Fractional Calculus & Applied Analysis
An International Journal for Theory and Applications
VOLUME 8, NUMBER 1 (2005) ISSN 1311-0454

RENEWAL PROCESSES OF MITTAG-LEFFLER AND WRIGHT TYPE

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Dedicated to Acad. Bogoljub Stanković, Prof. Emeritus,
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on the occasion of his 80-th birthday (November 1, 2004)

Abstract

After sketching the basic principles of renewal theory we discuss the classical Poisson process and offer two other processes, namely the renewal process of Mittag-Leffler type and the renewal process of Wright type, so named by us because special functions of Mittag-Leffler and of Wright type appear in the definition of the relevant waiting times. We compare these three processes with each other, furthermore consider corresponding renewal processes with reward and numerically their long-time behaviour.

2000 MSC: 26A33, 33E12, 33E20, 44A10, 44A35, 60G50, 60J05, 60K05
Key Words and Phrases: fractional derivative, Mittag-Leffler function, Wright function, renewal theory, Poisson process, fractional diffusion

1. Introduction

It is well known that the Poisson process (with and without reward) plays a fundamental role in renewal theory. In this paper, by means of functions of Mittag-Leffler and Wright type we provide a generalization of and a variant to this classical process and construct interesting subordinated stochastic processes of fractional diffusion.

The plan of the paper is as follows.

In Section 2 we recall the basic renewal theory including its fundamental concepts like waiting time between events, the survival probability, the renewal function.

If the waiting time is exponentially distributed we have the classical Poisson process, which is Markovian: this is the topic of Section 3. However, other waiting time distributions are also relevant in applications, in particular such ones with a fat tail caused by a power law decay of their density. In this context we analyze, respectively in Sections 4 and 5, two non-Markovian renewal processes with waiting time distributions described by functions of Mittag-Leffler and Wright type, that exhibit a similar power law decay. They both depend on a parameter $\beta \in (0,1)$ related to the common exponent in the power law. In the limit $\beta = 1$ the first becomes the Poisson process whereas the second goes over into the deterministic process producing its events at equidistant instants of time.

In Section 6, after sketching the basic differences between the renewal processes of Mittag-Leffler and Wright type, we compare numerically their survival functions and their probability densities in the special case $\beta = 1/2$ with respect to the corresponding functions of the classical Poisson process.

In Section 7 we discuss the general renewal process with reward (also called the *compound renewal process*), interpret it physically as a *continuous time random walk* (CTRW), and review how the resulting sojourn probability density function, evolving in time according to an integral equation, can analytically be represented as an infinite series. We then consider the *compound* renewal processes of Mittag-Leffler and Wright type including the limiting case $\beta = 1$.

In Section 8, we recall the conditions of the well-scaled transition to the diffusion limit according to which the CTRW integral equation of our compound processes reduces to the time fractional diffusion equation (TFDE). For all three cases (Poisson, Mittag-Leffler, Wright), by numerically summing the corresponding series for the cumulative probability function, we produce plots of its behaviour in time. These plots nicely illustrate that with increasing time the behaviour of time-fractional diffusion is approached.

Finally, concluding remarks are given in Section 9.

2. Essentials of renewal theory

For reader's convenience, we present a brief introduction to the renewal theory by using our notation. For more details see *e.g.* the classical treatises by Cox [8], Feller [11], and the more recent book by Ross [41].

We begin to recall that a stochastic process $\{N(t),\ t \geq 0\}$ is said to be a *counting process* if $N(t)$ represents the total number of "events" that have occurred up to time t. The concept of *renewal process* has been developed as a stochastic model for describing the class of counting processes for which the times between successive events, $T_1, T_2, \ldots$, are independent identically distributed (*iid*) non-negative random variables, obeying a given probability law. We call these times *waiting times* (or inter-arrival times) and the times

$$t_0 = 0\,, \quad t_k = \sum_{j=1}^{k} T_j\,, \quad k \geq 1\,. \tag{2.1}$$

renewal times. That is, $t_0 = 0$ is the starting time, $t_1 = T_1$ is the time of the first renewal, $t_2 = T_1 + T_2$ is the time of the second renewal, and so on; in general t_k denotes the time of the kth renewal.

Let the *waiting times* be distributed like T and let

$$\Phi(t) := P(T \leq t) \tag{2.2}$$

be the common probability distribution function. Here P stands for *probability*. We assume $\Phi(t)$ to be absolutely continuous, so that we can define its probability density function $\phi(t)$ as

$$\phi(t) = \frac{d}{dt}\Phi(t)\,, \quad \Phi(t) = \int_0^t \phi(t')\,dt'\,. \tag{2.3}$$

We recall that $\phi(t) \geq 0$ with $\int_0^\infty \phi(t)\,dt = 1$ and $\Phi(t)$ is a non-decreasing function in $\mathbf{R}^+$ with $\Phi(0) = 0$, $\Phi(+\infty) = 1$.

Let us remark that, as it is popular in Physics, we use the word density also for generalized functions that can be interpreted as probability measures. In these cases the function $\Phi(t)$ may lose its absolute continuity. Often, especially in Physics, the *probability density function* is abbreviated by *pdf*, so that, in order to avoid confusion, the probability distribution function is called the *probability cumulative function* (being the integral of the density) and abbreviated by *pcf*. When the nonnegative random variable represents the lifetime of technical systems, it is common to call $\Phi(t)$ the *failure probability* and

$$\Psi(t) := P\,(T > t) = \int_t^\infty \phi(t')\,dt' = 1 - \Phi(t)\,, \tag{2.4}$$

the *survival probability*, because $\Phi(t)$ and $\Psi(t)$ are the respective probabilities that the system does or does not fail in $(0,t]$. These terms, however, are commonly adopted for any renewal process.

As a matter of fact the *renewal process* is defined by the *counting process*

$$N(t) := \begin{cases} 0 & \text{for } \; 0 \le t < t_1 \,, \\ \max\{k | t_k \le t, \; k = 1,2,\ldots\} & \text{for } \; t \ge t_1 \,. \end{cases} \tag{2.5}$$

$N(t)$ is thus the random number of renewals occurring in $(0,t]$. We easily recognize that $\Psi(t) = P\,(N(t) = 0)$. For an example of a renewal process, suppose that we have an infinite supply of light-bulbs whose lifetimes are *iid* random variables. Suppose also that we use a single light-bulb at a time, and when it fails we *immediately* replace it with a new one. Under these conditions, $\{N(t),\, t \ge 0\}$ is a renewal process when $N(t)$ represents the number of light-bulbs that have failed by time t.

Continuing in the general theory, we set $F_1(t) = \Phi(t)$, $f_1(t) = \phi(t)$, and in general

$$F_k(t) := P\,(t_k = T_1 + \ldots + T_k \le t) \;, \; f_k(t) = \frac{d}{dt} F_k(t) \,, \; k \ge 1 \,. \tag{2.6}$$

$F_k(t)$ is the probability that the sum of the first k waiting times does not exceed t, and $f_k(t)$ is the corresponding density. $F_k(t)$ is normalized because $\lim\limits_{t\to\infty} F_k(t) = P\,(t_k = T_1 + \ldots + T_k < \infty) = \Phi(+\infty) = 1$. In fact, the sum of k random variables each of which is finite with probability 1 is finite with probability 1 itself.

We set for consistency $F_0(t) = \Theta(t)$, the Heaviside unit step function (with $\Theta(0) := \Theta(0^+)$) so that $F_0(t) \equiv 1$ for $t \ge 0$, and $f_0(t) = \delta(t)$, the Dirac delta generalized function.

A relevant quantity related to the counting process $N(t)$ is the function $v_k(t)$ that represents the probability that k events occur in the closed interval $[0,t]$. Using the basic assumption that the waiting times are *i.i.d.* random variables, we get, for any $k \ge 0$,

$$v_k(t) := P\,(N(t) = k) = P\,(t_k \le t \,,\, t_{k+1} > t) = \int_0^t f_k(t')\, \Psi(t-t')\, dt' \,. \tag{2.7}$$

We note that for $k = 0$ we recover $v_0(t) = \Psi(t)$.

