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FRACTIONAL DERIVATIVES AND FRACTIONAL POWERS AS TOOLS IN UNDERSTANDING WENTZELL BOUNDARY VALUE PROBLEMS FOR PSEUDO-DIFFERENTIAL OPERATORS GENERATING MARKOV PROCESSES [1]

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Dedicated to Acad. Bogoljub Stanković,
on the occasion of his 80-th birthday

Abstract

Wentzell boundary value problem for pseudo-differential operators generating Markov processes but not satisfying the transmission condition are not well understood. Studying fractional derivatives and fractional powers of such operators gives some insights in this problem. Since an L^p– theory for such operators will provide a helpful tool we investigate the L^p–domains of certain model operators.

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0. Introduction

This paper consists of two almost independent parts. There is a research paper (Section 4 and 5) with an outlook to some applications (Section 6), in which we determine in terms of function spaces the L^p-domain of the generator of certain classes of L^p-sub-Markovian semigroups. More precisely, let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous negative definite function satisfying

$$\sup_{\xi\in\mathbb{R}} \frac{|\xi\varphi'(\xi)|}{(1+\varphi^2(\xi))^{1/2}} \leq c.$$

We determine the domain of the L^p-extension of the pseudo-differential operator with symbol $(\varphi(\xi) + i\eta)^\alpha$, $(\xi, \eta) \in \mathbb{R}^2$, $0 < \alpha < 1$. By the general theory it is clear that this extension is the generator of an L_p-sub-Markovian semigroup $(T_t^{(p)})_{t\geq 0}$ which is in fact a semigroup of contraction operators on $L^p(\mathbb{R}^2)$ and the symbol of T_t is given by $(\xi, \eta) \mapsto e^{-t(\varphi(\xi)+i\eta)^\alpha}$. To derive our result we depend on an application of the Lizorkin Fourier multiplier theorem and we have to use ψ-Bessel potential spaces as introduced in [5].

Of course we have good reasons to handle this problem and to relate it to fractional derivatives and fractional powers of operators. But these reasons are not easy to explain to someone not knowing the relations of pseudo-differential operators having a symbol $q(x, \xi)$ with the property that $\xi \mapsto e^{-tq(x,\xi)}$ is a continuous positive definite function, i.e. in the standard notation, $\xi \mapsto q(x, \xi)$ must be a continuous negative definite function, and Markov processes. Therefore we have decided to give instead of an extended introduction a short survey setting the scene (Sections 1–3). In this survey the reader will also encounter several other applications of fractional derivatives to the theory of Markov processes. In fact we will indicate some new results which are obtained as corollaries of already published work.

Besides providing background material we hope that our survey will stimulate experts working in fractional calculus to take up some of the problems mentioned.

The authors would like to thank Professor S. Samko for his kindness to invite us for writing a contribution to this special volume and accepting our unconventional paper.

1. Setting the scene

It is maybe worth to start with a remark on H. Weyl's paper [29] published in 1917, i.e. long before the notion of a closed operator in a Banach space or the construction of fractional powers of closed operators had been known, and of course long before pseudo-differential operators had been introduced. Weyl discussed two definitions of fractional derivatives. First he had considered functions $f : [0, \infty) \to \mathbb{R}$, $f(0) = 0$, and defined for $\alpha > 0$ the fractional derivative of f by $D^\alpha f = \varphi$ if $f = I^\alpha \varphi$, where $I^\alpha \varphi$ is the Riemann-Liouville fractional integral

$$I^\alpha \varphi(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-y)^{\alpha-1} \varphi(y)\, dy. \tag{1.1}$$

He then turned to periodic functions (with period 1) having mean value zero and defined the fractional derivative now by modifying the Fourier coefficients:

$$(D_\pi^\alpha f)(x) \sim \sum_{k\in\mathbb{Z}} e^{\frac{\pi i \alpha}{2}} (2\pi k)^\alpha c_k e^{2\pi i k x} \tag{1.2}$$

where c_k is the k^{th} Fourier coefficient of f. Finally he gave conditions on f in order that $D^\alpha f$ and $D_\pi^\alpha f$, respectively, do exist.

Nowadays we consider the definition (1.2) more in the context of taking fractional power of the pseudo-differential operator

$$(D_\pi f)(x) \sim \sum_{k\in\mathbb{Z}} i(2\pi k) c_k e^{2\pi i k x}.$$

More generally, if $q(x, D)$ is a pseudo-differential operator with symbol $q(x, \xi)$ (either defined on $\mathbb{R}^n \times \mathbb{R}^n$ or on $T^n \times \mathbb{Z}^n$) we want to identify the operator with symbol $q(x, \xi)^\alpha$, $\alpha \in \mathbb{R}$, with some operator obtained from $q(x, D)$ by using some functional calculus. Of course, when working with a suitable domain we can also apply this point of view to $D^\alpha f$. For example we know, compare S. Samko et al. [22],

$$\begin{aligned} D_-^\alpha f(x) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^\infty \frac{f(t)}{(x-t)^\alpha}\, dt \\ &= (2\pi)^{-1/2} \int_{\mathbb{R}} e^{ix\xi} (-i\xi)^\alpha \hat{f}(\xi)\, d\xi. \end{aligned} \tag{1.3}$$

But

$$D^1 f(x) = \frac{df}{dx}(x) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{ix\xi} (-i\xi) \hat{f}(\xi)\, d\xi.$$

Thus the study of pseudo-differential operators having a symbol which is the fractional power of the symbol of a given pseudo-differential operator has its natural home in fractional calculus.

Next let us point out that generators of Markov processes are essentially pseudo-differential operators. We need a few definitions. Let $\psi : \mathbb{R}^n \to \mathbb{C}$ be a continuous function such that $\psi(0) \geq 0$ and for all $t > 0$ the function $\xi \mapsto e^{-t\psi(\xi)}$ is positive definite. Unfortunately such functions are called continuous **negative definite functions**. Since $\xi \mapsto e^{-t\psi(\xi)}$ is positive definite, by Bochner's theorem it is the Fourier transform of a (sub-)probability measure μ_t. In fact we have a family $(\mu_t)_{t\geq 0}$ of measures and the convolution theorem implies that $(\mu_t)_{t\geq 0}$ must be a convolution semigroup of sub-probabilities, in particular we have $\mu_t * \mu_s = \mu_{t+s}$. Now we may use the Kolmogorov theory to find that for each $x \in \mathbb{R}^n$ there is a canonical stochastic process $(X_t)_{t\geq 0}$ starting at x and with state space $\mathbb{R}^n$ such that

