) gives the analytical continuation of $\delta(1-x) \equiv \delta\left(1-\frac{1}{x}\right)$ for any complex x. The given identity can be proven by application of equation (4) or by the theorem by Mellin. The symmetry property (2) hereby also turns out to be provable. Dirac's statement about symmetry has been: "We can obviously take $\delta(-x) = \delta(x)$, $\delta'(x) = -\delta(x)$, etc." ([3], p. 626), which is obvious by the calculations above. Since there are still publications with asymmetrical delta "distributions" of real argument, we have given in this paper the symmetry (2) at the definitions to avoid confusion.

We also should mention, that the integrals with complex borders (18) are carried out by a straight line between the given limitations, whereby properties occur being similar to definition (3).

If the resulting function (12) would be the start of the action, the proportionality multiplicator unity would be found by this integral.

Example 2.3. The Mellin transform (7) of $\delta(x)$ due to definition (3) and equation (4) yields a Hausdorff dimension function (of dimension $z = 1$)

$$\int_0^\infty \delta(x)\, x^{z-1}\, \mathrm{d}x \;=\; \frac{0^{z-1}}{2}, \tag{20}$$

which behaves regularly for $\Re(z) > 1$ due to equation (6). In the same way the Mellin transform (7) of $\delta\left(\frac{1}{x}\right)$ yields an alternative Hausdorff dimension function (of dimension $z = -1$)

$$\int_0^\infty \delta\left(\frac{1}{x}\right) x^{z-1}\, \mathrm{d}x \;=\; \int_0^\infty \delta(x)\, x^{-z-1}\, \mathrm{d}x \;=\; \frac{0^{-z-1}}{2}, \tag{21}$$

which behaves regularly for $\Re(z) < -1$ due to equation (6).

The inverse Mellin transform of the results (20) and (21) is gained by an integration path ($\pm\infty$) of the following form with $a > 1$:

$$\frac{1}{4\,\pi\,\mathrm{i}} \left(\int\limits_{\pm\infty\pm\mathrm{i}\,\infty}^{\pm a\pm\mathrm{i}\,\infty} \frac{0^{\pm z-1}}{x^z}\, \mathrm{d}z + \int\limits_{\pm a\pm\mathrm{i}\,\infty}^{\pm a\mp\mathrm{i}\,\infty} \frac{0^{\pm z-1}}{x^z}\, \mathrm{d}z + \int\limits_{\pm a\mp\mathrm{i}\,\infty}^{\mp\infty\mp\mathrm{i}\,\infty} \frac{0^{\pm z-1}}{x^z}\, \mathrm{d}z \right). \tag{22}$$

Now for $x = 0$ there are more zeros in the denominator than in the numerator of the integrands, thus the result of the integral path ($\pm\infty$) indeed is ∞. For $x \neq 0$ the integrands can be reduced to the following form:

$$0^{\pm z-1}\, x^{-z} \;=\; \frac{\left(\frac{0}{x}\right)^{\pm z-1}}{x} \;=\; \frac{0^{\pm z-1}}{x}, \tag{23}$$

which leads to the result zero of the path integral (22) for $x \neq 0$ and $\Re(z) > 1$ or $\Re(z) < -1$ respectively, both according to equation (6). Thus the properties (1) and (2) of Dirac's delta function are confirmed for any complex x:

$$\frac{1}{2\,\pi\,\mathrm{i}} \int_{\pm2-\mathrm{i}\,\infty}^{\pm2+\mathrm{i}\,\infty} 0^{\pm z-1}\, x^{-z}\, \mathrm{d}z \;=\; 2\,\delta\left(x^{\pm1}\right). \tag{24}$$

If the resulting functions (20) or (21) would be the start of the action, the proportionality multiplicators unity would be found by these integrals.

3. Applications of the given examples

The theorem by Mellin is valid also for functions extending hypergeometric functions based on hypergeometric differential equations (see the examples in section 2). An application of the results is given to the fields of mathematics, physics, and computer algebra in the following.

