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LINEAR FRACTIONAL PDE, UNIQUENESS OF GLOBAL SOLUTIONS

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Dedicated to Acad. Bogoljub Stanković,
on the occasion of his 80-th birthday

Abstract

In this paper we treat the question of existence and uniqueness of solutions of linear fractional partial differential equations. Along examples we show that, due to the global definition of fractional derivatives, uniqueness is only sure in case of global initial conditions.

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1. Introduction

1.1. Notations

1. For brevity, we denote the partial differentiation by the index notation, common in mechanical engineering, i.e. for some function $u(x,t)$

$$u_{,xx} := \frac{\partial^2}{\partial x^2}\ , \quad u_{,xt} := \frac{\partial^2}{\partial x \partial t}\ , \quad u_{,xxt^q} := \frac{\partial^2}{\partial x^2}\frac{\partial^q}{\partial t^q}\ \ (q \in \mathbb{R}^+)\ , \ \dots$$

2. The *Fourier* transformation is denoted by $\mathcal{F}$, the *Fourier* transform of some function $x(t)$ is denoted by $\widehat{x}(\omega)$.

3. All noninteger powers s^q $(s \in \mathbb{C})$ are defined as principal branches, more precisely and in accordance with computer algebraic systems like *Mathematica, Maple, Mathlab*, etc., this is $-\pi < \arg(s) \leq \pi$.

1.2. Fractional differential operators

In several papers we have described and investigated an approach via a functional calculus technique which establishes fractional differential operators as pseudodifferential operators. This approach, briefly sketched now, is in rough based on the *Fourier* transforms on the space of square summable functions $\mathbf{L}_2$. An extension of the *Fourier* transforms to distributional spaces ($\mathcal{S}'$, $\mathcal{D}'$) (see e.g. [8,18]) yields an extension of fractional differentiation in these spaces.

Definition 1. *Let $\mathcal{F}$ denote the unitary Fourier transform on $\mathbf{L}_2$:*

$$\mathcal{F} : \mathbf{L}_2 \to \mathbb{C} : \; x(t) \mapsto \widehat{x}(\omega) = \frac{1}{\sqrt{2\pi}} \; p.v. \int_{-\infty}^{\infty} \mathrm{e}^{-i\omega t} x(t) \, \mathrm{d}t,$$

where "p.v." is the (Cauchy–) principal value. Then a fractional differential operator D^q, $(q \geq 0)$ is defined in $\mathbf{L}_2$ via

$$D^q := \mathcal{F}^{-1} \, (i\omega)^q \, \mathcal{F} \; ,$$

such that the q–th derivative of $x(t) \in \mathbf{L}_2$ is given as

$$D^q x(t) = \mathcal{F}^{-1} \left\{ (i\omega)^q \, \widehat{x}(\omega) \right\} \; , \tag{1}$$

if $(i\omega)^q \, \widehat{x}(\omega) \in \mathbf{L}_2$.

Remarks:

1. The integer order derivatives are included trivially, because the above definition is a generalization of the differentiation rule of the *Fourier* transforms.

2. As treated in previous papers, linear combinations of those symbols/-derivatives are defined in their entirety as one symbol/operator. The solutions of related differential equations can then be analyzed directly from the zeros of the symbol ([11]).

3. We have pointed out in [12] that our above definition of fractional derivatives coincides with the *Riemann–Liouville* integral as well as with the *Caputo*–integral with lower bounds $-\infty$, if the integrals exist. E.g. we have, for $0 \le q < 1$

$$D^q x(t) = {}_{-\infty}D_t^q\, x(t) = {}_{-\infty}^{\;\;C}D_t^q\, x(t) = \frac{1}{\Gamma\,(1-q)} \int\limits_{-\infty}^{t} \frac{x'(\tau)\, d\tau}{(t-\tau)^q}\,. \qquad (2)$$

1.3. The scope of the approach

For many applications, particularly in solid dynamics, the $\mathbf{L}_2$–approach comes out satisfactory: experiments on viscoelastic rods have shown that measurement and calculations are in very good accordance ([14], [11]).

The $\mathbf{L}_2$–approach fails by nature, if applied to unstable processes, which can (of course) not be modeled by $\mathbf{L}_2$–functions. But any extension of Fourier transforms leads via the above definition to an extension of fractional differential operators as far as $(i\omega)^q\, \widehat{x}(\omega)$ makes sense.