Another relevant quantity is the *renewal function* $m(t)$ defined as the expected value of the process $N(t)$, that is

$$m(t) := E\,(N(t)) = \langle N(t) \rangle = \sum_{k=1}^{\infty} P\,(t_k \le t) \;. \tag{2.8}$$

Thus this function represents the average number of events in the interval $(0, t]$ and can be shown to uniquely determine the renewal process [41]. It is related to the waiting time distribution by the so-called *Renewal Equation*,

$$m(t) = \Phi(t) + \int_0^t m(t-t')\,\phi(t')\,dt' = \int_0^t \left[1 + m(t-t')\right]\,\phi(t')\,dt'\,. \quad (2.9)$$

If the mean waiting time (the first moment) is finite, namely

$$\rho := \langle T \rangle = \int_0^\infty t\,\phi(t)\,dt < \infty\,, \quad (2.10)$$

it is known that, with probability 1, $t_k/k \to \rho$ as $k \to \infty$, and $N(t)/t \to 1/\rho$ as $t \to \infty$. These facts imply the *elementary renewal theorem*,

$$\frac{m(t)}{t} \to \frac{1}{\rho} \quad \text{as} \quad t \to \infty\,. \quad (2.11)$$

We shall also consider renewal processes in which the mean waiting time is infinite because the waiting time laws have fat tails with power law asymptotics:

$$\phi(t) \sim \frac{A_\infty}{t^{(1+\beta)}}\,,\ \Psi(t) \sim \frac{A_\infty}{\beta\, t^\beta}\,,\ \text{for } t \to \infty\,, \quad 0 < \beta < 1\,,\ A_\infty > 0\,. \quad (2.12)$$

It is convenient to use the common $*$ notation for the Laplace convolution of two causal well-behaved (generalized) functions $f(t)$ and $g(t)$,

$$\int_0^t f(t')\,g(t-t')\,dt' = (f * g)\,(t) = (g * f)\,(t) = \int_0^t f(t-t')\,g(t')\,dt'\,.$$

Being $f_k(t)$ the *pdf* of the sum of the k *iid* random variables $T_1, \ldots, T_k$ with *pdf* $\phi(t)$, we recognize that $f_k(t)$ is the k-fold convolution of $\phi(t)$ with itself,

$$f_k(t) = \left(\phi^{*k}\right)(t)\,, \quad (2.13)$$

so that Eq. (2.7) simply reads:

$$v_k(t) := P\,(N(t) = k) = \left(\Psi * \phi^{*k}\right)(t)\,. \quad (2.14)$$

We note that in the convolution notation the *renewal equation* (2.9) reads

$$m(t) = \Phi(t) + (m * \phi)(t)\,. \quad (2.15)$$

The presence of Laplace convolutions allows us to treat a renewal process by the Laplace transform.

Throughout we will denote by $\widetilde{f}(s)$ the Laplace transform of a sufficiently well-behaved (generalized) function $f(t)$ according to

$$\mathcal{L}\{f(t);s\} = \widetilde{f}(s) = \int_0^{+\infty} \mathrm{e}^{-st}\, f(t)\, dt\,, \quad s > s_0\,,$$

and for $\delta(t)$ consistently we will have $\widetilde{\delta}(s) \equiv 1$. Note that for our purposes we agree to take s real.

We recognize that (2.13)-(2.14) reads in the Laplace domain

$$\widetilde{f}_k(s) = \left[\widetilde{\phi}(s)\right]^k\,, \quad \widetilde{v}_k(s) = \left[\widetilde{\phi}(s)\right]^k\, \widetilde{\Psi}(s)\,, \tag{2.16}$$

where, using (2.4),

$$\widetilde{\Psi}(s) = \frac{1-\widetilde{\phi}(s)}{s}\,. \tag{2.17}$$

Then, in the Laplace domain, the *renewal equation* reads

$$\widetilde{m}(s) = \widetilde{\Phi}(s) + \widetilde{m}(s)\,\widetilde{\phi}(s)\,, \quad \text{with} \quad \widetilde{\Phi}(s) = \frac{\widetilde{\phi}(s)}{s}\,, \tag{2.18}$$

from which

$$\widetilde{m}(s) = \frac{\widetilde{\phi}(s)}{s\left[1-\widetilde{\phi}(s)\right]}\,, \quad \widetilde{\phi}(s) = \frac{s\,\widetilde{m}(s)}{1+s\,\widetilde{m}(s)}\,. \tag{2.19}$$

3. The Poisson process as a renewal process

The most celebrated renewal process is the *Poisson process* (with parameter $\lambda > 0$). It is characterized by a survival function of exponential type:

$$\Psi(t) = \mathrm{e}^{-\lambda t}\,, \quad t \geq 0\,. \tag{3.1}$$

As a consequence, the corresponding density for the waiting times is exponential as well:

$$\phi(t) = \lambda\, \mathrm{e}^{-\lambda t}\,, \quad t \geq 0\,, \tag{3.2}$$

with moments

$$\langle T \rangle = \frac{1}{\lambda}\,, \quad \langle T^2 \rangle = \frac{1}{\lambda^2}\,, \quad \dots\,, \quad \langle T^n \rangle = \frac{1}{\lambda^n}\,, \quad \dots\,, \tag{3.3}$$

and the renewal function is linear:

$$m(t) = \lambda\, t\,, \quad t \ge 0\,. \tag{3.4}$$

The Poisson process is Markovian because the exponential distribution is characteristic for a process without memory. We know that the probability that k events occur in an interval of length t is the celebrated *Poisson distribution* with parameter λt,

$$v_k(t) := P\left(N(t) = k\right) = \frac{(\lambda t)^k}{k!}\, \mathrm{e}^{-\lambda t}\,, \quad t \ge 0\,, \quad k = 0, 1, 2, \dots\,. \tag{3.5}$$

The probability distribution related to the sum of k *iid* exponential random variables is known to be the so-called *Erlang distribution* (of order $k \ge 1$). The corresponding density (the *Erlang pdf*) is thus

$$f_k(t) = \lambda\, \frac{(\lambda t)^{k-1}}{(k-1)!}\, \mathrm{e}^{-\lambda t}\,, \quad t \ge 0\,, \quad k = 1, 2, \dots\,, \tag{3.6}$$

and the corresponding *Erlang pcf* is

$$F_k(t) = \int_0^t f_k(t')\, dt' = 1 - \sum_{n=0}^{k-1} \frac{(\lambda t)^n}{n!}\, \mathrm{e}^{-\lambda t} = \sum_{n=k}^{\infty} \frac{(\lambda t)^n}{n!}\, \mathrm{e}^{-\lambda t}\,, \quad t \ge 0\,. \tag{3.7}$$

In the limiting case $k = 0$ we recover $f_0(t) = \delta(t)$ and $F_0(t) = \Theta(t)$.

The results (3.4)-(3.7) can be easily obtained by the Laplace transform technique. In fact, starting from the Laplace transforms of the probability laws (3.1)-(3-2)

$$\widetilde{\Psi}(s) = \frac{1}{\lambda + s}\,, \quad \widetilde{\phi}(s) = \frac{\lambda}{\lambda + s}\,, \tag{3.8}$$

and, using (2.16)-(2.19), we have,

$$\widetilde{m}(s) = \frac{\widetilde{\phi}(s)}{s\left[1 - \widetilde{\phi}(s)\right]} = \frac{\lambda}{s^2}\,, \quad \widetilde{v}_k(s) = [\widetilde{\phi}(s)]^k\, \widetilde{\Psi}(s) = \frac{\lambda^k}{(\lambda + s)^{k+1}}\,, \tag{3.9}$$

hence (3.4)-(3.5), and

$$\widetilde{f}_k(s) = [\widetilde{\phi}(s)]^k = \frac{\lambda^k}{(\lambda + s)^k}\,, \quad \widetilde{F}_k(s) = \frac{[\widetilde{\phi}(s)]^k}{s} = \frac{\lambda^k}{s(\lambda + s)^k}\,, \tag{3.10}$$

hence (3.6)-(3.7).

4. The renewal process of Mittag-Leffler type

A "fractional" generalization of the renewal Poisson process has been recently proposed by Mainardi, Gorenflo and Scalas [29]. Noting that the survival probability for the Poisson renewal process (with parameter $\lambda > 0$) obeys the ordinary differential equation (of relaxation type)

$$\frac{d}{dt}\Psi(t) = -\lambda\Psi(t)\,, \quad t \geq 0\,; \quad \Psi(0^+) = 1\,. \tag{4.1}$$

the required generalization is obtained by replacing in (4.1) the first derivative by the fractional derivative (in Caputo's sense [1]) of order $\beta \in (0,1]$. We thus write, taking for simplicity $\lambda = 1$,

$$_tD_*^\beta\,\Psi(t) = -\Psi(t)\,, \quad t \geq 0\,, \quad 0 < \beta \leq 1\,; \quad \Psi(0^+) = 1\,. \tag{4.2}$$

The solution of Eq. (4.2) can be obtained by using the technique of the Laplace transforms. We have for $t \geq 0$:

$$\Psi(t) = E_\beta(-t^\beta)\,, \quad \text{from} \quad \widetilde{\Psi}(s) = \frac{s^{\beta-1}}{1+s^\beta}\,, \quad 0 < \beta \leq 1\,, \tag{4.3}$$

hence :

$$\phi(t) = -\frac{d}{dt}\Psi(t) = -\frac{d}{dt}E_\beta(-t^\beta), \text{ corresponding to } \widetilde{\phi}(s) = \frac{1}{1+s^\beta}. \tag{4.4}$$

Here E_β denotes the Mittag-Leffler function [2] of order β. We call this

[1] The Caputo derivative of order $\beta \in (0,1]$ of a well-behaved function $f(t)$ in $\mathbf{R}^+$ is

$$_tD_*^\beta\, f(t) := \begin{cases} \dfrac{1}{\Gamma(1-\beta)}\displaystyle\int_0^t \frac{f^{(1)}(\tau)}{(t-\tau)^\beta}\,d\tau\,, & 0 < \beta < 1\,, \\ \dfrac{d}{dt}f(t)\,, & \beta = 1\,. \end{cases}$$

Its Laplace transform turns out as

$$\mathcal{L}\{\,_tD_*^\beta\, f(t); s\} = s^\beta\,\widetilde{f}(s) - s^{\beta-1}\,f(0^+)\,.$$

For more information on the theory and the applications of the Caputo fractional derivative (of any order $\beta > 0$), see *e.g.* [6, 16, 27, 40].

