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FRACTIONAL CALCULUS OF THE GENERALIZED WRIGHT FUNCTION

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Dedicated to Acad. Bogoljub Stanković,
on the occasion of his 80-th birthday

Abstract

The paper is devoted to the study of the fractional calculus of the generalized Wright function ${}_p\Psi_q(z)$ defined for $z \in \mathbf{C}$, complex $a_i, b_j \in \mathbf{C}$ and real $\alpha_i,\ \beta_j \in \mathbf{R}$ $(i = 1, 2, \cdots p;\ j = 1, 2, \cdots, q)$ by the series

$$ {}_p\Psi_q(z) = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k)} \frac{z^k}{k!}\,. $$

It is proved that the Riemann-Liouville fractional integrals and derivative of the Wright function are also the Wright functions but of greater order. Special cases are considered.

Mathematics Subject Classification: 26A33, 33C20

Key Words and Phrases: Riemann-Liouville fractional integrals and derivatives, generalized Wright function, Wright and Bessel-Maitland functions

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1. Introduction

The paper deals with the generalized Wright function defined for $z \in \mathbf{C}$, complex $a_i, b_j \in \mathbf{C}$ and real $\alpha_i,\ \beta_j \in \mathbf{R} = (-\infty, \infty)$ $(\alpha_i, \beta_j \neq 0;\ i = 1, 2, \cdots p;\ j = 1, 2, \cdots, q)$ by the series

$$ {}_p\Psi_q(z) \equiv {}_p\Psi_q \left[\begin{array}{c|c} (a_i, \alpha_i)_{1,p} & \\ & z \\ (b_j, \beta_j)_{1,q} & \end{array} \right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k)} \frac{z^k}{k!}. \tag{1} $$

Here $\Gamma(z)$ is the Euler gamma-function [3, Section 1.1]. The function in (1) was introduced by Wright [21] and is called the generalized Wright function, see [3, Section 4.1]. Conditions for the existence of the generalized Wright function (1) together with its representation in terms of the Mellin-Barnes integral and of the H-function were established in [6].

The special case of the function (1) in the form

$$ \phi(\beta, b; z) \equiv {}_0\Psi_1 \left[\begin{array}{c|c} & \\ & z \\ (b, \beta) & \end{array} \right] = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\beta k + b)} \frac{z^k}{k!} \tag{2} $$

with complex $z, b \in \mathbf{C}$ and real $\beta \in \mathbf{R}$, known as the Wright function [4, Section 18.1], was introduced by Wright in [19]. When $\beta = \delta$, $b = \nu + 1$ and z is replaced by $-z$, the function $\phi(\delta, \nu + 1; -z)$ is denoted by $J_\nu^\delta(z)$:

$$ J_\nu^\delta(z) \equiv \phi(\delta, \nu + 1; -z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\delta k + \nu + 1)} \frac{(-z)^k}{k!}, \tag{3} $$

and such a function is known as the Bessel-Maitland function, or the Wright generalized Bessel function, see [7, p. 352] and [14, (8.3)]. Some other particular cases of the generalized Wright function (1), generalizing the classical Mittag-Leffler function, were presented in [6, Section 6].

Wright in [20], [24] investigated the asymptotic expansions of the function $\phi(\beta, b; z)$ for large values of z in the cases $\beta > 0$ and $-1 < \beta < 0$, respectively, making use of the "steepest descent" method. In [20] he gives an application of the obtained results to the asymptotic theory of partitions. In [21]-[23] Wright extended the last results to the generalized Wright function (1) and proved several theorems on the asymptotic expansion of ${}_p\Psi_q(z)$ for all values of the argument z under the condition

$$ \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > -1. \tag{4} $$

The properties of the Wright function (2) were studied in a series of papers. Some of them can be found in [4, Section 18.1]. We also mention that some fractional integral relations for the function (2) were presented in [2], asymptotic relations for zeros of the Wright function $\phi(\beta, b; z)$ were established in [8], and distributions of these zeros were investigated in [9]. Applications of the Wright function (2) to the operational calculus were given in [15], to integral transforms of Hankel type - in [5] and [18], to partial differential equations of fractional order - in [1] and [10]-[13], see also [16, Section 4.1.2]. We also note [2], where solution in closed form of the integral equation of the first with the Wright function as a kernel was obtained.