Example 3.1. Since there is a strong connection between Mellin and Fourier transformation ([10];[16]), the Fourier transformation of the symmetrical function unity leads to the cosine transformation of unity according to [2], which contains the integral

$$\int_0^\infty \cos(k\,x)\,\mathrm{d}x \;=\; \begin{cases} \infty & k \;=\; 0, \\ \sum_{n=0}^{\infty}\int_{2\,\pi\,n}^{2\,\pi\,(n+1)}\cos(k\,x)\,\mathrm{d}x \;=\; 0 & k \;\neq\; 0. \end{cases} \tag{25}$$

If the delta function would be derived from this integral (25) only, the discussion about symmetry and complex k might start again. More interesting is the result of the formal integration

$$\int_0^\infty \cos(x)\,\mathrm{d}x \;=\; \sin(x)|_0^\infty \;=\; \sin(\infty) \;=\; \frac{\mathrm{e}^{\mathrm{i}\,\infty} - \mathrm{e}^{-\mathrm{i}\,\infty}}{2\,\mathrm{i}} \;=\; 0, \tag{26}$$

which leads to the fundamental results

$$\cos(\infty) \;=\; \exp(\pm \mathrm{i}\,\infty) \;=\; |\exp(\pm \mathrm{i}\,\infty)| \;=\; 1 \;=\; 0^0. \tag{27}$$

Also in this case there are publications (e.g. [13], section 5.4, p. 32) discussing an undetermined interval $[-1, 1]$ to be the result of $\sin(\infty)$ or $\cos(\infty)$. Since there is a relation $\cos(x) \;=\; \sin(\frac{\pi}{2} \pm x)$, it is important to give a *further calculation rule* before discussing terms like "$\cos(\infty) \;=\; \sin(\pm\infty) \;=\; ...???$", which also seem to be obvious. The further calculation rule to avoid such fallacies is:

> If a limit of a trigonometric function is searched for the argument ∞, the result can be given only, if the value $\infty + b$ is not evaluated until it is the argument of the trigonometric function. Then it is possible to use the addition theorems to evaluate a trigonometric function of a sum correctly by use of the results (26) and (27).

This calculation rule may be a future way to avoid fallacies of computer algebra, whereby warnings about correctness will be essential anyway.

Since results (26) and (27) are depending on the hierarchy of the calculation rules, we did not yet give the uniform result $\exp(\pm \mathrm{i}\,\infty) \;=\; 1$ when discussing the values (6) of the Hausdorff dimension function.

Example 3.2. A physical application of the function $\delta(1-x)$ is found when discussing the propagator of a fractional wave equation due to Schneider/Wyss [15], here given for one dimension only, but with a generalized sound velocity c:

$$u(x,t) \;=\; \sum_{k=0}^{1} \frac{t^k}{k!}\,\frac{\partial^k u}{\partial t^k}(x,0) \;+\; \frac{c^2}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\,\frac{\partial^2 u(x,\tau)}{\partial x^2}\,\mathrm{d}\tau \tag{28}$$

with $1 \;<\; \alpha \;\leq\; 2$.

The solution propagator $P(x,t)$ for the initial value problems $\frac{\partial^k u}{\partial t^k}(x,0) \;=\; \delta(x)$ yields in the notation of Schneider/Wyss [15] for $c \;=\; 1$,

$$P(x,t) \;=\; \pi^{-1/2}\, 2^{-1-2\,k/\alpha}\, x^{-1+2\,k/\alpha} \tag{29}$$
$$\times\, H^{2,0}_{1,2}\left[\frac{1}{2}\, x\, t^{-\alpha/2} \left| \begin{matrix} () & | & (1, \frac{\alpha}{2}) \\ \left(\frac{1}{2}-\frac{k}{\alpha},\, \frac{1}{2}\right),\, \left(1-\frac{k}{\alpha},\, \frac{1}{2}\right) & | & () \end{matrix} \right.\right].$$