[2] The Mittag-Leffler function with parameter β is defined as

$$E_\beta(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta\,n+1)}\,, \quad \beta > 0\,, \quad z \in \mathbf{C}\,.$$

It is an entire function of order β and reduces for $\beta = 1$ to $\exp(z)$. For detailed information on the functions of Mittag-Leffler type the reader may consult *e.g.* [10, 16, 24, 28, 40, 43] and references therein.

process the *renewal process of Mittag-Leffler type* (of order β).

Hereafter, we find it convenient to summarize the most relevant features of the functions $\Psi(t)$ and $\phi(t)$ when $0 < \beta < 1$. We begin to quote their expansions in power series of t^β (convergent for $t \ge 0$) and their asymptotic representations for $t \to \infty$,

$$\Psi(t) = \sum_{n=0}^{\infty} (-1)^n \frac{t^{\beta n}}{\Gamma(\beta\, n + 1)} \sim \frac{\sin(\beta\pi)}{\pi} \frac{\Gamma(\beta)}{t^\beta}\,, \tag{4.5}$$

$$\phi(t) = \frac{1}{t^{1-\beta}} \sum_{n=0}^{\infty} (-1)^n \frac{t^{\beta n}}{\Gamma(\beta\, n + \beta)} \sim \frac{\sin(\beta\pi)}{\pi} \frac{\Gamma(\beta+1)}{t^{\beta+1}}\,. \tag{4.6}$$

We recognize that for $0 < \beta < 1$ both functions $\Psi(t)$, $\phi(t)$ keep the complete monotonicity [3], characteristic for the Poissonian case $\beta = 1$.

In contrast to the Poissonian case, in the case $0 < \beta < 1$ the mean waiting time is infinite because the waiting time laws no longer decay exponentially but exhibit power law asymptotics according to (2.12) where the constant A_∞ derived from (4.5)-(4.6) is

$$A_\infty = \Gamma(\beta+1)\, \sin(\beta\pi)/\pi = \beta\,\Gamma(\beta)\, \sin(\beta\pi)/\pi\,. \tag{4.7}$$

As a consequence the process turns out to be no longer Markovian but of long-memory type.

The renewal function of this process can be deduced from the Laplace transforms in (2.19) and (4.4); we find

$$\widetilde{m}(s) = \frac{1}{s^{1+\beta}}\,, \quad \text{hence} \quad m(t) = \frac{t^\beta}{\Gamma(1+\beta)}\,, \quad t \ge 0\,, \quad 0 < \beta \le 1\,. \tag{4.8}$$

Thus, for $\beta < 1$ the renewal function turns out super-linear for small t and sub-linear for large t.

[3] Complete monotonicity of a function $f(t)$ means, for $n = 0, 1, 2, \ldots$, and $t \ge 0$, $(-1)^n \dfrac{d^n}{dt^n} f(t) \ge 0$, or equivalently, its representability as (real) Laplace transform of a non-negative function or measure. Recalling the theory of the Mittag-Leffler functions of order less than 1, we obtain for $0 < \beta < 1$ the following representations, see *e.g.* [16],

$$\Psi(t) = \frac{\sin(\beta\pi)}{\pi} \int_0^\infty \frac{r^{\beta-1}\, \mathrm{e}^{-rt}}{r^{2\beta} + 2\, r^\beta\, \cos(\beta\pi) + 1}\, dr\,, \quad t \ge 0\,,$$

$$\phi(t) = \frac{\sin(\beta\pi)}{\pi} \int_0^\infty \frac{r^{\beta}\, \mathrm{e}^{-rt}}{r^{2\beta} + 2\, r^\beta\, \cos(\beta\pi) + 1}\, dr\,, \quad t \ge 0\,.$$

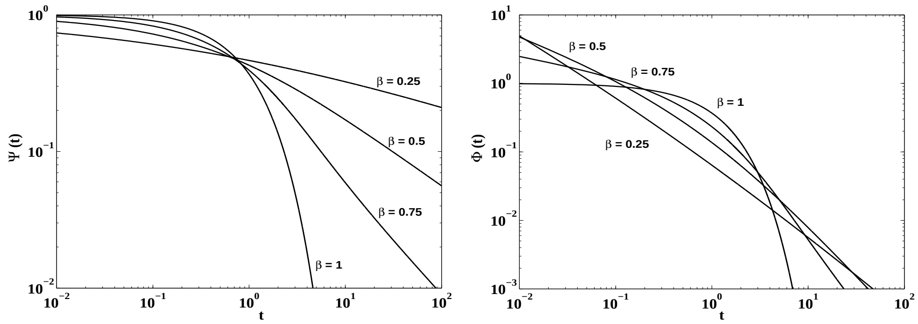

Fig. 1

The functions $\Psi(t)$ (left) and $\phi(t)$ (right) versus t ($10^{-2} < t < 10^2$) for the renewal processes of Mittag-Leffler type with $\beta = 0.25, 0.50, 0.75, 1$.

For the generalization of Eqs (3.5)-(3.7), concerning the Poisson and the Erlang distributions, we give the Laplace transform formula

$$\mathcal{L}\{t^{\beta\, k}\, E_\beta^{(k)}(-t^\beta); s\} = \frac{k!\, s^{\beta-1}}{(1+s^\beta)^{k+1}}\,, \quad \beta > 0\,, \quad k = 0, 1, 2, \ldots\,. \tag{4.9}$$

with $E_\beta^{(k)}(z) := \dfrac{d^k}{dz^k} E_\beta(z)$, that can be deduced from the book by Podlubny, see (1.80) in [40].

Then we get, with $0 < \beta < 1$ and $k = 0, 1, 2, \ldots$,

$$v_k(t) = \frac{t^{k\,\beta}}{k!}\, E_\beta^{(k)}(-t^\beta)\,, \text{ from } \widetilde{v}_k(s) = \widetilde{\Psi}(s)\, [\widetilde{\psi}(s)]^k = \frac{s^{\beta-1}}{(1+s^\beta)^{k+1}}\,, \tag{4.10}$$

as generalization of the Poisson distribution (with parameter t), what we call the *β-fractional Poisson distribution*. Similarly, with $0 < \beta < 1$ and $k = 1, 2, \ldots$, we obtain the *β-fractional Erlang pdf* (of order $k \geq 1$):

$$f_k(t) = \beta\, \frac{t^{k\beta-1}}{(k-1)!}\, E_\beta^{(k)}(-t^\beta)\,, \text{ from } \widetilde{\phi}_k(s) = [\widetilde{\psi}(s)]^k = \frac{1}{(1+s^\beta)^k}\,, \tag{4.11}$$

and the corresponding *β-fractional Erlang pcf*:

$$F_k(t) = \int_0^t f_k(t')\, dt' = 1 - \sum_{n=0}^{k-1} \frac{t^{n\beta}}{n!}\, E_\beta^{(n)}(-t^\beta) = \sum_{n=k}^{\infty} \frac{t^{n\beta}}{n!}\, E_\beta^{(n)}(-t^\beta)\,. \tag{4.12}$$

5. The renewal process of Wright type

A possible choice for obtaining an analytically treatable variant to the Poisson process has been suggested by Mainardi et al. [33]. It is based on the assumption that the *waiting-time pdf* $\phi(t)$ is the density of an extremal, unilateral, Lévy stable distribution with index $\beta \in (0,1)$, which exhibits, as we shall show, the same power law asymptotics as the corresponding *pdf* of the previous renewal process. In this case, however, the transition to the limit $\beta = 1$ is singular, and the Poisson process is no longer obtained. Now we have for $t \geq 0$,

$$\Psi(t) = \begin{cases} 1 - \Phi_{-\beta,1}\left(-\frac{1}{t^\beta}\right), & 0 < \beta < 1, \\ \Theta(t) - \Theta(t-1), & \beta = 1, \end{cases} \quad \text{from } \widetilde{\Psi}(s) = \frac{1 - \mathrm{e}^{-s^\beta}}{s}, \tag{5.1}$$

$$\phi(t) = \begin{cases} \frac{1}{t}\,\Phi_{-\beta,0}\left(-\frac{1}{t^\beta}\right), & 0 < \beta < 1, \\ \delta(t-1), & \beta = 1, \end{cases} \quad \text{from } \widetilde{\phi}(s) = \mathrm{e}^{-s^\beta}, \tag{5.2}$$

where $\Phi_{-\beta,1}$ and $\Phi_{-\beta,0}$ denote Wright functions [4].

In view of the presence of the Wright function in (5-1)-(5.2), we call this process the *renewal process of Wright type*.