The extension of *Fourier* transforms to the space of tempered distributions $\mathcal{S}'$ as well as to $\mathcal{D}'$ is well understood (see e.g. [8], [18], [7]). In this way, we can establish fractional derivatives in this spaces also via formula (1).

2. The reference problem

We consider a linear fractional PDE for $u(x,t)$, $(a \le x \le b)\,,\ (-\infty \le t)$ together with boundary conditions (BC) and global initial conditions (IC)

$$\begin{aligned} \eta\, u_{,xxt^q} + E\, u_{,xx} - \rho\, u_{,tt} &= r(x,t) \\ u(a,t) = g_a(t)\ ,\ \ u(b,t) &= g_b(t) \\ u(x,-\infty) = u_{,t}\,(x,-\infty) &= 0. \end{aligned} \qquad (3)$$

The noninteger order of differentiation is $0 \le q < 1$ and $\eta, E, \rho > 0$.

2.1. Existence

Because it is not the crux of the paper, we will keep short and sketch the verification. Following our definition (1) of fractional derivatives we solve the problem via *Fourier* transform with respect to t which yields for convenient $r(x,t)$ a solution $\widehat{u}(x,\omega)$ such that the inverse *Fourier* transform exists. This is exactly the same procedure that we have treated in one dimension in several previous papers (see e.g. our summary paper [11]).

2.2. Uniqueness

If there are two different solutions u_1, u_2 of this problem, then the difference $v := u_1 - u_2$ solves the totally homogeneous problem, i.e., $r = g_a = g_b = 0$. We conclude

$$v_{,t}(a,t) = v_{,t}(b,t) = 0. \tag{4}$$

Thus, to show that problem (3) has a unique solution, we have to show that $v(x,t) = 0 \quad \forall x, t$.

We denote by $\mathcal{J}(t)$ the energy of the homogeneous system and keep in mind that $\mathcal{J}(-\infty) = 0$. Splitting into kinetic, potential and dissipative parts, we get

$$\begin{aligned} \mathcal{J}(T) &= \int_a^b \int_{-\infty}^T \rho\, v_{,t}\, v_{,tt}\, dt\, d\xi + \int_a^b \int_{-\infty}^T E\, v_{,x} v_{,xt}\, dt\, d\xi + \int_a^b \int_{-\infty}^T \eta\, v_{,xt} v_{,xt^q}\, dt\, d\xi \\ &= \int_a^b \frac{\rho}{2}\,(v_{,T})\; d\xi + \int_a^b \frac{E}{2}\,(v_{,x})^2\; d\xi + \int_a^b \int_{-\infty}^T \eta\, v_{,xt}\, v_{,xt^q}\, dt\, d\xi\,. \end{aligned} \tag{5}$$

Whereas the first two integrals are trivially ≥ 0 this cannot be seen easily for the third integral. From a physical point of few we may argue that this term models the dissipative energy and hence must be also positive definite. But, from a mathematical point of view, we have to show this independently, to confirm the physical consistency of the approach. But, not to lose the plot, we postpone the proof in Section 4.

Assuming the correctness, let us investigate $\mathcal{J}(T)$. We differentiate with respect to t and apply partial integration with respect to x to the last two integrals

$$\begin{aligned} D\mathcal{J}(t) &= \int_a^b \rho\, v_{,t} v_{,tt}\, dx + \int_a^b E\, v_{,x} v_{,xt}\; dx + \int_a^b \eta\, v_{,xt} v_{,xt^q}\, dx \\ &= \int_a^b v_{,t}\,(\rho\, v_{,tt} - E\, v_{,xx} - \eta\, v_{,xxt^q})\, dx + \left[v_{,t}\,(E\, v_{,x} + \eta\, v_{,xt^q})\right]_a^b\,. \end{aligned}$$

Now the first term vanishes via the homogeneous PDE and the second via (4). We remark that this term models the energy flow at the boundaries.

Consequently, we have $D\mathcal{J}(t) = 0$ and thus from $\mathcal{J}(-\infty) = 0$ further $\mathcal{J}(t) = 0$. Hence, all integrals in the second line of (5) vanish and we get

$$v_{,t}(x,t) = v_{,x}(x,t) = 0 \quad \forall x, t\,.$$

Together with the homogeneous boundary and initial conditions we conclude that the homogeneous problem has only the trivial solution $v(x,t) = 0$.