$$E^x\left(e^{i\xi(X_t-x)}\right) = e^{-t\psi(\xi)} \tag{1.4}$$

holds. The process $(X_t)_{t\geq 0}$ is a Lévy process, it has stationary and independent increments. Probabilitists know ψ under the name **characteristic exponent**. The famous **Lévy-Khinchin formula** states that a continuous function $\psi : \mathbb{R}^n \to \mathbb{C}$ is a characteristic exponent if and only if it holds

$$\psi(\xi) = c_0 + id\cdot\xi + \sum_{k,l=1}^{n} q_{kl}\xi_k\xi_l + \int_{\mathbb{R}^n\setminus\{0\}} \left(1 - e^{-iy\cdot\xi} - \frac{iy\cdot\xi}{1+|y|^2}\right)\nu(dy), \tag{1.5}$$

where $c_0 \geq 0$, $d \in \mathbb{R}^n$, $q_{kl} = q_{lk} \in \mathbb{R}$ such that $\sum_{k,l=1}^{n} q_{kl}\xi_k\xi_l \geq 0$, and the Lévy measure ν integrates $y \mapsto 1 \wedge |y|^2$.

Do not underestimate (1.5), this formula determines the structure of objects we have to handle as we will see soon: they will be symbols of the pseudo-differential operators we are interested in.

Using the processes $(X_t)_{t\geq 0}$ associated with ψ and the starting points x by (1.4) we may introduce on $S(\mathbb{R}^n)$ the operators

$$(T_t u)(x) = E^x(u(X_t)) = \int_{\mathbb{R}} u(x-y)\mu_t(dy). \tag{1.6}$$

For $1 < p < \infty$ this family of operators extends to a strongly continuous contraction semigroup $(T_t^{(p)})_{t\geq 0}$ on $L^p(\mathbb{R}^n)$ with the additional property that $0 \leq u \leq 1$ a.e. always implies that $0 \leq T_t u \leq 1$ a.e. Such a semigroup is called an **L^p-sub-Markovian semigroup**. Using the Fourier transform, on $S(\mathbb{R}^n)$ we find

$$T_t u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} e^{-t\overline{\psi(\xi)}} \hat{u}(\xi)\, d\xi, \tag{1.7}$$

and for the L^p-generator $A^{(p)}$ it follows that $S(\mathbb{R}^n) \subset D(A^{(p)})$, and for $u \in S(\mathbb{R}^n)$ we have

$$A^{(p)}u(x) = -\overline{\psi}(D)u(x) = -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi}\overline{\psi(\xi)}\hat{u}(\xi)\, d\xi. \tag{1.8}$$

Thus, $A^{(p)}$ is the extension of a pseudo-differential operator with symbol $-\overline{\psi}$, $\overline{\psi}$ being a continuous negative definite function. Since $\overline{\psi}$ is a continuous negative definite function too, and since we will later on not depend on (1.4), we will now work with $\overline{\psi}$ instead ψ. The trouble is caused by the fact that in probability and analysis different signs in defining the Fourier transform are used.

To proceed further we need some knowledge on continuous negative definite functions. First, note that in general a continuous negative definite function need not be differentiable nor is it decomposable into a series of homogeneous functions with decreasing degrees of homogeneity α. – Bad news, because this tells us that in general the standard theory of pseudo-differential operators does not apply. Some concrete examples are given in Section 4, more in [11]. But here are some simple and important examples:

$$\xi \mapsto |\xi|^2, \quad \xi \mapsto |\xi|^{2\alpha}, \quad 0 < \alpha < 1, \quad \xi \mapsto -i\xi \cdot b, \quad b \in \mathbb{R}^n \quad \text{fixed.}$$

The first refers to the Laplacian, the second refers to $-(-\triangle)^\alpha$ and the third is just a drift operator $\sum_{j=1}^n b_j \frac{\partial}{\partial x_j}$. Of course these are examples related to a classical pseudo-differential calculus. S. Bochner's theory of **subordination** gives a possibility to construct new continuous negative definite function out of a given one, say ψ. We call $f \in C^\infty(0,\infty)$ a **Bernstein function** if $f \geq 0$ and $(-1)^k f^{(k)} \leq 0$, $k \in \mathbb{N}$. A fact is that $f \circ \psi$ is always a continuous negative definite function if ψ is. Now, for $0 < \alpha \leq 1$ the function $s \mapsto s^\alpha$ is a Bernstein function and it follows that $\xi \mapsto |\xi|^{2\alpha}$ as well as the symbol of the fractional derivative in (1.3), i.e. $(-i\xi)^\alpha$, are obtained by subordination. There is a nice functional or operational calculus related to Bernstein functions and generators of semigroups, we refer to F. Hirsch [6] and R. Schilling [23] and the references given there.

Note that there is also a probabilistic counterpart to subordination, in fact this is the core of the theory, but we do not need this here.

The problem, given a continuous negative definite function ψ, define $-\psi(D)$ by (1.7) on $S(\mathbb{R}^n)$ and try to extend this operator to become a generator of an L_p-sub-Markovian semigroup is already solved. But let

$G \subset \mathbb{R}^n$ be a set with smooth boundary $\partial G \neq \varnothing$ and consider $-\psi(D)$ as an operator with domain $C_0^\infty(G) \subset S(\mathbb{R}^n)$. Here is a non-trivial

Problem: Determine the extensions of $(-\psi(D), C_0^\infty(G))$ which generate a sub-Markovian semigroup on $L^p(\overline{G})$.

In the case $\psi(\xi) = |\xi|^2$, i.e. $-\psi(D) = \triangle$, we know that we need now boundary conditions, and different boundary conditions will lead (in general) to different extensions. But not all conditions on the boundary which will lead to an extension generating a strongly continuous contraction semigroup will give a sub-Markovian semigroup. The answer to this sub-problem is known: we do have to impose Wentzell boundary conditions. These conditions include Dirichlet and Neumann conditions as well as conditions described by an integro-differential operator on the boundary or operators modelling transitions from the boundary to the interior or vice versa. K. Taira in [25] or [26] gave a readable discussion of these conditions and in [10] we borrowed from him.

A final remark: Of course, there is no need to restrict our considerations to translation invariant operators. A result due to Ph. Courrège [3] states roughly that whenever $(A, C_0^\infty(G))$ extends to a generator of a semigroup we can associate a Markov process with, then A has the structure

$$A\,u = -q(x,D)u(x) = -(2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} q(x,\xi)\hat{u}(\xi)\,d\xi, \tag{1.9}$$

where $q : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ has the property that $\xi \mapsto q(x,\xi)$ is a continuous negative definite function.