Hereafter, taking $0 < \beta < 1$, we quote for $\Psi(t)$ and $\phi(t)$ their expansions in powers series of $t^{-\beta}$ (*convergent for* $t > 0$)

$$\Psi(t) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(\beta\, n)}{n!} \, \frac{\sin(\pi\beta n)}{t^{\beta n}}\,, \quad 0 < \beta < 1\,, \tag{5.3}$$

$$\phi(t) = \frac{1}{\pi t} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\Gamma(\beta\, n + 1)}{n!} \, \frac{\sin(\pi\beta n)}{t^{\beta n}}\,, \quad 0 < \beta < 1\,. \tag{5.4}$$

We indeed note that the first term of the above series (5.3)-(5.4) is identical to the asymptotic representation of the corresponding functions $\Psi(t)$ and

[4] The Wright function with parameters λ, μ is defined as

$$\Phi_{\lambda,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\lambda\, n + \mu)}\,, \quad \lambda > -1\,, \quad \mu \in \mathbf{C}\,, \quad \mathbf{z} \in \mathbf{C}\,.$$

It is an entire function of order $\rho = 1/(1+\lambda)$. For detailed information on the functions of Wright type the reader may consult *e.g.* [12, 14, 23, 24, 27, 49], and references therein. We note that in the classical handbook of the Bateman Project [10], see Vol. 3, Ch. 18, presumably for a misprint, the Wright function is considered with λ restricted to be non-negative.

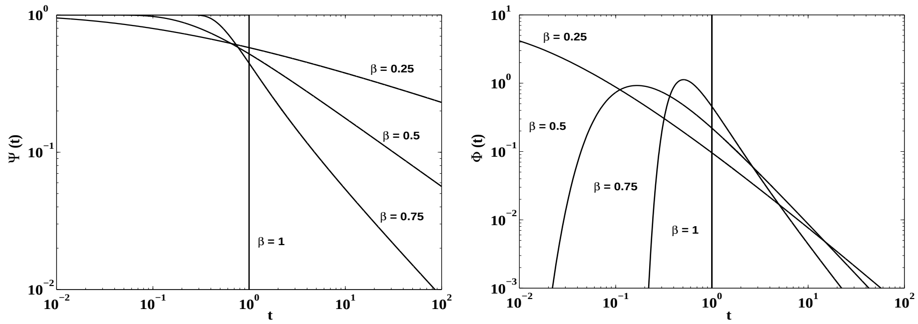

Fig. 2

The functions $\Psi(t)$ (left) and $\phi(t)$ (right) versus t ($10^{-2} < t < 10^2$) for the renewal processes of Wright type with $\beta = 0.25, 0.50, 0.75, 1$. For $\beta = 1$ the reader would recognize the Box function (extended up to $t = 1$) at left and the delta function (centered in $t = 1$) at right.

$\phi(t)$ of the renewal process of the Mittag-Leffler type, see (4.5)-(4.7). The behaviour near $t = 0$ of these functions is provided by the first term of their asymptotic expansions as $t \to 0$, namely from [33],

$$\Psi(t) \sim 1 - A\, t^{b/2} \exp\left(-B\, t^{-b}\right), \quad \phi(t) \sim C\, t^{-c} \exp\left(-B\, t^{-b}\right), \quad (5.5)$$

where [5]:

$$A = \left[\frac{1}{2\pi(1-\beta)\,\beta^{1/(1-\beta)}}\right]^{1/2}, \quad B = (1-\beta)\,\beta^b, \quad C = \left[\frac{\beta^{1/(1-\beta)}}{2\pi(1-\beta)}\right]^{1/2},$$
$$b = \frac{\beta}{1-\beta}, \quad c = \frac{2-\beta}{2(1-\beta)}. \quad (5.6)$$

As far as the functions $v_k(t)$, see (2.14), are concerned, we have

$$\widetilde{v}_0(s) = \widetilde{\Psi}(s) = \frac{1 - \mathrm{e}^{-s^\beta}}{s}, \quad (5.7)$$

[5] We take this occasion to point out a misprint (sign error) in the paper [30]. Noting that the asymptotic representation as $x \to 0^+$ of the unilateral extremal density of index $\alpha \in (0,1)$ in Eq. (4.15) of [30] is equivalent to ours of $\phi(t)$ as $t \to 0$, the coefficient c_1 (corresponding to our b in (5-5)-(5-6)), must be taken with the *minus* sign.

so,

$$v_0(t) = \Psi(t) = \begin{cases} 1 - \Phi_{-\beta,1}\left(-\frac{1}{t^\beta}\right), & 0 < \beta < 1, \\ \Theta(t) - \Theta(t-1), & \beta = 1, \end{cases} \tag{5.8}$$

and

$$\widetilde{v}_k(s) = \widetilde{\Psi}(s)\,[\widetilde{\psi}(s)]^k = \frac{\mathrm{e}^{-ks^\beta}}{s} - \frac{\mathrm{e}^{-(k+1)s^\beta}}{s}, \quad k = 1, 2, \dots, \tag{5.9}$$

from which, in view of the scaling property of the Laplace transform,

$$v_k(t) = \begin{cases} \Phi_{-\beta,1}\left(-\dfrac{k}{t^\beta}\right) - \Phi_{-\beta,1}\left(-\dfrac{k+1}{t^\beta}\right), & 0 < \beta < 1, \\ \Theta(t-k) - \Theta(t-k-1), & \beta = 1. \end{cases} \tag{5.10}$$

Let us close this section with a discussion on the renewal function $m(t)$. Whereas for the process of Mittag-Leffler type we have an explicit expression, namely (4.8), we could not find such one for the process of Wright type in the case $0 < \beta < 1$ from the Laplace transforms (2.19) and (5.2), i.e.

$$\widetilde{m}(s) = \frac{\widetilde{\phi}(s)}{s\left[1 - \widetilde{\phi}(s)\right]} = \frac{1}{s}\,\frac{\mathrm{e}^{-s^\beta}}{1 - \mathrm{e}^{-s^\beta}} = \frac{1}{s}\sum_{k=1}^{\infty} \mathrm{e}^{-k\,s^\beta}. \tag{5.11}$$

In the special case $\beta = 1$ we have (the term by term inversion is allowed!)

$$m(t) = \sum_{k=1}^{\infty} \Theta(t-k) = [t]\,, \tag{5.12}$$

where $[t]$ denotes the greatest integer less than or equal to t.

In the case $0 < \beta < 1$ we do not know how to invert (5.11). We can, however, see the asymptotics near zero and near infinity and apply the Tauber theory: $s \to 0$ gives $\widetilde{m}(s) = 1/s^{1+\beta}$, hence,

$$m(t) \sim t^\beta/\Gamma(1+\beta) \quad \text{for} \quad t \to \infty\,, \tag{5.13}$$

$s \to \infty$ gives $\widetilde{m}(s) = \exp(-s^\beta)/s = \widetilde{\Phi}(s)$, hence (if this Tauber trick is allowed for such fast decay towards zero), in view of (5.5)-(5.6),

$$m(t) \sim \Phi(t) \sim A\,t^{b/2} \exp\left(-B\,t^{-b}\right) \quad \text{for} \quad t \to 0\,. \tag{5.14}$$

6. The Mittag-Leffler and Wright processes in comparison

In this section we intend to compare the renewal processes of Mittag-Leffler and Wright type introduced in the Sections 4 and 5, respectively. In this comparison we agree to use the upper indices a and b to distinguish the relevant functions characterizing these processes.

We begin by pointing out the major differences between the survival functions $\Psi^a(t)$ and $\Psi^b(t)$, provided by Eqs (4.3) and (5.1), respectively, for a common index β when $0 < \beta \leq 1$. These differences, visible from a comparison of the plots (with logarithmic scales) in the left plates of Figures 1 and 2, can be easily inferred by analytical arguments, as previously pointed out (in a preliminary way) in the paper by Mainardi et al. [33].

Here we stress again the different behaviour of the two processes in the limit $\beta \to 1$ for $t \geq 0$.

Whereas $\Psi^a(t)$ and $\phi^a(t)$ tend to the exponential $\exp(-t)$, $\Psi^b(t)$ tends to the box function $\Theta(t) - \Theta(t-1)$ and the corresponding *waiting-time pdf* $\phi^b(t)$ tends to the shifted Dirac delta function $\delta(t-1)$.

The first of these processes is thus a direct generalization of the Poisson process since, for the limiting value $\beta = 1$, the Poisson process is recovered. In distinct contrast, the second process changes its character from stochastic to deterministic: for $\beta = 1$ at every instant $t = n$, n a natural number, an event happens (and never at other instants) so that $N(t) = [t]$, hence trivially $m(t) := E(N(t)) = [t]$ as we had already found in (5.12) by summation. We refer to this peculiar counting process as the *clock process* because of its similarity with the tick-tick of a (perfect) clock.

We also note that in the limit $\beta = 1$ the densities of the Mittag-Leffler and Wright processes have an identical finite first moment since $\rho = \int_0^\infty t \exp(-t)\, dt = \int_0^\infty t\, \delta(t-1)\, dt = 1$.