This result can be simplified according to the multiplication formula of the gamma function [6] and the rules given in Mathai/Saxena [9], to the form [16]

$$P(x,t) \;=\; \frac{t^k}{|x|}\, H^{1,0}_{1,1}\left[\frac{|x|^2}{c^2\, t^\alpha} \left| \begin{matrix} () & | & (1+k,\alpha) \\ (1,2) & | & () \end{matrix} \right.\right], \tag{30}$$

which due to definition (9) and result (16) reduces in the special case $\alpha \;=\; 2$ and $k \;=\; 0$ to the ordinary wave propagator

$$P(x,t) \;=\; \frac{H^{0,0}_{0,0}\left[\left(\frac{|x|}{c\,t}\right)^2 \left| \begin{matrix} () & | & () \\ () & | & () \end{matrix} \right.\right]}{|x|} \;=\; \frac{\delta\left(1-\left(\frac{|x|}{c\,t}\right)^2\right)}{|x|}$$
$$=\; \frac{\delta\left(1-\frac{|x|}{c\,t}\right)}{2\,c\,t} \;=\; \frac{\delta(x-c\,t)\;+\;\delta(x+c\,t)}{2}\,, \tag{31}$$

whereby the initial value problem $P(x,0) \;=\; \delta(x)$ is obvious for $t \;\rightarrow\; 0$. The handling of these calculations needs some experience and the knowledge about the property, that the parameters t and c are positive, while x is a real valued number. Some properties of the delta function turn out to be properties of Fox's H–function [9] or of an inverse Mellin transform [12]. These calculations can also be done by computer algebra, whereby a special hierarchy of the possible operations is essential.

Since d'Alembert [8] did not know of fractional calculus or Fourier convolution in 1748, we now in a mathematical sufficient way have found back to the roots of partial differential equations.

EXAMPLE 3.3. Some difficulties occur when programming the inverse Mellin transform of the Hausdorff dimension function for computer algebra systems. Our *Mathematica* package *FractionalCalculus* includes a function called InverseMellinTransform. The application to the Hausdorff dimension function looks like

InverseMellinTransform[$0^{\beta+z}$, z , x] := $2\, x^{\beta+1}\, \delta(x)$,

InverseMellinTransform[$0^{\beta-z}$, z , x] := $2\, \left(\frac{1}{x}\right)^{\beta+1}\, \delta\left(\frac{1}{x}\right)$.

An integral representation would loose the information of the variable name x within the integral (23), since x is not dependent of z. The implementation of this function InverseMellinTransform obviously uses standardized integration paths for each function that is listed in the corresponding database of Mellin transforms [12]. An arbitrary integration path called $\mathcal{L}$ like in definition (9) does not make sense within computer algebra, because a computer is unable to reason.

Also the inverse Mellin transform of a product [12] is interesting with the Hausdorff dimension function, because the inverse Mellin transform of the Hausdorff dimension functions gives Dirac's delta function. Now this causes a severe problem of an infinite loop in computer algebra, if these formulae are used to find the function value of the propagator (30) for $t \to 0$ in general:

$$H_{1,1}^{1,0}\left[\left(\frac{|x|}{0}\right)^2 \middle| \begin{array}{c|c} () & (k+1,\alpha) \\ (1,2) & () \end{array}\right] \overset{?}{=} \tag{32}$$
$$\int_0^\infty H_{1,1}^{1,0}\left[\frac{|x|}{u} \middle| \begin{array}{c|c} () & (k+1,\frac{\alpha}{2}) \\ (1,1) & () \end{array}\right] \frac{\delta(u)}{2}\,\mathrm{d}u.$$

Maybe the loop (32) is a reason for discussing asymmetrical delta distributions (of real argument?), because due to definition (3) would occur an imploding term by iteration. However this proves nothing as long as the integral (32) cannot be solved by (formal) use of traditional calculation rules. The function values 0 and ∞ of the resulting delta function (33) indeed are not changed by the iterative factors at all!