Thus, problem (3) has a unique solution.

3. Local initial conditions

We use the same PDE for $u(x,t)$, $(a \le x \le b)$, $(-\infty \le t)$ as above but change the initial conditions from $-\infty$ to the finite boundary 0:

$$\begin{aligned} \eta\, u_{,xxt^q} + E\, u_{,xx} - \rho\, u_{,tt} &= r(x,t) \\ u(a,t) = g_a(t)\ ,\ \ u(b,t) &= g_b(t) \\ u(x,0) = u_0(x)\ ,\ \ u_{,t}\,(x,0) &= z_0(x). \end{aligned} \tag{6}$$

This problem includes the solution for the 'past' of the system, i.e. $u(x,t)$ for $t < 0$. As the past of a solution of a fractional DE cannot be described via a set of local (initial) conditions (see our proof in [10]) we expect that, concerning uniqueness, we run now into difficulties for noninteger q.

3.1. The energy of the system

We proceed analogously as to the first problem. We have to investigate the solution $v(x,t)$ of the half–homogeneous problem, i.e., all right–hand sides in (6) vanish for $t \ge 0$. The same consideration of the energy as done above leads again to

$$\mathcal{J}(T) = \int_a^b \frac{\rho}{2}\,(v_{,T})^2\,d\xi + \int_a^b \frac{E}{2}\,(v_{,x})^2\,d\xi + \int_a^b \int_{-\infty}^{T} \eta\, v_{,xt}\, v_{,xt^q}\,dt\,d\xi\,, \tag{7}$$

and again we have $\mathcal{J}(-\infty) = 0$. For $D\mathcal{J}$ we cannot conclude as above, because we have no pre–information about $v(x,t)$ for $t < 0$. Of course, for $t \le 0$ the first two integrals vanish $D\mathcal{J}(t) \ge 0$. If we assume as above in Section 2 the positiveness of the last integral in (7), we get on the way

$$\mathcal{J}(0) = 0 + 0 + \int_a^b \int_{-\infty}^{0} \eta\, v_{,xt}\, v_{,xt^q}\,dt\,d\xi \ge 0. \tag{8}$$

But, from the homogenous conditions for $t = 0$ we have $D\mathcal{J}(t) = 0$ for $t \geq 0$ such that $\mathcal{J}(t) = \mathcal{J}(0)$ for $t \geq 0$. Equation (7) yields

$$\mathcal{J}(T) = \int_a^b \frac{\rho}{2}\,(v_{,t})^2\;dt\,d\xi + \int_a^b \frac{E}{2}\,(v_{,x})^2\;dt\,d\xi + \mathcal{J}(0) + \int_a^b \int_0^T \eta\, v_{,xt}\, v_{,xt^q}\,dt\,d\xi.$$

Consequently, the sum of the three integrals must vanish.

Now we come to the above suggested crucial point.

1. For $q = 1$ we see that the last integral is trivially ≥ 0 such that we conclude
$$v_{,t}(x,t) = v_{,x}(x,t) = 0 \;\Rightarrow\; v(x,t) = 0$$
and we have uniqueness, as it has to be in case of an ordinary PDE.

2. For $0 < q < 1$ we cannot conclude as above in (7) that the questionable last integral is ≥ 0. So, we cannot conclude that $v_{,t}$, $v_{,x}$ and thus v vanish. And we cannot conclude uniqueness. This is what we have expected, due to the fact that the global character of fractional derivatives is not compatible with local initial conditions.

3.2. The existence of other solutions

Consider for $a \leq x \leq b,\ t \geq 0,\ 0 < q < 1$ the BVP

$$\eta\, v_{,xxt^q} + E\, v_{,xx} - \rho\, v_{,tt} = \begin{cases} 0 & \text{for} \quad t \geq 0 \\ g(x,t) & \text{for} \quad t < 0 \end{cases}$$
$$v(a,t) = 0\,,\quad v(b,t) = 0\,, \tag{9}$$
$$v(x,0) = 0\,,\quad v_{,t}\,(x,0) = 0\,.$$