It is instructive to consider the special value $\beta = 1/2$ because in cases (a) and (b) we have an explicit representation of the corresponding survival functions and waiting-time densities in terms of well known functions.

For the renewal process of Mittag-Leffler type with $\beta = 1/2$ we have

$$\Psi^a(t) = E_{1/2}(-\sqrt{t}) = \mathrm{e}^{\,t}\, \mathrm{erfc}(\sqrt{t}) = \mathrm{e}^{\,t}\, \frac{2}{\sqrt{\pi}} \int_{\sqrt{t}}^\infty \mathrm{e}^{-u^2}\, du\,, \quad t \geq 0\,, \quad (6.1)$$

where erfc denotes the *complementary error* function [6], and

$$\phi^a(t) = -\frac{d}{dt}E_{1/2}(-\sqrt{t}) = \frac{1}{\sqrt{\pi t}} - \mathrm{e}^{\,t}\,\mathrm{erfc}(\sqrt{t})\,, \quad t \geq 0\,. \tag{6.2}$$

For the renewal process of Wright type with $\beta = 1/2$ we obtain for $t \geq 0$,

$$\Psi^b(t) = 1 - \Phi_{-1/2,1}(-1/\sqrt{t}) = 1 - \mathrm{erfc}\left(\frac{1}{2\sqrt{t}}\right) = \mathrm{erf}\left(\frac{1}{2\sqrt{t}}\right), \tag{6.3}$$

and

$$\phi^b(t) = \frac{1}{t}\,\Phi_{-1/2,0}(-1/\sqrt{t}) = \frac{1}{2\sqrt{\pi}}\,t^{-3/2}\exp\left(-\frac{1}{4t}\right)\,. \tag{6.4}$$

Observe that for this particular value of β the expression for the density [7] is obtained not only from the the sum of the convergent series (5.4) but also exactly from its asymptotic representation for $t \to 0$, see (5.5)-(5.6).

We easily note the common asymptotic (power-law) behaviour of the survival and density functions as $t \to \infty$ in the cases (a) and (b), that is, indicating by the index ∞ that we mean the leading asymptotic term:

$$\Psi_\infty(t) = \frac{t^{-1/2}}{\sqrt{\pi}}\,, \tag{6.5}$$

$$\phi_\infty(t) = \frac{t^{-3/2}}{2\sqrt{\pi}}\,. \tag{6.6}$$

It is now interesting to compare numerically the survival functions and the density functions of the two processes, that is (6.1)-(6.2) with (6.3)-(6.4), respectively.

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[6] We remind for $z \in \mathbf{C}$, see *e.g.* [1],

$$\mathrm{erf}\,(z) := \frac{2}{\sqrt{\pi}}\int_0^z \mathrm{e}^{-\zeta^2}\,d\zeta = \sum_{n=0}^{\infty}\frac{2^n}{(2n+1)!!}\,z^{2n+1}\,,\ \mathrm{erfc}\,(z) := 1 - \mathrm{erf}\,(z),$$

$$\mathrm{erfc}\,(z) \sim \frac{1}{\sqrt{\pi}}\,\frac{\mathrm{e}^{-z^2}}{z}\left(1 + \sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\,\frac{(2m)!}{(2z)^{2m}}\right)\,,\ |z| \to \infty\,,\ |\arg z| < \frac{3\pi}{4}\,.$$

[7] We point out $\mathcal{L}\left\{\phi^b(t) = \frac{t^{-3/2}}{2\sqrt{\pi}}\exp\left(-\frac{1}{4t}\right); s\right\} = \exp(-s^{1/2})$. This Laplace transform pair was noted by Lévy with respect to the unilateral stable density of order 1/2 and later, independently, by Smirnov. In the probability literature the distribution corresponding to this *pdf* is known as the *Lévy-Smirnov* stable distribution. We take this occasion to point pout a misprint in the paper [33]. There, in Eq. (3.25) giving the expression of the Lévy-Smirnov density, the factor 2 in front of $\sqrt{\pi}$ was missed.

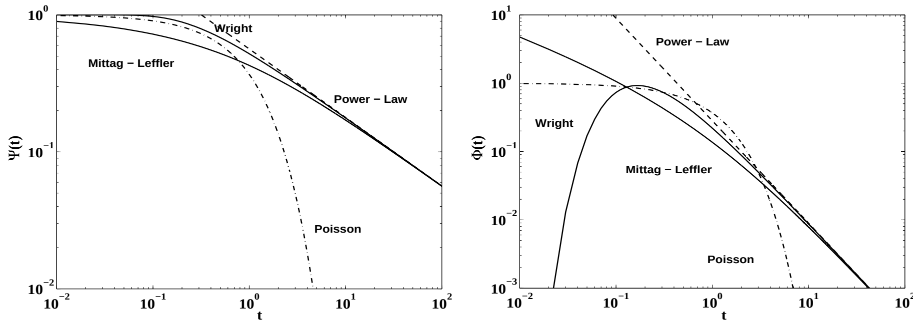

Fig. 3

Comparison versus time of the survival functions $\Psi(t)$ (left plate) and of the corresponding probability densities $\phi(t)$ (right plate).

We find it worthwhile to add in the comparison their asymptotic expression (6.5)-(6.6) and also the corresponding functions of the Poisson process, namely (3.1)-(3.2).

In Figure 3, we display the plots concerning the above comparison for the survival functions (left plate) and for the density functions (right plate) adopting the continuous line for the exact functions of the two long-memory processes and the dashed line both for the asymptotic power-law functions and for the exponential functions of the Poisson process.

The comparison is made by using logarithmic scales for $10^{-1} \leq t \leq 10^{1}$, just before the onset of the common power law regime; we note that, at least for the case $\beta = 1/2$ under consideration, the Wright process reaches the asymptotic power-law regime a little bit earlier than the corresponding Mittag-Leffler process.

In Tables I and II we show for some values of time t the corresponding values of the above survival and density functions in comparison, abbreviating power law by P-L and Mittag-Leffler by M-L.

Time	P-L (6.5)	M-L (6.1)	Wright (6.3)	Poisson (3.1)
0.1	$1.78\,10^{0}$	$7.24\,10^{-1}$	$9.74\,10^{-1}$	$9.05\,10^{-1}$
0.5	$7.98\,10^{-1}$	$5.23\,10^{-1}$	$6.83\,10^{-1}$	$6.07\,10^{-1}$
1	$5.64\,10^{-1}$	$4.28\,10^{-1}$	$5.21\,10^{-1}$	$3.68\,10^{-1}$
2	$3.99\,10^{-1}$	$3.36\,10^{-1}$	$3.83\,10^{-1}$	$1.35\,10^{-1}$
5	$2.52\,10^{-1}$	$2.32\,10^{-1}$	$2.48\,10^{-1}$	$6.74\,10^{-3}$
10	$1.78\,10^{-1}$	$1.71\,10^{-1}$	$1.77\,10^{-1}$	$4.54\,10^{-5}$
20	$1.26\,10^{-1}$	$1.23\,10^{-1}$	$1.26\,10^{-1}$	$2.06\,10^{-9}$
50	$7.98\,10^{-2}$	$7.90\,10^{-2}$	$7.97\,10^{-2}$	–
100	$5.64\,10^{-2}$	$5.61\,10^{-2}$	$5.64\,10^{-2}$	–

Table I: Comparison among the survival functions at different times.

Time	P-L (6.6)	M-L (6.2)	Wright (6.4)	Poisson (3.2)
0.1	$8.92\,10^{0}$	$1.06\,10^{-1}$	$7.32\,10^{-1}$	$9.05\,10^{-1}$
0.5	$7.98\,10^{-1}$	$2.75\,10^{-1}$	$4.84\,10^{-1}$	$6.07\,10^{-1}$
1	$2.82\,10^{-1}$	$1.37\,10^{-1}$	$2.20\,10^{-1}$	$3.68\,10^{-1}$
2	$9.97\,10^{-2}$	$6.27\,10^{-2}$	$8.80\,10^{-2}$	$1.35\,10^{-1}$
5	$2.52\,10^{-2}$	$2.00\,10^{-2}$	$2.40\,10^{-2}$	$6.74\,10^{-3}$
10	$8.92\,10^{-3}$	$7.83\,10^{-3}$	$8.70\,10^{-3}$	$4.54\,10^{-5}$
20	$3.15\,10^{-3}$	$2.94\,10^{-3}$	$3.11\,10^{-3}$	$2.06\,10^{-9}$
50	$7.98\,10^{-4}$	$7.75\,10^{-4}$	$7.94\,10^{-4}$	–
100	$2.82\,10^{-4}$	$2.78\,10^{-4}$	$2.81\,10^{-2}$	–

Table II: Comparison among the density functions at different times.