The solution to solve the problem of the limit $t \to 0$ correctly is the application of the *fundamental lemma of integral theory*:

An integral of a pure real function is real valued,
because the imaginary part is null.

Thus the integral of Mellin transform (7) of the propagator (30) must give a real value for any real z. The only real z being interesting with a product of the Hausdorff dimension function (6) now is $z = 0$. This demonstrates, that z is to be set to the Hausdorff dimension in each factor being multiplied by a Hausdorff dimension function. In our case this yields by using equations (30), (9), (24), and a simplification rule of an inverse Mellin transform [12],

$$P(x, t \to 0) = \frac{t^k}{2\pi\,\mathrm{i}\,|x|}\int_{\mathcal{L}} \frac{\Gamma(1+2z)}{\Gamma(k+1+\alpha z)}\left(\frac{|x|}{0}\right)^{-2z}\mathrm{d}z$$
$$\overset{!}{=} \frac{t^k}{2\pi\,\mathrm{i}\,|x|}\int_{\mathcal{L}} \frac{\Gamma(1)}{\Gamma(k+1)}\left(\frac{0}{|x|}\right)^{2z}\mathrm{d}z = \frac{2\,t^k\,|x|\,\delta\left(\sqrt{|x|^2}\right)}{2\,|x|\,\Gamma(k+1)} = \frac{t^k}{k!}\,\delta(x), \tag{33}$$

which finishes our considerations about the mathematical and physical relevance of Dirac's delta function.

Until now the inverse Mellin transform of $\delta\left(\frac{1}{z}\right)$ has been too heavy to us, because for $x = 0$ results a term like $0^{-2-\frac{1}{0}}$ concerning the Hausdorff dimension function in a critical case. Until the correct solution is found, the result of computer algebra further on should be called Indeterminate, etc. Since the Mellin transform of the function $\delta\left(\frac{1}{x}\right)$ yields a correct pair of reciprocal functions due to the theorem by Mellin, the theorem by Mellin proofs, that the inverse Mellin transform of $\delta\left(\frac{1}{z}\right)$ is not identical to null.

At the end there is still a discussion about a Barnes integral of the delta function. From the identity $\delta(1-(1\pm x)) \equiv \delta(x)$ it follows

$$\delta(x) = \frac{1}{2\,\pi\,\mathrm{i}} \int_{\mathcal{L}} (1\pm x)^{-z}\,\mathrm{d}z = \frac{1}{2\,\pi\,\mathrm{i}} \int_{2-\mathrm{i}\,\infty}^{2+\mathrm{i}\,\infty} \frac{0^{z-1}}{2}\, x^{-z}\,\mathrm{d}z \tag{34}$$

by use of the equations (16) and (20). However the identity due to Barnes [1]; [10])

$$(1\pm x)^{-z} = \frac{1}{2\,\pi\,\mathrm{i}} \int_{\mathcal{L}} \frac{\Gamma(z-s)\,\Gamma(s)}{\Gamma(z)}\,(\pm\,x)^{-s}\,\mathrm{d}s \tag{35}$$

cannot be used to give a transformation from one to the other identity by swapping the integrals, because the singularities of 0^z are too heavy to allow the application of the corresponding Fubini theorem.

Thus Dirac's delta function is a Mellin, but not a Barnes integral.

We showed that function theory can be enlarged to handle non–continuous functions of complex argument. In the case of Dirac's delta function and the Hausdorff dimension function we also found a reciprocity connection due to the theorem by Mellin.

The possibilities of enlarging distributions to be functions of complex argument have not yet been fathomed, and there is still a sensible field of research on Mellin integrals, that are not Barnes integrals. The hierarchy of calculation rules is also an interesting field when implementing function theory to computer algebra systems.

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