The present paper is devoted to the study of the Riemann-Liouville fractional integration and differentiation of the Wright function (1). For $\alpha \in \mathbf{C}$ ($\mathrm{Re}(\alpha) > 0$), such a left- and right-hand sided fractional integration operators are defined by

$$(I_{0+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt \ (x > 0); \tag{5}$$

and

$$(I_{-}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} \frac{f(t)}{(t-x)^{1-\alpha}} dt \ (x > 0), \tag{6}$$

respectively [17, Section 5.1], and the corresponding fractional differentiation operators have the forms

$$(D_{0+}^{\alpha} f)(x) = \left(\frac{d}{dx}\right)^{[\mathrm{Re}(\alpha)+1} (I_{0+}^{1-\alpha+[\mathrm{Re}(\alpha)]} f)(x)$$

$$= \left(\frac{d}{dx}\right)^{[\mathrm{Re}(\alpha)+1} \frac{1}{\Gamma(1-\alpha+[\mathrm{Re}(\alpha)])} \int_0^x \frac{f(t)}{(x-t)^{\alpha-[\mathrm{Re}(\alpha)]}} dt \ (x > 0) \tag{7}$$

and

$$(D_{-}^{\alpha} f)(x) = \left(-\frac{d}{dx}\right)^{[\mathrm{Re}(\alpha)+1} (I_{-}^{1-\alpha+[\mathrm{Re}(\alpha)]} f)(x)$$

$$= \left(-\frac{d}{dx}\right)^{[\mathrm{Re}(\alpha)+1} \frac{1}{\Gamma(1-\alpha+[\mathrm{Re}(\alpha)])} \int_x^{\infty} \frac{f(t)}{(t-x)^{\alpha-[\mathrm{Re}(\alpha)]}} dt \ (x > 0), \tag{8}$$

respectively, where $[\mathrm{Re}(\alpha)]$ is the integral part of $\mathrm{Re}(\alpha)$.

The paper is organized as follows. Some known results are presented in Section 2. The fractional integration and differentiation of the generalized

Wright function (1) is established in Sections 3 and 4, respectively. The corresponding results for the Wright function (2) and the Bessel-Maitland function (3) are presented in Section 5.

2. Preliminaries

In this section we present the conditions for the existence of the generalized Wright function ${}_p\Psi_q(z)$ in (1) proved in [6], and the known formulas for the fractional integration (5) and (6) of a power function [17]. To formulate the first result we use the following notation:

$$\Delta = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i,$$

$$\delta = \prod_{i=1}^{p} |\alpha_i|^{-\alpha_i} \prod_{j=1}^{q} |\beta_j|^{\beta_j},$$

$$\mu = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p-q}{2}.$$

THEOREM 1. *Let* $a_i, b_j \in \mathbf{C}$ *and* $\alpha_i,\ \beta_j \in \mathbf{R}$ $(i = 1, 2, \cdots p;\ j = 1, 2, \cdots, q)$.

(a) If $\Delta > -1$, *then the series in* (1) *is absolutely convergent for all* $z \in \mathbf{C}$.

(b) If $\Delta = -1$, *then the series in (1) is absolutely convergent for all values of* $|z| < \delta$ *and of* $|z| = \delta$, $\Re(\mu) > 1/2$.

COROLLARY 1.1. *Let* $a_i, b_j \in \mathbf{C}$ *and* $\alpha_i,\ \beta_j \in \mathbf{R}$ $(i = 1, 2, \cdots p;\ j = 1, 2, \cdots, q)$ *be such that the condition in* (4) *is satisfied. Then the generalized Wright function* ${}_p\Psi_q(z)$ *is an entire function of* z.

COROLLARY 1.2. *Let* $\alpha \in \mathbf{R}$ *and* $\beta \in \mathbf{C}$.

(a) If $\alpha > -1$, *then the series in* (2) *is absolutely convergent for all* $z \in \mathbf{C}$.

(b) If $\alpha = -1$, *then the series in* (2) *is absolutely convergent for all values of* $|z| < 1$ *and of* $|z| = 1$, $\Re(\beta) > 1$.

COROLLARY 1.3. *If* $\alpha > -1$ *and* $\beta \in \mathbf{C}$, *then the Wright function* $\phi(\alpha, \beta; z)$ *is an entire function of* z.