We finally consider the functions $v_k(t)$ for the two long-memory processes. For the functions $v_k^a(t)$ of the Mittag-Leffler process, see (4.10), we can take profit of the recurrence relations for repeated integrals of the error functions, see *e.g.* [1], §7.2, pp 299-300, to compute the derivatives of the Mittag-Leffler functions in Eqs. (4.10). We recall for $n = 0, 1, 2, \dots$,

$$\frac{d^n}{dz^n}\left(\mathrm{e}^{z^2}\,\mathrm{erfc}\,(z)\right) = (-1)^n\, 2^n\, n!\,\mathrm{e}^{z^2}\, I^n\,\mathrm{erfc}\,(z)\,, \tag{6.7}$$

where $I^n\,\mathrm{erfc}\,(z) = \int_z^\infty I^{n-1}\,\mathrm{erfc}\,(\zeta)\,d\zeta$ and $I^{-1}\,\mathrm{erfc}\,(z) = 2\exp(-z^2)/\sqrt{\pi}$.

For the Wright process, see (5.10), we have $v_0^b(t) = \Psi^b(t)$ as in (6.3) and

$$v_k^b(t) = \left[\mathrm{erfc}\left(\frac{k}{2\sqrt{t}}\right) - \mathrm{erfc}\left(\frac{k+1}{2\sqrt{t}}\right)\right]\,, \quad k = 1, 2, \dots\,. \tag{6.8}$$

7. Renewal processes with reward

The renewal process can be accompanied by reward that means that at every renewal instant a space-like variable makes a random jump from its actual position to a new point in "space". "Space" is here meant in a very general sense. In the insurance business, e.g., the renewal points are instants where the company receives a payment or must give away money to some claim of a customer, so space is money. In such process occurring in time and in space, also referred to as *compound renewal process*, the probability distribution of jump widths is as relevant as that of waiting times.

Let us denote by X_n the jumps occurring at instants t_n , $n = 1, 2, 3, \ldots$, and assume that they are *iid* (real, not necessarily positive) random variables with probability density $w(x)$, independent of the *waiting time* density $\phi(t)$. In a physical context the X_ns represent the jumps of a diffusing particle (the walker), and the resulting random walk model is known as *continuous time random walk*, abbreviated as CTRW [8], in that the waiting time is assumed to be a random variable with a *continuous* probability distribution function. The position x of the walker at time t is

$$x(t) = x(0) + \sum_{k=1}^{N(t)} X_k \,, \qquad (7.1)$$

with $N(t)$ as in (2.5).

Let us now denote by $p(x,t)$ the probability density (density with respect to x) of finding the random walker at the position x at time instant t. We assume the initial condition $p(x,0) = \delta(x)$, meaning that the walker is initially at the origin, $x(0) = 0$. We look for the evolution equation for $p(x,t)$ of the compound renewal process.

Based upon the previous probabilistic arguments we arrive at

$$p(x,t) = \delta(x)\,\Psi(t) + \int_0^t \phi(t-t')\,\left[\int_{-\infty}^{+\infty} w(x-x')\,p(x',t')\,dx'\right]\,dt' \,, \qquad (7.2)$$

[8] The name CTRW became popular in physics after Montroll, Weiss and Scher (just to cite the pioneers) in the 1960's and 1970's published a celebrated series of papers on random walks for modelling diffusion processes on lattices, see *e.g.* [38, 39], and the book by Weiss [51] with references therein. CTRWs are rather good and general phenomenological models for diffusion, including processes of anomalous transport. However, it should be noted that the idea of combining a stochastic process for waiting times between two consecutive events and another stochastic process which associates a reward or a claim to each event dates back at least to the first half of the twentieth century with the so-called Cramér–Lundberg model for insurance risk, see [9] for a review.

called the *integral equation of the CTRW.* From Eq. (7.2) we recognize the role of the survival probability $\Psi(t)$ and of the densities $\phi(t)$, $w(x)$. The first term in the RHS of (7.2) expresses the persistence (whose strength decreases with increasing time) of the initial position $x = 0$. The second term (a space-time convolution) gives the contribution to $p(x,t)$ from the walker sitting in point $x' \in \mathbf{R}$ at instant $t' < t$ jumping to point x just at instant t, after stopping (or waiting) time $t - t'$.

The integral equation (7.2) can be solved by using the machinery of the transforms of Laplace and Fourier. Having introduced the notation for the Laplace transform in Sect. 1, we now quote our notation for the Fourier transform, $\mathcal{F}\{f(x);\kappa\} = \widehat{f}(\kappa) = \int_{-\infty}^{+\infty} \mathrm{e}^{i\kappa x} f(x)\,dx$ ($\kappa \in \mathbf{R}$), and for the corresponding Fourier convolution of (generalized) functions

$$(f_1 * f_2)(x) = \int_{-\infty}^{+\infty} f_1(x')\, f_2(x - x')\, dx' .$$

Then, applying the transforms of Fourier and Laplace in succession to (7.2) and using the well-known operational rules, we arrive at the famous Montroll-Weiss equation, see [39],

$$\widehat{\widetilde{p}}(\kappa, s) = \frac{\widetilde{\Psi}(s)}{1 - \widetilde{\phi}(s)\, \widehat{w}(\kappa)} . \tag{7.3}$$

As pointed out in [13], this equation can alternatively be derived from the Cox formula, see [8], Chapter 8, formula (4), describing the process as subordination of a random walk to a renewal process.

By inverting the transforms one can, in principle, find the evolution $p(x,t)$ of the sojourn density for time t running from zero to infinity. In fact, recalling that $|\widehat{w}(\kappa)| < 1$ and $|\widetilde{\phi}(s)| < 1$, if $\kappa \neq 0$ and $s \neq 0$, Eq. (7.3) becomes

$$\widehat{\widetilde{p}}(\kappa, s) = \widetilde{\Psi}(s) \sum_{k=0}^{\infty} [\widetilde{\phi}(s)\, \widehat{w}(\kappa)]^k = \sum_{k=0}^{\infty} \widetilde{v}_k(s)\, \widehat{w}_k(\kappa) , \tag{7.4}$$

and we promptly obtain the series representation

$$p(x,t) = \sum_{k=0}^{\infty} v_k(t)\, w_k(x) = \Psi(t)\, \delta(x) + \sum_{k=1}^{\infty} v_k(t)\, w_k(x) , \tag{7.5}$$

where the functions $v_k(t)$ and $w_k(x)$ are obtained by repeated convolutions in time and in space, $v_k(t) = (\Psi * \phi^{*k})(t)$, see also (2.14), and $w_k(x) =$

$(w^{*k})(x)$, respectively. In particular, $v_0(t) = (\Psi * \delta)(t) = \Psi(t)$, $v_1(t) = (\Psi * \phi)(t)$, $w_0(x) = \delta(x)$, $w_1(x) = w(x)$. In the R.H.S of Eq (7.5) we have isolated the first singular term related to the initial condition $p(x,0) = \Psi(0)\,\delta(x) = \delta(x)$.

A special case of the integral equation (7.2) is obtained for the *compound Poisson process* where $\phi(t) = \mathrm{e}^{-t}$ (as in (3.2), with $\lambda = 1$ for simplicity). Then, the corresponding equation reduces after some manipulations, that best are carried out in the Laplace-Fourier domain, to the *Kolmogorov-Feller equation*:

$$\frac{\partial}{\partial t}\, p(x,t) = -p(x,t) + \int_{-\infty}^{+\infty} w(x-x')\, p(x',t)\, dx' \,, \qquad (7.6)$$

which is the *master equation of the compound Poisson process.* In view of Eqs (3.5) and (7.5) the solution reads

$$p(x,t) = \sum_{k=0}^{\infty} \frac{t^k}{k!}\, \mathrm{e}^{-t}\, w_k(x)\,. \qquad (7.7)$$

When the survival probability is the Mittag-Leffler function introduced in (4.3), the master equation of the corresponding *compound process of Mittag-Leffler type* can be shown to be

$${}_tD_*^{\beta}\, p(x,t) = -p(x,t) + \int_{-\infty}^{+\infty} w(x-x')\, p(x',t)\, dx' \,, \quad 0 < \beta < 1 \,, \qquad (7.8)$$

where ${}_tD_*^{\beta}$ denotes the time fractional derivative of order β in the Caputo sense. For a (detailed) derivation of Eq (7.8) we refer to the paper by Mainardi et al. [33], in which the results have been obtained by an approach different from that adopted in a previous paper by Hilfer and Anton [22]. In this case, in view of Eqs (4.10) and (7.5), the solution of the *master equation* (7.8) reads:

$$p(x,t) = E_\beta(-t^\beta)\,\delta(x) + \sum_{k=1}^{\infty} \frac{t^{\beta k}}{k!}\, E_\beta^{(k)}(-t^\beta)\, w_k(x)\,, \quad 0 < \beta < 1\,. \qquad (7.9)$$

When the survival probability is the Wright function introduced in (5.1), we resist the temptation to work out an explicit form for the the master equation of the corresponding *compound process of Wright type*, but we content ourselves to give its series solution derived from (7.5) using Eqs (5.8),(5.10), as

$$p(x,t) = \left[1 - \Phi_{-\beta,1}\left(-\frac{1}{t^\beta}\right)\right]\, \delta(x) + \sum_{k=1}^{\infty}\left[\Phi_{-\beta,1}\left(-\frac{k}{t^\beta}\right) - \Phi_{-\beta,1}\left(-\frac{k+1}{t^\beta}\right)\right]\, w_k(x)\,, \quad 0 < \beta < 1\,. \tag{7.10}$$