Now we have stated our problem; in the next section we discuss surprises when trying to solve it.

2. Surprises

The Lévy-Khinchin formula allows us to decompose an operator $q(x,D)$ with symbol such that $\xi \mapsto q(x,\xi)$ is a continuous negative definite function into a sum of a second order differential operator with non-negative characteristic form and a non-local operator,

$$q(x,D) = L(x,D) + S(x,D).$$

We are interested in the case where $L(x,D)$ does not appear. There is a lot of work done and highly appreciated where $L(x,D)$ is a second order (degenerate) elliptic differential operator, or where $q(x,D)$ is a classical pseudo-differential operator satisfying the transmission condition. We refer to K. Taira [25]-[27] and the references given there. But our goal is to try to understand the case where the classical theory does not work and this is already interesting for Dirichlet and Neumann conditions. — In fact, knowing the Poisson and Green operator would lead to results for more general Wentzell boundary conditions.

The simplest problem of our interest could be:

Extend $-\psi_\alpha(D)$, $\sigma(\psi_\alpha(D))(\xi) = |\xi|^{2\alpha}$, $0 < \alpha < 1$, from $C_0^\infty(G)$ under Dirichlet or Neumann conditions to a generator of an L^p-sub-Markovian semigroup. Here $G \subset \mathbb{R}^n$ is open, $\partial G \neq \varnothing$ and smooth, and $\sigma(q(x,D))$ denotes the symbol of a given pseudo-differential operator.

The first surprise is that we have two choices to approach the problem leading to different results! We may consider the operator

$$\big((-\triangle)^\alpha u\big)(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\xi} |\xi|^{2\alpha} \hat{u}(\xi)\, d\xi. \tag{2.1}$$

Even for $u \in C_0^\infty(G) \subset S(\mathbb{R}^n)$, there is no need to restrict x in (2.1) to G. In fact, this would be artificial. Thus we should work with $((-\triangle)^\alpha, S(\mathbb{R}^n))$ or $((-\triangle)^\alpha, H^{2\alpha}(\mathbb{R}^n))$ $(p = 2)$ and try to solve

$$\begin{aligned} &(-\triangle)^\alpha u(x) = 0 \quad \text{on } G \\ &u\Big|_{\partial G} = f \quad \text{on} \quad \partial G \end{aligned} \tag{2.2}$$

for a suitable function f. It is well-known that this problem is not a reasonable one, see M. Riesz [21], N. Landkof [19], J. Bliedtner, W. Hansen [2], or in a more general case W. Hoh et al. [7]. A reasonable problem would be

$$\begin{aligned} &(-\triangle)^\alpha u(x) = 0 \quad \text{in } G \\ &u\Big|_{G^c} = f \quad \text{in} \quad G^c \end{aligned} \tag{2.3}$$

for a suitable function f. For a large class of pseudo-differential operators the problem

$$\begin{aligned} &q(x,D)u = 0 \quad \text{in } G \\ &u\Big|_{G^c} = f \end{aligned} \tag{2.4}$$

can be solved using balayage theory or Hilbert space methods or probabilistic methods. But this is not anymore a Wentzell boundary value problem.

We have to work in $\mathbb{R}^n$ not only in $\overline{G}$! In fact, the probabilistic solution gives the best insight: $-(-\triangle)^\alpha$ generates a jump process and this process may jump from G into G^c, i.e. we need "all of $\mathbb{R}^n$" to understand the corresponding process. Another way to understand this result is to realize that the associated harmonic measure is supported in G^c. Thus balayage theory does not give a clue how to handle our problem — it is a misleading approach, referring to a different type of problem.

Now, there is a second approach. Clearly we can prove that the Dirichlet problem and the Neumann problem for $-\triangle$ in G is solvable (∂G sufficiently smooth) and $(-\triangle_D^{(p)}, D(-\triangle_D^{(p)}))$ as well as $(-\triangle_N^{(p)}, D(-\triangle_N^{(p)}))$ are generators of L^p-sub-Markovian semigroups. In [15] R. Schilling and coauthor studied the problem to subordinate these operators in case $p = 2$, i.e. the Hilbert space case under homogeneous boundary conditions for simplicity. In fact they studied more general second order elliptic operators and in [4] W. Farkas and coauthor extended these results to non-smooth boundaries. Only the Bernstein function $s \mapsto s^\alpha$, $0 < \alpha < 1$, had been considered. Clearly, for the L^2-closures we get $D(-\triangle_D^{(2)}) = H^2(G) \cap H_0^1(G)$ and $D(-\triangle_N^{(2)}) = H^2(G)$. Further we find in his case that $D(-(-\triangle_D^{(2)})^\alpha)$ and $D(-(-\triangle_N^{(2)})^\alpha)$ can be determined by complex interpolation and we get for the Dirichlet problem

$$D((-\triangle_D^{(2)})^\alpha) = H^{2\alpha}(G), \quad 0 < \alpha < \frac{1}{4}, \tag{2.5}$$

and for the Neumann problem

$$D((-\triangle_N^{(2)})^\alpha) = H^{2\alpha}(G), \quad 0 < \alpha < \frac{3}{4}, \tag{2.6}$$

if ∂G is smooth, for the non-smooth case see [4]. (The other cases for α are the nice cases and not listed here, see however [15]). The surprise is that for certain values of α after subordination we cannot recover the boundary behaviour. Clearly in $H_0^1(G)$ are functions with trace zero, but elements in $H^\alpha(G)$, $\alpha < \frac{1}{4}$, do not have a trace! Analogous is the situation with the Neumann problem. Thus solving our original problem by considering subordinated Dirichlet or Neumann Laplacians (just as example) may lead to a situation where we loose all boundary behaviour!

This result has a probabilistic counterpart: the associated subordinate process does not "see" ∂G since it has capacity zero!

Thus, two approaches, different solutions and in the more interesting case we may loose control on the boundary behaviour! We may even enlarge the confusion: The results (2.5) and (2.6) are obtained by applying a result of R. Seeley [24]. Compare also H. Triebel [28], in case $p = 2$. But Seeley's result does hold also for $1 < p < \infty$. Thus in the more general case we will end up with some $\alpha_{D/N}(p)$ where for $\alpha < \alpha_{D/N}(p)$ boundary data for the Dirichlet/Neumann problem will "disappear" after subordination. Clearly, for $p \to \infty$ we may end up with the Feller case, i.e. a semigroup on $C_b(\overline{G})$ and elements in $C_b(\overline{G})$, ∂G smooth, do have a boundary behaviour!