COROLLARY 1.4. *If $\delta > -1$ and $\nu \in \mathbf{C}$, then the Bessel-Maitland function $J_\nu^\delta(z)$ is an entire function of z.*

The next assertion is well known, see [17, (2.44) and Table 9.3, formula 1].

LEMMA 1. *Let $\alpha \in \mathbf{C}$ $(\mathrm{Re}(\alpha) > 0)$ and $\gamma \in \mathbf{C}$.*
(a) *If $\mathrm{Re}(\gamma) > 0$, then*

$$\left(I_{0+}^{\alpha} t^{\gamma-1}\right)(x) = \frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} x^{\alpha+\gamma-1}. \tag{9}$$

(b) *If $\mathrm{Re}(\gamma) > \mathrm{Re}(\alpha) > 0$, then*

$$\left(I_{-}^{\alpha} t^{-\gamma}\right)(x) = \frac{\Gamma(\gamma-\alpha)}{\Gamma(\gamma)} x^{\alpha-\gamma}. \tag{10}$$

3. Fractional integration of the generalized Wright function

In this section we establish a formula for the fractional integration of the generalized Wright function (1). We begin with the left-hand sided fractional integral (5).

THEOREM 2. *Let $\alpha, \gamma \in \mathbf{C}$ be complex numbers such that $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\gamma) > 0$, and let $a \in \mathbf{C}$, $\mu > 0$. If the condition (4) is satisfied, then the fractional integration I_{0+}^{α} of the generalized Wright function (1) is given for $x > 0$ by*

$$\left(I_{0+}^{\alpha}\left(t^{\gamma-1}\ {}_p\Psi_q\left[\begin{array}{c|c} (a_i,\alpha_i)_{1,p} & \\ & at^{\mu} \\ (b_j,\beta_j)_{1,q} & \end{array}\right]\right)\right)(x)$$

$$= x^{\gamma+\alpha-1}\ {}_{p+1}\Psi_{q+1}\left[\begin{array}{c|c} (a_i,\alpha_i)_{1,p},(\gamma,\mu) & \\ & ax^{\mu} \\ (b_j,\beta_j)_{1,q},(\gamma+\alpha,\mu) & \end{array}\right]. \tag{11}$$

P r o o f. According to (4) and Corollary 1.1, the generalized Wright functions in both sides of (11) exist for $x > 0$. By (5) and (1) we have

$$\left(I_{0+}^{\alpha}\left(t^{\gamma-1}\ {}_p\Psi_q\left[\begin{array}{c|c} (a_i,\alpha_i)_{1,p} & \\ & at^{\mu} \\ (b_j,\beta_j)_{1,q} & \end{array}\right]\right)\right)(x)$$

$$= \left(I_{0+}^{\alpha} \left[t^{\gamma-1} \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k)} \frac{(at^{\mu})^k}{k!} \right] \right)(x). \tag{12}$$

According to [17, Lemma 15.1] a term-by-term integration of a series in the right-hand side of (12) is possible. Carrying out such an integration and using (9) we obtain

$$\left(I_{0+}^{\alpha} \left(t^{\gamma-1} \, {}_p\Psi_q \left[\begin{array}{c|c} (a_i, \alpha_i)_{1,p} & \\ & at^{\mu} \\ (b_j, \beta_j)_{1,q} & \end{array} \right] \right) \right)(x)$$

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k)} \frac{a^k}{k!} \left(I_{0+}^{\alpha} t^{\gamma+\mu k-1} \right)(x)$$

$$= x^{\gamma+\alpha-1} \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k)} \frac{\Gamma(\gamma + \mu k)}{\Gamma(\gamma + \alpha + \mu k)} \frac{(ax^{\mu})^k}{k!}.$$

According to (1) from here we deduce (11), which completes the proof of theorem. ■

The following result yields the right-hand sided fractional integration (6) of the generalized Wright function (1).