Recalling what we have observed in Section 6 on the contrasting behaviours of the Mittag-Leffler process and the Wright process in the limit $\beta = 1$ we recognize that *with rewards* the first of these processes becomes the *compound Poisson process* ((7.9) with $\beta = 1$ reduces to (7.7)), whereas the second goes over into the process, whose sojourn *pdf* is directly obtained from the Laplace inversion of Eqs (5.7) and (5.9) for $\beta = 1$. Recalling the natural convention $w_0(x) = \delta(x)$, we obtain

$$p(x,t) = \sum_{k=0}^{\infty} \left[\Theta(t-k) - \Theta(t-k-1)\right]\, w_k(x)\,. \tag{7.11}$$

This simply means: $p(x,t) = w_k(x)$ for $k \le t < k+1$ $(k = 0, 1, 2, \ldots)$, or, more concisely, $p(x,t) = w_{[t]}(x)$. This limiting process is nothing but a simple *random walk, discrete in time and Markovian on the set of integer time-values*: at every instant $[t]$ a jump occurs with *pdf* $w(x)$, hence $[t]$ jumps have occurred exact up to instant t, which leads to $[t]$-fold convolution $w_{[t]}(x)$. We have rigorously investigated such random walks in [17]. However, in the framework of the CTRW theory (where the resulting random walk, discontinuous or continuous in space, is implicitly intended to be *continuous in time*), this case appears a peculiar (non-Markovian) process, as discussed by Weiss [51] in his book at p.47, see also [33].

8. Time fractional diffusion as limit of CTRW

There are several ways to generalize the classical diffusion equation by introducing space and/or time derivatives of fractional order. We mention here the seminal papers by Schneider & Wyss [46] of 1989 for the time fractional diffusion equation and the likewise influential paper by Saichev & Zaslavsky [42] of 1997 for diffusion fractional in time as well in space. In the recent literature several authors stress the viewpoint of *subordination*, see *e.g.* [2, 5, 32, 35, 52], and special attention is being paid to diffusion equations with *distributed orders* of fractional temporal or/and spatial differentiation, see *e.g.* [6, 7, 19, 48]. The transition from CTRW to such generalized types of diffusion has been investigated by different methods, not only from mathematical interest but also due to applications in Physics, Chemistry (for generalized Fokker-Planck and Liouville equations see *e.g.* [3, 4, 21, 22, 47]), and other Applied Sciences including Economics and Finance (for *financial markets* see *e.g.* [20, 33, 44]). For a good list of references on these topics we recommend the review papers by Metzler and Klafter [36, 37].

Gorenflo and Mainardi, see *e.g.* [18, 19], have, under proper conditions on the tails of the probability distributions for jumps and waiting times, investigated for Eq. (7.2) the so-called *well scaled transition to the diffusion limit*. This transition is obtained by making smaller all jumps by a positive factor h and all waiting times by a positive factor τ related to h by a *proper scaling relation*, and then letting h and τ tend to zero. An alternative interpretation is that we look at the same process from far away and after long time, so that spatial distances and time intervals of normal size appear very small but always related through *proper scaling*. In this limit the integral equation (7.2) has been shown to reduce to a partial differential equation with fractional derivatives in space and/or in time, referred to as a *space-time fractional diffusion equation*. This is also the topic of the recent paper by Scalas et al. [45] and, in a more general framework, of the paper by Gorenflo and Abdel-Rehim [13]. For a variant of well-scaled transition (there not called so) see [50]. In this paper we limit ourselves to recall the well scaled transition from the CTRW integral equation (7.2) to the *time fractional diffusion equation* (TFDE),

$$ {}_tD_*^{\beta}\, u(x,t) := \frac{1}{\Gamma(1-\beta)} \int_0^t \left[\frac{\partial}{\partial \tau}\, u(x,\tau) \right] \, \frac{d\tau}{(t-\tau)^{\beta}} = \frac{\partial^2}{\partial x^2}\, u(x,t)\,, \qquad (8.1) $$

with $0 < \beta \le 1$, subjected to the initial condition $u(0,t) = \delta(x)$. For $\beta = 1$ we recover the standard diffusion equation, for which the fundamental solution is the Gaussian probability density evolving in time with variance $\sigma^2 = 2t$,

$$ u(x,t) = \frac{1}{2\sqrt{\pi}}\, t^{-1/2}\, \mathrm{e}^{-x^2/(4t)}\,. \qquad (8.2) $$

This probability law is known to govern the classical Brownian model for the phenomenon of normal diffusion. Also for $0 < \beta < 1$ the fundamental solution of t (8.1) can be interpreted as a (spatially-symmetric) density evolving in time, but exhibiting stretched exponential tails with a variance proportional to t^β, implying a phenomenon of *slow anomalous diffusion*. We have, see for details [14, 15, 26, 27, 30, 31],

$$u(x,t) = \frac{1}{2\,t^{\beta/2}}\, M_{\beta/2}(|x|/t^{\beta/2})\,, \tag{8.3}$$

where

$$M_{\beta/2}(y) := \sum_{n=0}^{\infty} \frac{(-y)^n}{n!\,\Gamma[-\beta n/2 + (1-\beta/2)]}\,, \tag{8.4}$$

$|x|/t^{\beta/2}$ acting as a *similarity variable*. We stress that, according to footnote (4), $M_{\beta/2}(y) = \Phi_{-\beta/2,1-\beta/2}(y)$, hence the fundamental solution of the TFDE is a Wright function [9]. The TFDE (8.1) can be derived from the CTRW integral equation (7.2), by properly rescaling the waiting time and the jump widths and passing to the diffusion limit. Making smaller all waiting times by a positive factor τ, all jumps by a positive factor h, we get, for $n \in \mathbf{N}$, the jump instants $t_n(\tau) = \tau T_1 + \tau T_2 + \ldots + \tau T_n$ and the jump sums, $x_0(h) = 0$, $x_n(h) = hX_1 + hX_2 + \ldots + hX_n$. The reduced waiting times τT_n all have the *pdf* $\phi_\tau(t) = \phi(t/\tau)/\tau$, $t > 0$, analogously the reduced jumps hX_n all have the *pdf* $w_h(x) = w(x/h)/h$, $x \in \mathbf{R}$. The probability density $p(x,t)$ so obtained is denoted by $p_{h,\tau}(x,t)$. Then, the transition to the diffusion limit consists in sending $\tau \to 0$ and $h \to 0$ under an appropriate relation between τ and h and deriving the evolution equation satisfied by the limiting *pdf* $p_{0,0}(x,t)$.

Let us now resume the approach by Gorenflo and Mainardi, for giving the conditions on $w(x)$, $\phi(t)$ and the *scaling relation* between h and τ, which allow $p_{0,0}(x,t)$ to be identified with the fundamental solution $u(x,t)$

[9] In his first 1993 analysis of the time fractional diffusion equation, one of the present authors (F.M.) [25] introduced the two *auxiliary functions* (of Wright type), $F_\nu(z) := \Phi_{-\nu,0}(-z)$ and $M_\nu(z) := \Phi_{-\nu,1-\nu}(-z)$ with $0 < \nu < 1$, inter-related through $F_\nu(z) = \nu\, z\, M_\nu(z)$. Being in that time only aware of the Bateman project where the parameter λ of the Wright function $\Phi_{\lambda,\mu}(z)$ was erroneously restricted to non-negative values, F.M. thought to have extended the original Wright function. It was just Professor Stanković during the presentation of the paper [34] at the Conference *Transform Methods and Special Functions, Sofia 1994*, who informed F.M. that this extension for $-1 < \mu < 0$ was already made just by Wright himself in 1940 (following his previous papers in 1930's). On this special occasion F.Mainardi wants to renew his personal gratitude to Professor Stanković for this earlier information, that has induced him to study the original papers by Wright and work (also in collaboration) on the functions of the Wright type for further applications.

of our TFDE (8.1). Assuming the jump *pdf* $w(x)$ to be symmetric with finite variance σ^2 and the waiting time *pdf* $\phi(t)$ either with finite mean ρ (relevant in the case $\beta = 1$), or, with $c > 0$ and some $\beta \in (0,1)$, $\phi(t) \sim c\, t^{-(\beta+1)}$ for $t \to \infty$, and setting $\mu = \sigma^2$, $\lambda = \rho$ if $\beta = 1$, $\lambda = c\,\Gamma(1-\beta)/\beta$ if $0 < \beta < 1$, the required *scaling relation* for the diffusion limit is $\lambda\, \tau^\beta = \mu\, h^2$.