3. Searching for examples to get some insights

Obviously, to get some insides we may and we shall simplify the geometry of G. The most reasonable choice is to work in a half-space $\mathbb{R}^n_+ := \{(x_1, \ldots, x_{n+1});\ (x_1, \ldots, x_n) \in \mathbb{R}^n \text{ and } x_{n+1} > 0\}$ with boundary $\Gamma = \{(x_1, \ldots, x_n, 0);\ (x_1, \ldots, x_n) \in \mathbb{R}^n\}$ and to consider operators which decompose into a normal and tangential part with respect to the boundary

$$Au = A_{(n)}u + A_{(n+1)}u, \tag{3.1}$$

where $A_{(n)}$ acts only on the coordinates $x_1, \ldots, x_n$ and $A_{(n+1)}$ acts on the coordinate $x_{(n+1)}$. If $A_{(n+1)}$ satisfies the transmission condition we should be able to reduce the problem to a problem which has a chance to be handled in a classical setting (Boutet de Monvel approach). This is exactly the frame of K. Taira's work, but this is the case we want to exclude. In order to study the boundary behaviour we may ignore $A_{(n)}u$ for some time and consider only $A_{(n+1)}$. When longing for a Feller smigroup or an L_p-sub-Markovian semigroup $A_{(n+1)}$ should satisfy the positive maximum principle

$$\sup u(y) = u(y_0) \geq 0 \quad \text{implies} \quad (A_{(n+1)}u)(y_0) \leq 0. \tag{3.2}$$

Clearly, if $A_{(n+1)} = -\big(-\frac{d^2}{dx_{n+1}^2}\big)^\alpha$, $0 < \alpha < 1$, we run into the trouble mentioned in the last section. In [14] A. Krägeloh et.al. studied the case where $A_{(n+1)}$ is a fractional derivative of some order $0 < \alpha < 1$ defined on the half axis. Thus the problem was to find when the Dirichlet or the Neumann problem (or some other problems) for $\big(-\frac{d}{dx}\big)^\alpha$ on the half axis is solvable and the operator $-\big(-\frac{d}{dx}\big)^\alpha$ with a "nice" domain incorporating the boundary condition satisfies the positive maximum problem.

Of course on $C_0^\infty(\mathbb{R}_+)$ we may represent $\big(-\frac{d}{dx}\big)^\alpha$ using the Fourier transform, see (1.3). But we should not expect to work with extensions having such a representation for all elements in its domain. The main result in [14], see also A. Krägeloh [18], was that not all extensions give operators satisfying the positive maximum principle. In particular it was proved that the Caputo form yields "good" operators in our context: Denote by

$$D_C^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-y)^{-\alpha} f'(y)\,dy, \quad x > 0, \tag{3.3}$$

the Caputo form of the fractional derivative of order $\alpha \in (0,1)$, and set

$$D_D^1 := \{f \in \dot{C}_\infty(\mathbb{R}_0^+) \cap C^1(\mathbb{R}_0^+); \quad f' \in \dot{C}_\infty(\mathbb{R}_0^+)\} \tag{3.4}$$

as well as

$$D_N^1 := \{f \in C_\infty(\mathbb{R}_0^+) \cap C^1(\mathbb{R}_0^+); \quad f' \in C_\infty(\mathbb{R}_0^+)\} \tag{3.5}$$

where $\dot{C}_\infty(\mathbb{R}_0^+) = \{f \in C_\infty(\mathbb{R}_0^+);\ f(0) = 0\}$. Then, $(-D_C^\alpha, D_D^1)$ and $(-D_C^\alpha, D_N^1)$ extend to generators of Feller semigroups. It is interesting to note that a similar result for the Robin or 3^{rd} boundary condition does not hold. In [18] A. Krägeloh then handled the case where $A_{(n)} = -p(x, D_x)$ and $p(x, D_x)$ was a pseudo-differetial operator as handle in [9] (or related papers, see [12], Section 2.6).

We want to switch to an L^p-theory, take $p(x, D_x)$ to be independent of x (for simplicity in the begining) and compare $-D_C^\alpha - p(D_x)$ with the operator being subordinate to the operator with symbol $i\xi_{n+1} + \psi(\xi')$, $\xi' = (\xi_1, \ldots, \xi_n)$ and $p^{1/\alpha}(\xi') = \psi(\xi')$ (just as an example), subjected to some boundary conditions. Our first task should be to determine the L^p-domain of $A_\pm$, $\sigma(A_\pm)(\xi_1, \ldots, \xi_{n+1}) = \pm i\xi_{n+1} + \psi(\xi')$ when acting on functions defined on $\mathbb{R}^{n+1}$ or $\mathbb{R}_+^{n+1}$.

4. Preliminary remarks and results for determining $D(A^{(p)})$

Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous negative definite function which is one time continuously differentiable and satisfies the following condition

$$\left|\left(\frac{d}{d\xi}\right)^j \varphi(\xi)\right| \le c_j (1 + |\xi|^2)^{\frac{m-|j|}{2}}, \quad j = 0, 1, \tag{4.1}$$

and

$$1 + \varphi(\xi) \ge c_2 (1 + |\xi|^2)^{m/2} \tag{4.2}$$

for all $\xi \in \mathbb{R}$. Clearly we must have $0 \leq m \leq 2$. From (4.1) and (4.2) it follows that

$$\frac{|\varphi'(\xi)||\xi|}{(1+\varphi^2(\xi))^{1/2}} \leq \frac{2|\varphi'(\xi)||\xi|}{1+\varphi(\xi)} \leq \frac{2c_0(1+|\xi|^2)^{\frac{m-1}{2}}(1+|\xi|^2)^{1/2}}{c_2(1+|\xi|^2)^{m/2}} = \frac{2c_0}{c_2},$$

i.e.

$$\sup_{\xi\in\mathbb{R}} \frac{|\varphi'(\xi)||\xi|}{(1+\varphi^2(\xi))^{1/2}} \leq const. \tag{4.3}$$

Examples of such functions are:

$$\xi \mapsto (1+|\xi|^2)^{\beta} \quad \text{for} \quad 0<\beta\leq 1, \tag{4.4}$$

$$\xi \mapsto 1+\delta\ln\Big(cosh^2\big(\frac{a\xi}{2}\big)-\sin^2\big(\frac{b}{2}\big)\Big)-2\delta\ln\big(\cos\frac{b}{2}\big), \tag{4.5}$$

for $\delta > 0$, $a > 0$ and $-\pi < b < \pi$, and

$$\xi \mapsto 1+\delta\Big(\sqrt{(\alpha^2-\beta^2+\xi^2)^2+(\beta\xi)^2}\cos\big(\frac{1}{2}arctan\frac{-\beta\xi}{\alpha^2-\beta^2+\xi^2}\big)-\sqrt{\alpha^2-\beta^2}\Big), \tag{4.6}$$

where $\delta > 0$ and $0 < |\beta| < \alpha$.