For simplicity the righthand side of the PDE may be continuous and e.g. $g \in \mathbf{L}_2$ such that a solution exists. If we restrict for the moment the problem on the halfspace $t \geq 0$, (what is the fullspace problem with $g(x,t) = 0$), the trivial solution $v(x,t) = 0$ solves obviously the BVP. Of course, $v(x,t) = 0$ for all t cannot be the global solution for $g(x,t) \neq 0$. So we ask for a non trivial $v^*(x,t)$ which vanishes for $t \geq 0$, such that the PDE is fulfilled, i.e., the lefthand side must add to $g(x,t)$ for $t \leq 0$. We get

$$\eta\, v^*_{,xxt^q} + E\, v^*_{,xx} - \rho\, v^*_{,tt} = \begin{cases} \eta v^*_{,xxt^q}(x,0) & \text{for} \quad t \geq 0 \\ =: g(x,t) & \text{for} \quad t < 0 \end{cases}.$$

And from the *Riemann–Liouville* representation (2) we have

$$v^*_{,xxt^q}(x,0) = \frac{1}{\Gamma(1-q)} \frac{d}{dt} \int\limits_{-\infty}^{0} (t-\tau)^{-q}\, v^*_{,xx}(x,\tau)\, d\tau \neq 0\,.$$

Thus, the halfspace solution $v(x,t) = 0$ for $t \geq 0$ is not part of a global solution which of course also solves the halfspace problem. Consequently, the halfspace problem has no unique solution.

This demonstrates that in the case of fractional DE the whole past is needed as (global) initial condition.

Now, we proceed with the proof of the inequality which establishes the energy balance in formula (5).

4. The questionable inequality

In this section we deal with functions of one real variable. Hence, we write $w'(t)$ for integer ordered differentiation and $D^q w(t)$ for noninteger differentiation, respectively.

We will show now that in case of existence the above occuring integrals for the dissipative energy are positive definite. We show that in general we have

$$\int\limits_{-\infty}^{T} w'(t)\, D^q w(t)\, dt \geq 0, \tag{10}$$

if the integral exists.

Similar inequalities have been considered by other authors. *M. Enelund* et al. proved the inequality for the lower bound 0 and consequently for another definition of D^q. Recently, *B. Stankovič* and *T. Atanackovič* proved $\int\limits_0^T {}_0D_t^q w(t)\cdot w(t)\, dt \geq 0$, which needs even some more efforts (see [16]). The proofs use instruments due to *S. Bochner* and *L. Schwartz* which represent positive definite (generalized) functions as Fourier transforms of positive measures.

Let us now consider (10). From the second representation in formula (2), the inequality (10) is equivalent to

$$\mathcal{J}(T) = \frac{1}{\Gamma(1-q)} \int\limits_{-\infty}^{T} \int\limits_{-\infty}^{t} (t-\tau)^{-q}\, v'(\tau) v'(t)\, d\tau\, dt \geq 0\,,$$

if the integral exists. Reflecting the inner integral on the line $t = \tau$ and adding it, we arrive at the conjecture

$$\mathcal{J}(T) = \frac{1}{2\Gamma(1-q)} \int_{-\infty}^{T} \int_{-\infty}^{T} |t-\tau|^{-q} \, v'(\tau) v'(t) \, d\tau \, dt \geq 0 \, .$$

Similarly to the argumentation in [16],[15] we make now use of the so called *Bochner–Schwartz* theorem (see e.g. [7],[5]) which briefly confirms the above conjecture, if the Fourier transform of the kernel function is also of positive type. Indeed, we have (e.g. with *Mathematica*)

$$\mathcal{F}\left\{|t|^{-q}\right\} = \sqrt{\frac{2}{\pi}}\,\Gamma(1-q)\,\sin\left(\frac{\pi q}{2}\right)\,|\omega|^{1-q} \; .$$

5. Other problems and survey

We will not bore the reader with exactly the same calculations and only inform about the fact that many other BVPs can be treated in the same way with same results. E.g., the *telegraph* equation

$$E\,u_{,xx} - \eta\,u_{,t^q} - \rho\,u_{,tt} = r(x,t)$$

which degenerates for $\rho = 0$ to the diffusion equation. An example of higher order is the *Bernoulli* equation

$$E\,u_{,xxxx} + \eta\,u_{,xxxxt^q} + \rho\,u_{,tt} = 0 \, .$$

In this and other cases the result for noninteger q is always the same:

1. in case of global IC a solution of stable problems (if existing) is always unique, whereas
2. in case of local IC (e.g. for $t = 0$) uniqueness is not given.

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