We now will compare, at fixed times, the spatial probability distributions provided by the fundamental solutions of the integral equation of the CTRW as given by the series (7.5) with the fundamental solutions of the corresponding limiting TFDE as given by the function (8.3) of Wright type. This will enable us, for the compound Poisson, Mittag-Leffler and Wright processes, to investigate from a numerical view the increasing quality of approximation to the diffusion limit with advancing time, independently from the scaling relation. In this numerical comparison, in order to avoid the singular terms originated by delta functions, we prefer to consider, instead of the spatial densities $p(x,t)$, $w(x)$ and $u(x,t)$, the corresponding *probability cumulative functions* that we denote by capital letters, namely $P(x,t)$, $W(x)$ and $U(x,t)$. Thus, by integrating in space (7.5) from $-\infty$ to x, and denoting by $\Theta(x)$ the unit step Heaviside function, we have

$$P(x,t) = \Psi(t)\, \Theta(x) + \sum_{k=1}^{\infty} v_k(t)\, W_k(x)\,. \tag{8.5}$$

In an analogue way, for the TFDE we have

$$U(x,t) = \frac{1}{2}\left[1 + \int_0^{x/t^{\beta/2}} M_{\beta/2}(y)\, dy\right]\,, \quad 0 < \beta \le 1\,, \tag{8.6}$$

that in the case $\beta = 1$ reduces to

$$U(x,t) = \frac{1}{2}\left[1 + \text{erf}\left(\frac{x}{2\sqrt{t}}\right)\right]\,. \tag{8.7}$$

In our case studies we have chosen $t = 1$ and $t = 10$, as typical short and long times. As waiting time distributions we have used the exponential distribution for the compound Poisson process so that the $v_k(t)$ are given by (3.5), and the Mittag-Leffler and Wright distributions for $\beta = 1/2$, for the corresponding compound processes so that $v_k(t)$ are given by (4.10) with (6.7), and (6.8), respectively. For all the processes we have used a *symmetric Gaussian* for the common jump probability distribution, for which we have

$$w_k(x) = \frac{1}{2\sqrt{\pi}}\,\frac{\mathrm{e}^{-x^2/(4k)}}{\sqrt{k}}\,, \quad W_k(x) = \frac{1}{2}\left[1 + \text{erf}\left(\frac{x}{2\sqrt{k}}\right)\right]\,. \tag{8.8}$$

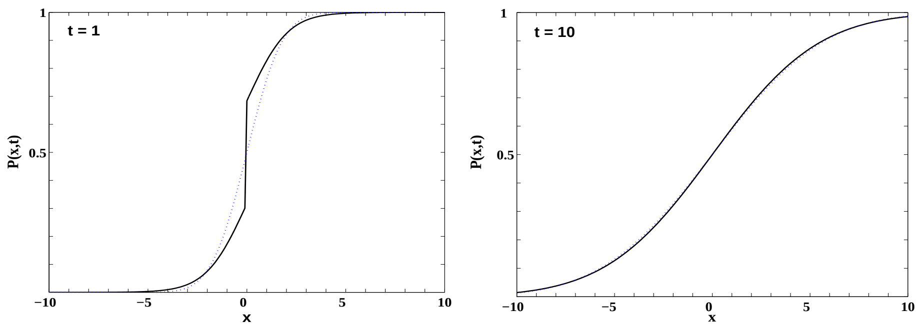

Fig. 4
Compound Poisson process.

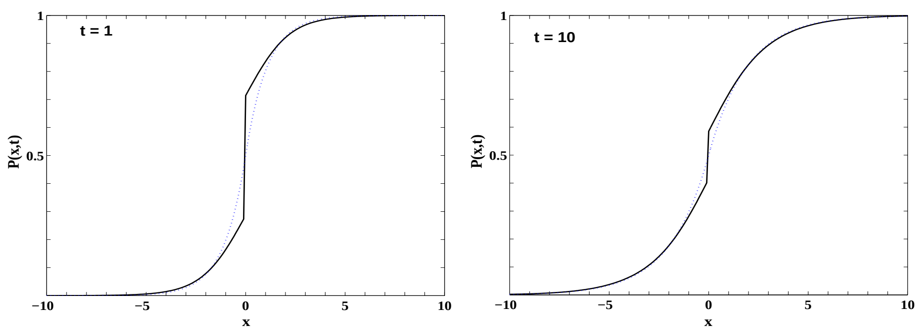

Fig. 5
Compound Mittag-Leffler process.

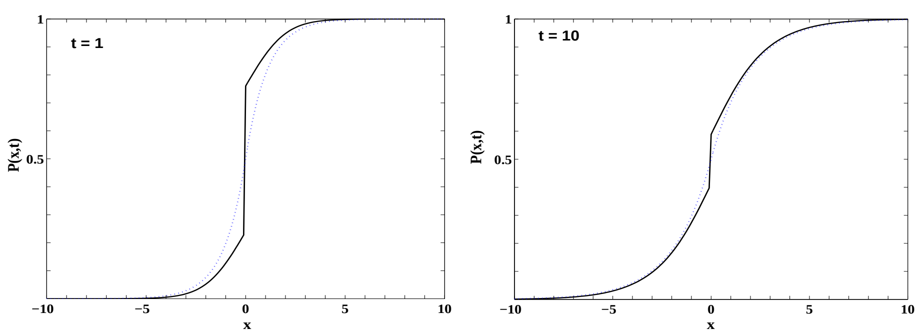

Fig. 6
Compound Wright process.

In Figures 4,5,6, regarding respectively, the compound Poisson, Mittag-Leffler and Wright processes, we report in continuous line the *pcf* for the CTRW whereas in dashed line the corresponding cumulative functions of the limiting TFDE. The values of the survival function $\Psi(t)$ for $t = 1$ and $t = 10$ are given in Table I; however, they are clearly visible as the height of the vertical part (at $x = 0$) of the continuous line.

We note that for $\beta = 1/2$ there is no substantial difference between the cases of Mittag-Leffler and Wright compound processes: at these times the dominant effect is due to the common asymptotic behaviour of the corresponding distributions resulting from the diffusion limit. In other words: the results are consistent with the asymptotic properties of the diffusion limit. We also note that in this comparison we cannot take direct profit of the scaling relation of the diffusion limit, because the fundamental solution of the CTRW integral equation has no self-similarity property at variance to the fundamental solution of the corresponding TFDE.

We point out that also Barkai [3] has investigated the diffusion limit for the CTRW process of Wright type but considering different case studies and using different notation and terminology.

9. Conclusions

After sketching the basic principles of renewal theory we have discussed the classical Poisson process. It is well known that in this process the sum of k waiting times obeys the so-called *Erlang distribution*, named after the Danish telecommunication engineer Agner Krarup *Erlang* [10] (1878-1929).

Because the exponential function is just a special case of the *Mittag-Leffler function*, one may expect that an analogous explicit representation exists for the situation of waiting times distributed according to the Mittag-Leffler law $\Psi(t) = E_\beta(-t^\beta)$, $0 < \beta \leq 1$, with $\Psi(t)$ denoting the survival function (the probability of no event occurring in an interval of length t just after an event). We have thus presented such explicit representations, for the above renewal process that we have called *renewal process of Mittag-Leffler type*.

We have also considered another process where the survival function, being related to the unilateral Lévy stable law of index β, is given as an expression involving the *Wright function*. This process has been called *renewal process of Wright type*.

[10] See `http://plus.maths.org/issue2/erlang/` for information on Erlang.

In both of these renewal processes the waiting time probability distribution is long-tailed, and there is a parameter β, $0 < \beta < 1$. The first of these processes is a direct generalization of the Poisson process as for the limiting value $\beta = 1$ the Poisson process is recovered. In distinct contrast, the second process changes its character from stochastic to deterministic: for $\beta = 1$ at every instant $t = n$, n a natural number, an event happens (and never for other instants). Then, we have reviewed the general theory of renewal processes with reward, the so called *compound renewal processes*, known in physics and chemistry literature as *continuous time random walks*.

Using the fact that the sojourn probability density of a compound renewal process can be written as an infinite series involving the convolution powers of the waiting times and the rewards (the jumps), we have simulated these processes (all three cases: first *Poisson*, then, with $\beta = 1/2$, *Mittag-Leffler* and *Wright*). This comparison has allowed us to make visible how in the large time regime the behaviour of the probability cumulative functions approaches that of the corresponding functions for *time-fractional diffusion* processes, to which our compound renewal processes are known to weakly converge in the well scaled transition to the diffusion limit. Nicely one can see that the survival function decays towards zero very fast (exponentially) in the Poisson process, but slowly (like a power with negative exponent) in the other two processes. Incidentally we have also stressed, from a theoretical view-point, the remarkable contrast in the behaviour of the non-Markovian compound processes of Mittag-Leffler and of Wright type in the limit $\beta = 1$, yielding the classical compound Poisson process (which is continuous in time and Markovian) for the first type, but a process discrete in time (Markovian on the set of the integer time-values, but non Markovian on the set of the real time-values) for the second.

Acknowledgements

This work has been carried out in the framework of a joint research project for *Fractional Calculus Modelling* (`www.fracalmo.org`), and is partially supported by the Italian COFIN-MIUR project *Order and Chaos in Nonlinear Extended Systems: Coherent Structures, Weak Stochasticity and Anomalous Transport* (`www.physycom.unibo.it`). R.G. and F.M. appreciate the support of the EU ERASMUS-SOCRATES program for visits to Bologna and Berlin that, besides for teaching, were also useful for this research activity.

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Received: November 1, 2004

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