It is interesting to note that the class of continuous negative definite functions $\varphi : \mathbb{R} \to \mathbb{R}$, satisfying (4.3) is invariant under subordination, i.e. composition with a Bernstein function. Indeed, let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous negative definite function such that $\varphi(0) > 0$ and let $f : (0,\infty) \to \mathbb{R}$ be a Bernstein function. Since for every Bernstein function it holds

$$\frac{f'(s)}{f(s)} \leq \frac{1}{s}, \quad s>0, \tag{4.7}$$

compare [11], Lemma 3.9.34, a straightforward calculation yields

$$\Big|\xi\frac{d}{d\xi}f(\varphi(\xi))\Big| = |\xi\varphi'(\xi)f'(\varphi(\xi))| \leq \frac{|\xi\varphi'(\xi)|f(\varphi(\xi))}{\varphi(\xi)},$$

implying

$$\frac{\Big|\xi\frac{d}{d\xi}f(\varphi(\xi))\Big|}{(1+f^2(\varphi(\xi)))^{1/2}} \leq \frac{|\xi\varphi'(\xi)|}{|\varphi(\xi)|}. \tag{4.8}$$

Since $\xi \mapsto f(\varphi(\xi))$ is again a continuous negative definite function and since for $\varphi(0) > 0$ we may replace (4.2) by

$$\varphi(\xi) \geq c_2'(1 + |\xi|^2)^{m/2}, \tag{4.9}$$

we are now in position to provide a large class of non-trivial examples of continuous negative definite functions satisfying condition (4.3).

Next let $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous negative definite function. Then by $(\xi, \eta) \mapsto \varphi(\xi) + i\eta$ a continuous negative definite function on $\mathbb{R}^2$ is defined. Further, since for $0 < \alpha < 1$ the function $s \mapsto s^\alpha$, $s \geq 0$, is a Bernstein function, it follows that

$$\psi_\alpha(\xi, \eta) := (1 + \varphi(\xi) + i\eta)^\alpha \tag{4.10}$$

is a (complex-valued) continuous negative definite function on $\mathbb{R}^2$. We may decompose ψ_α according to

$$\begin{aligned} \psi_\alpha(\xi, \eta) &= ((1 + \varphi(\xi))^2 + \eta^2)^{\alpha/2} e^{i\alpha\, arg(1+\varphi(\xi)+i\eta)} \\ &= ((1 + \varphi(\xi))^2 + \eta^2)^{\alpha/2} \Big(\cos\big(\alpha\, arctan \frac{\eta}{1+\varphi(\xi)}\big) \\ &\quad + i \sin\big(\alpha\, arctan \frac{\eta}{1+\varphi(\xi)}\big)\Big). \end{aligned} \tag{4.11}$$

Further we find

$$\begin{aligned} \frac{|Im\, \psi_\alpha(\xi, \eta)|}{|Re\, \psi_\alpha(\xi, \eta)|} &= \frac{\big|\sin\big(\alpha\, arctan \frac{\eta}{1+\varphi(\xi)}\big)\big|}{\big|\cos\big(\alpha\, arctan \frac{\eta}{1+\varphi(\xi)}\big)\big|} \\ &= \big|\tan\big(\alpha\, arctan \frac{\eta}{1+\varphi(\xi)}\big)\big|. \end{aligned}$$

Since $-\infty < \frac{\eta}{1+\varphi(\xi)} < \infty$ for $(\xi, \eta) \in \mathbb{R}^2$, we get first

$$-\frac{\pi}{2} \leq arctan \frac{\eta}{1+\varphi(\xi)} \leq \frac{\pi}{2},$$

and therefore we derive

$$-\frac{\alpha\pi}{2} \leq \alpha\, arctan \frac{\eta}{1+\varphi(\xi)} \leq \frac{\alpha\pi}{2},$$

which implies for $0 < \alpha < 1$ that

$$\big|\tan(\alpha \arctan \frac{\eta}{1+\varphi(\xi)})\big| \leq \tan \frac{\alpha\pi}{2}$$

or

$$|Im\,\psi_\alpha(\xi,\eta)| \le \tan(\frac{\alpha\pi}{2}) Re\,\psi_\alpha(\xi,\eta). \tag{4.12}$$

Hence the continuous negative definite function ψ_α satisfies the sector condition, compare Chr. Berg and G. Forst [1], [11] or [16]. Consequently, we may associate a non-symmetric Dirichlet form $(\mathcal{E}^{\psi_\alpha}, D(\mathcal{E}^{\psi_\alpha}))$ with ψ_α. The domain of $D(\mathcal{E}^{\psi_\alpha})$ is of course $H_2^{Re\psi_\alpha,1}(\mathbb{R}^2)$ defined by

$$H_2^{Re\,\psi_\alpha,1}(\mathbb{R}^2) := \{u \in L^2(\mathbb{R}^2) \quad \|u\|_{H_2^{Re\,\psi_\alpha,1}} < \infty\} \tag{4.13}$$

where

$$\|u\|^2_{H_2^{Re\,\psi_\alpha,1}} = \int_{\mathbb{R}}\int_{\mathbb{R}} (1 + (Re\,\psi_\alpha(\xi,\eta)))|\hat{u}(\xi,\eta)|^2\, d\xi d\eta. \tag{4.14}$$

Since

$$Re\,\psi_\alpha(\xi,\eta) = ((1+\varphi(\xi))^2 + \eta^2)^{\alpha/2} \cos(\alpha \arctan \frac{\eta}{1+\varphi(\xi)})$$

and

$$0 < \cos\frac{\alpha\pi}{2} \le \cos(\alpha \arctan \frac{\eta}{1+\varphi(\xi)}) \le 1$$

we may take instead of (4.14) the equivalent norm

$$\Big(\int_{\mathbb{R}}\int_{\mathbb{R}} \big(1 + (\varphi^2(\xi) + \eta^2)^{\alpha/2}\big)|\hat{u}(\xi,\eta)|^2\, d\xi d\eta\Big)^{1/2} \tag{4.15}$$

and we denote this norm once again by $\|u\|_{H_2^{Re\,\psi_\alpha,1}}$. More generally we introduce for $s > 0$ the norm

$$\|u\|^2_{H_2^{Re\,\psi_\alpha,s}} = \int_{\mathbb{R}}\int_{\mathbb{R}} \big(1 + (\varphi^2(\xi) + \eta^2)^{\alpha/2}\big)^s |\hat{u}(\xi,\eta)|^2\, d\xi d\eta \tag{4.16}$$

which gives the Hilbert space

$$H_2^{Re\,\psi_\alpha,s}(\mathbb{R}^2) = \{u \in L^2(\mathbb{R}^2); \quad \|u\|_{H_2^{Re\,\psi_\alpha,s}} < \infty\}.$$

By standard arguments using the Plancherel theorem we arrive at the following

Proposition 4.1. *The pseudo-differential operator $-\psi_\alpha(D)$ defined on $S(\mathbb{R}^2)$ by*

$$-\psi_\alpha(D)u(x,y) = -(2\pi)^{-1}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(x\xi+y\eta)}\psi_\alpha(\xi,\eta)\hat{u}(\xi,\eta)\,d\xi d\eta \tag{4.17}$$

extends to the generator $(A^{(2)}, D(A^{(2)}))$ of the non-symmetric Dirichlet form $(\mathcal{E}^{\psi_\alpha}, H_2^{Re\,\psi_\alpha,1}(\mathbb{R}^2))$, where

$$\mathcal{E}^{\psi_\alpha}(u,v) = \int_{\mathbb{R}}\int_{\mathbb{R}} \psi_\alpha(\xi,\eta)\hat{u}(\xi,\eta)\overline{\hat{v}(\xi,\eta)}\,d\xi\,d\eta. \tag{4.18}$$

The domain of $A^{(2)}$ is given by

$$D(A^{(2)}) = H_2^{Re\,\psi_\alpha,2}(\mathbb{R}^2) \tag{4.19}$$

and for $u \in D(A^{(2)})$, $v \in D(\mathcal{E}^{\psi_\alpha})$ we have

$$\mathcal{E}^{\psi_\alpha}(u,v) = (-A^{(2)}u, v)_{L^2}.$$

(For a proof of this proposition compare Chr. Berg and G. Forst [1]), or [16]).

The operator $-\psi_\alpha(D)$ with domain $S(\mathbb{R}^2)$ is known to have an extension $A^{(p)}$ which generates an L^p-sub-Markovian semigroup $(T_t^{(p)})_{t\geq 0}$. On $S(\mathbb{R}^2)$ we have for $1 \leq p < \infty$ the representation

$$T_t^{(p)}u(x,y) = (2\pi)^{-1}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{i(x\xi+y\eta)}e^{-t\psi_\alpha(\xi,\eta)}\hat{u}(\xi,\eta)\,d\xi d\eta. \tag{4.20}$$

Our problem is to determine $D(A^{(p)})$, i.e. the domain of the L^p-generator of $(T_t^{(p)})_{t\geq 0}$, in terms of function spaces.

5. On the domain of $A^{(p)}$

In order to study the operator $\psi_\alpha(D)$ as an L^p-operator we prove first a Fourier multiplier theorem.

Theorem 5.1. *Let ψ_α be defined by* (4.10) *and suppose that φ satisfies* (4.3). *Then the function*

$$(\xi,\eta) \mapsto e^{i\alpha arg(1+\varphi(\xi)+i\eta)} \tag{5.1}$$

is an L^p-Fourier multiplier, $1 < p < \infty$.

P r o o f. Clearly $(\xi,\eta)\mapsto e^{i\alpha arg(1+\varphi(\xi)+i\eta)}$ belongs to $L^\infty(\mathbb{R}^2)$, and in order to apply the Lizorkin Fourier multiplier theorem, compare P.F. Lizorkin [20] or [11], p.241, we estimate

$$\eta\xi\frac{\partial^2}{\partial\eta\partial\xi}\Big(e^{i\alpha arg(1+\varphi(\xi)+i\eta)}\Big).$$

An elementary, but lengthy calculations gives first

$$\begin{aligned}
&\frac{\partial^2}{\partial\eta\partial\xi}\Big(e^{i\alpha arg(1+\varphi(\xi)+i\eta)}\Big)\\
&=\frac{\alpha\varphi'(\xi)(\eta^2-(1+\varphi(\xi))^2)}{(\eta^2+(1+\varphi(\xi))^2)^2}\Big(-\sin\big(\alpha\arctan\frac{\eta}{1+\varphi(\xi)}\big)\\
&\qquad+i\cos\big(\alpha\arctan\frac{\eta}{1+\varphi(\xi)}\big)\Big)\\
&\quad+\frac{\alpha^2\varphi'(\xi)(1+\varphi(\xi))\eta}{(\eta^2+(1+\varphi(\xi))^2)^2}\Big(\cos\big(\alpha\arctan\frac{\eta}{1+\varphi(\xi)}\big)\\
&\qquad+i\sin\big(\alpha\arctan\frac{\eta}{1+\varphi(\xi)}\big)\Big),
\end{aligned}$$

and now we may estimate

$$\begin{aligned}
&\Big|\eta\xi\frac{\partial^2}{\partial\eta\partial\xi}\Big(e^{i\alpha arg(1+\varphi(\xi)+i\eta)}\Big)\Big|\\
&\quad\le\frac{\alpha|\varphi'(\xi)||\xi||\eta|}{\eta^2+(1+\varphi(\xi))^2}+\frac{\alpha^2|\varphi'(\xi)|(1+\varphi(\xi))|\eta|^2|\xi|}{(\eta^2+(1+\varphi(\xi))^2)^2}\\
&\le\frac{2|\varphi'(\xi)||\xi||\eta|}{\eta^2+(1+\varphi(\xi))^2}\\
&=\frac{2|\eta|}{(\eta^2+(1+\varphi(\xi))^2)^{1/2}}\cdot\frac{|\varphi'(\xi)||\xi|}{(\eta^2+(1+\varphi(\xi))^2)^{1/2}}\le C',
\end{aligned}$$

for all $\xi,\eta\in\mathbb{R}$ where we used (4.3) to estimate the second term in the last line.

From our calculations it is easy to derive also the bounds for

$$\eta\frac{\partial}{\partial\eta}\big(e^{i\alpha\, arg(1+\varphi(\xi)+i\eta)}\big)\quad\text{and}\quad\xi\frac{\partial}{\partial\xi}\big(e^{i\alpha\, arg(1+\varphi(\xi)+i\eta)}\big).$$

Thus, by the Lizorkin Fourier multiplier theorem the proposition is proved. ∎

For a real-valued continuous negative defined function $\psi : \mathbb{R}^n \to \mathbb{R}$ the space $H_p^{\psi,s}(\mathbb{R}^n)$ had been introduced and discussed in detail in [5]. On $S(\mathbb{R}^n)$ which turns out to be a dense subspace of each of spaces $H_p^{\psi,s}(\mathbb{R}^n)$, $s \geq 0$, $1 < p < \infty$, the norm $\|\cdot\|_{H_p^{\psi,s}}$ is given by

$$\|u\|_{H_p^{\psi,s}} = \|F^{-1}((1+\psi(\cdot))^{s/2}\hat{u}(\cdot))\|_{L^p}. \tag{5.2}$$

Now, in general for a continuous negative definite function $\varphi : \mathbb{R} \to \mathbb{R}$ the function $\tilde{\psi}_\alpha(\xi,\eta) := (1+\varphi^2(\xi)+\eta^2)^{\alpha/4}$ need not to be a continuous negative definite function since φ^2 need not be negative definite. However, by inspection one may prove that the norms

$$\|u\|_{H_p^{\tilde{\psi}_\alpha,s}} = \|F^{-1}((1+\psi_\alpha)^{s/2}\hat{u})\|_{L^p} \tag{5.3}$$

lead to Banach space $H_p^{\tilde{\psi}_\alpha,s}(\mathbb{R}^2)$ having all properties as the spaces $H_p^{\psi,s}(\mathbb{R}^2)$, $\psi : \mathbb{R}^2 \to \mathbb{R}$ being a continuous negative definite function, $s \geq 0$ and $1 < p < \infty$, except that the contractions need not to operate on $H_p^{\tilde{\psi}_\alpha,s}(\mathbb{R}^2)$ for $0 < s \leq 1$.

PROPOSITION 5.2. *The operator $\psi_\alpha(D)$ maps the space $H_p^{\tilde{\psi}_\alpha,2}(\mathbb{R}^2)$ continuously into $L^p(\mathbb{R}^2)$, $1 < p < \infty$, where*

$$\tilde{\psi}_\alpha(\xi,\eta) = (1+\varphi^2(\xi)+\eta^2)^{\alpha/2}, \tag{5.4}$$

i.e. we have the estimate

$$\|\psi_\alpha(D)u\|_{L^p} \leq c\|u\|_{H_p^{\tilde{\psi}_\alpha,2}} \tag{5.5}$$

for all $u \in H_p^{\tilde{\psi}_\alpha,2}(\mathbb{R}^2)$.

P r o o f. For $u \in S(\mathbb{R}^2)$ we find (with some abuse of notation which is however selfexplaining)

$$\begin{aligned}
&\|\psi_\alpha(D)u\|_{L^p} = \|F^{-1}(\psi_\alpha\hat{u})\|_{L^p}\\
&= \|F^{-1}\Big((1+\varphi^2(\xi)+\eta^2)^{\alpha/2}e^{i\alpha arg(1+\varphi(\xi)+i\eta)}\hat{u}(\xi,\eta)\Big)\|_{L^p}\\
&= \|F^{-1}\Big(e^{i\alpha arg(1+\varphi(\xi)+i\eta)}F(F^{-1}((1+\varphi^2(\xi)+\eta^2)^{\alpha/2}\hat{u}(\xi,\eta)))\Big)\|_{L^p}\\
&\leq c\|F^{-1}\Big((1+\varphi^2(\xi)+\eta^2)^{\alpha/2}\hat{u}(\xi,\eta)\Big)\|_{L^p} = c\|u\|_{H_p^{\tilde{\psi}_\alpha,2}},
\end{aligned}$$

where we used for the crucial estimate of course Theorem 5.1. ∎

PROPOSITION 5.3. *For all $u \in H_p^{\tilde{\psi}_\alpha,2}(\mathbb{R}^2)$, $1 < p < \infty$, the lower estimate*

$$\|\psi_\alpha(D)u\|_{L^p} \geq c\|u\|_{H_p^{\tilde{\psi}_\alpha,2}} \tag{5.6}$$

is satisfied.

P r o o f. It is sufficient to prove (5.6) for all $u \in S(\mathbb{R}^2)$. For $u, v \in S(\mathbb{R}^2)$ we find using the estimate

$$\|F^{-1}\Big(e^{i\arg\psi_\alpha(\xi,\eta)}\hat{v}\Big)\|_{L^{p'}} \leq c\|v\|_{L^{p'}}$$

that

$$\begin{aligned}
&\|\psi_\alpha(D)u\|_{L^p}\|v\|_{L^{p'}} \geq c\|F^{-1}(\psi_\alpha\hat{u})\|_{L^p}\|F^{-1}\Big(e^{i\arg\psi_\alpha(\xi,\eta)}\hat{v}\Big)\|_{L^{p'}} \\
&= c\|F^{-1}\Big(|\psi_\alpha|e^{i\arg\psi_\alpha(\xi,\eta)}\hat{u}\Big)\|_{L^p}\|F^{-1}\Big(e^{i\arg\psi_\alpha(\xi,\eta)}\hat{v}\Big)\|_{L^{p'}} \\
&\geq c\Big|\int_{\mathbb{R}}\int_{\mathbb{R}} F^{-1}\Big(|\psi_\alpha|e^{i\arg\psi_\alpha(\xi,\eta)}\hat{u}\Big)\overline{F^{-1}\Big(e^{i\arg\psi_\alpha(\xi,\eta)}\hat{v}\Big)}\, d\xi d\eta\Big| \\
&= c\Big|\int_{\mathbb{R}}\int_{\mathbb{R}} |\psi_\alpha|\hat{u}\overline{\hat{v}}\, d\xi d\eta\Big| \\
&= c\Big|\int_{\mathbb{R}}\int_{\mathbb{R}} (1+\varphi(\xi)^2+\eta^2)^{\alpha/2}\hat{u}(\xi,\eta)\overline{\hat{v}(\xi,\eta)}\, d\xi d\eta\Big| \\
&= c\Big|\int_{\mathbb{R}}\int_{\mathbb{R}} F^{-1}\big((1+\varphi(\xi)^2+\eta^2)^{\alpha/2}\hat{u}\big)\overline{F^{-1}\hat{v}} d\xi d\eta\Big| \\
&= c\Big|\Big(F^{-1}\big((1+\varphi(\xi)^2+\eta^2)^{\alpha/2}\hat{u}\big), v\Big)_{L^2}\Big|,
\end{aligned}$$

implying

$$\|\psi_\alpha(D)u\|_{L^p} \geq c' \frac{\big|\big(F^{-1}\big((1+\varphi(\xi)^2+\eta^2)^{\alpha/2}\hat{u}\big), v\big)_{L^2}\big|}{\|v\|_{L^{p'}}}.$$

Taking the supremum over all $v \in S(\mathbb{R}^2) \subset L^{p'}(\mathbb{R}^2)$ we arrive at

$$\|\psi_\alpha(D)u\|_{L^p} \geq c'\|F^{-1}((1+\varphi(\xi)^2+\eta^2)^{\alpha/2}\hat{u})\|_{L^p} = c'\|u\|_{H_p^{\tilde{\psi}_\alpha,2}}$$

proving the proposition. ∎

Combining Proposition 5.2 and Proposition 5.3, we finally derive

THEOREM 5.4. *The domain of the closure of $(\psi_\alpha(D), S(\mathbb{R}^2))$ in $L^p(\mathbb{R}^2)$, $1 < p < \infty$, is the space $H_p^{\tilde{\psi}_\alpha,2}(\mathbb{R}^2)$.*

P r o o f. This follows from the estimates

$$\gamma_0 \|u\|_{H_p^{\tilde{\psi}_\alpha,2}} \leq \|\psi_\alpha(D)u\|_{L^p} \leq \gamma_1 \|u\|_{H_p^{\tilde{\psi}_\alpha,2}} \tag{5.7}$$

and the obvious estimate $\|u\|_{L^p} \leq \|u\|_{H_p^{\tilde{\psi}_\alpha,2}}$. ■

REMARK 5.5. As pointed out in the introduction, the behaviour of $(\xi,\eta) \mapsto (\psi(\xi)+i\eta)^\alpha$ with respect to η is considered to play a different rôle compared with that to ξ. The y - or η - direction should be seen as a direction normal to the boundary, the x - or ξ - direction should be seen as a direction tangent to a boundary. For this reason, considering the one-dimensional case with respect to x (or ξ) is sufficient for our case study. However, there in no doubt that the higher dimensional case (with respect to ξ) is of course also of interest, especially when thinking on applications. In her PhD-thesis, the second author is considering this question in detail for some larger classes of continuous negative definite functions. While her general approach is along the lines of the considerations in this paper, calculations are more involved and of course condition (4.3) must be extended in order to apply the Lizorkin Fourier multiplier theorem. For details we refer to [17].

6. Back to the boundary value problem

In this section we outline recent results contained in the second author's PhD-thesis. We do not long to contain her most general results but we want to give some ideas.

Clearly we may study a "Dirichlet" problem

$$\begin{aligned} -\frac{\partial u(x,y)}{\partial y} + \varphi(D_x)u(x,y) = 0, \quad x \in \mathbb{R}, \quad y > 0, \\ u(x,0) = g(x), \quad x \in \mathbb{R}, \end{aligned} \tag{6.1}$$

where

$$\varphi(D_x)u(x,y) = (2\pi)^{-1/2} \int_{\mathbb{R}} e^{ix\xi} \varphi(\xi) \big(F_{x\mapsto\xi} u(\cdot,y)\big)(\xi,y)\, d\xi,$$

and $\varphi : \mathbb{R} \to \mathbb{R}$ is as in Section 5. Further we may consider $\frac{\partial}{\partial y} - \varphi(D_x)$ as pregenerator of an L^p-sub-Markovian semigroup on $L^p(\mathbb{R}^2)$. In fact we have more

Theorem 6.1. *In the situation of Theorem 5.4 the operator $\psi_\alpha(D)$ extends from $S(\mathbb{R}^2)$ to a generator of an L^p-sub-Markovian semigroup with domain $H_p^{\tilde{\psi}_\alpha,2}(\mathbb{R}^2)$.*

The question is whether we can get some type of "fractional power version" for (6.1).

For this we introduce the space

$$\tilde{H}_{p,+}^{\tilde{\psi}_\alpha,s} := \{f; \quad f \in H_p^{\tilde{\psi}_\alpha,s} \quad \text{and} \quad supp\, f \subset \mathbb{R}^2_{0,+}\} \tag{6.2}$$

Now with some efforts, mainly by following the classical case as discussed in H. Triebel [28], one can prove, compare V. Knopova [17],

Theorem 6.2. *Let φ satisfy the assumptions of Theorem 6.1. Then the operator with symbol $(1+i\eta+\varphi(\xi))^\alpha$ is an isomorphism between $\tilde{H}_{p,+}^{\tilde{\psi}_\alpha,t}$ and $\tilde{H}_{p,+}^{\tilde{\psi}_\alpha,t-2}$, $t \geq 2$.*

Using this result as well as the fact that $-\big(\frac{\partial}{\partial y} + \varphi(D_x)\big)^\alpha$ extends from $H^1_{p,+}(\mathbb{R}_{0,+}) \otimes H_p^{\varphi,2\alpha}(\mathbb{R})$ to a generator of an L^p-sub-Markovian yields finally

Theorem 6.3. *Let φ be as in Theorem 5.4. The operator $\Big(-\big(\frac{\partial}{\partial y} + \varphi(D_x)\big)^\alpha, \tilde{H}_{p,+}^{\tilde{\psi}_\alpha,2}\Big)$ generates an L^p-sub-Markovian semigroup by*

$$T_t u(x,y) = \int_{\mathbb{R}} \Big(\int_0^y u(x-x', y-s)\mathcal{W}_s(x')\sigma^{(\alpha)}(s)\,ds\Big)\,dx'$$

where $\sigma^{(\alpha)}(s)$ satisfies

$$e^{-tz^\alpha} = \int_0^\infty e^{-zs}\sigma^{(\alpha)}(s)\,ds$$

and $\mathcal{W}_s(\cdot)$ is the inverse Fourier transform of $e^{-\varphi(\xi)s}$.

This final theorem is just a proto-type result of those obtained in [17] by V. Knopova. These results partly extend those obtained by A. Krägeloh [18], see also [13] and [14], in the setting of Feller semigroups to the L^p-sub-Markovian semigroup setting. Once again, by constructing semigroups using fractional derivatives and fractional powers we obtain solutions to some Wentzell problems by characterizing domains of generators of semigroups. A detailed analysis of the boundary behaviour of elements in $\tilde{H}_{p,+}^{\tilde{\psi}_\alpha,s}$ of course requires some restriction and extension results for these spaces — some results are given in [17]. In this approach we could ignore the lack of having no transmission condition at our disposal by using the fact that we work with generators of semigroups which are obtained as subordinate operators of operators satisfying (a type of) the transmission condition. (Note that Y. Ishikawa [8] proved that the transmission condition in general is violated after subordination.)

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