THEOREM 3. *Let* $\alpha, \gamma \in \mathbf{C}$ *be complex numbers such that* $\mathrm{Re}(\gamma) > \mathrm{Re}(\alpha) > 0$, *and let* $a \in \mathbf{C}$, $\mu > 0$. *If the condition* (4) *is satisfied, then the fractional integration* I_{-}^{α} *of the generalized Wright function* (1) *is given by*

$$\left(I_{-}^{\alpha} \left(t^{-\gamma} \, {}_p\Psi_q \left[\begin{array}{c|c} (a_i, \alpha_i)_{1,p} & \\ & at^{-\mu} \\ (b_j, \beta_j)_{1,q} & \end{array} \right] \right) \right)(x)$$

$$= x^{\alpha-\gamma} \, {}_{p+1}\Psi_{q+1} \left[\begin{array}{c|c} (a_i, \alpha_i)_{1,p}, (\gamma - \alpha, \mu) & \\ & ax^{-\mu} \\ (b_j, \beta_j)_{1,q}, (\gamma, \mu) & \end{array} \right]. \tag{13}$$

P r o o f. According to (4) and Corollary 1.1, the generalized Wright functions in both sides of (13) exist for $x > 0$. The fractional integrals (5) and (6) are connected by the relation

$$\left(I_{-}^{\alpha} f \left[\frac{1}{t} \right] \right)(x) = x^{\alpha-1} \left(I_{0+}^{\alpha} [t^{-\alpha-1} f(t)] \right) \left(\frac{1}{x} \right).$$

Using this formula and taking into account (11) with γ replaced by $\gamma-\alpha$, we have

$$\left(I_{-}^{\alpha}\left(t^{-\gamma}\ {}_p\Psi_q\left[\begin{array}{c|c}(a_i,\alpha_i)_{1,p} & \\ & at^{-\mu}\\ (b_j,\beta_j)_{1,q} & \end{array}\right]\right)\right)(x)$$

$$= x^{\alpha-1}\left(I_{0+}^{\alpha}\left(t^{\gamma-\alpha-1}\ {}_p\Psi_q\left[\begin{array}{c|c}(a_i,\alpha_i)_{1,p} & \\ & at^{\mu}\\ (b_j,\beta_j)_{1,q} & \end{array}\right]\right)\right)\left(\frac{1}{x}\right)$$

$$= x^{\alpha-\gamma}\ {}_{p+1}\Psi_{q+1}\left[\begin{array}{c|c}(a_i,\alpha_i)_{1,p},(\gamma-\alpha,\mu) & \\ & ax^{-\mu}\\ (b_j,\beta_j)_{1,q},(\gamma,\mu) & \end{array}\right],$$

and (13) is proved. ■

4. Fractional differentiation of the generalized Wright function

In this section we establish a formula for the fractional differentiation of the generalized Wright function (1). As in Section 3, we begin with the left-hand sided fractional differentiation (7).

THEOREM 4. *Let $\alpha,\gamma\in\mathbf{C}$ and $\mathrm{Re}(\alpha)>0$ and $\mathrm{Re}(\gamma)>0$, and let $a\in\mathbf{C}$, $\mu>0$. If condition (4) is satisfied, then the fractional differentiation D_{0+}^{α} of the generalized Wright function (1) is given for $x>0$ by*

$$\left(D_{0+}^{\alpha}\left(t^{\gamma-1}\ {}_p\Psi_q\left[\begin{array}{c|c}(a_i,\alpha_i)_{1,p} & \\ & at^{\mu}\\ (b_j,\beta_j)_{1,q} & \end{array}\right]\right)\right)(x)$$

$$= x^{\gamma-\alpha-1}\ {}_{p+1}\Psi_{q+1}\left[\begin{array}{c|c}(a_i,\alpha_i)_{1,p},(\gamma,\mu) & \\ & ax^{\mu}\\ (b_j,\beta_j)_{1,q},(\gamma-\alpha,\mu) & \end{array}\right]. \tag{14}$$

P r o o f. According to (1) and Corollary 1.1, the generalized Wright functions on both sides of (14) exist for $x>0$. Let $n=[\mathrm{Re}(\alpha)]+1$, where $[\mathrm{Re}(\alpha)]$ is an integer part of $\mathrm{Re}(\alpha)$. Using (7) and (1) and taking into account (11), with α replaced by $n-\alpha$, we have

$$\left(D_{0+}^{\alpha}\left(t^{\gamma-1}\ {}_p\Psi_q\left[\begin{array}{c|c}(a_i,\alpha_i)_{1,p} & \\ & at^{\mu}\\ (b_j,\beta_j)_{1,q} & \end{array}\right]\right)\right)(x)$$

$$= \left(\frac{d}{dx}\right)^n \left(I_{0+}^{n-\alpha}\left(t^{\gamma-1}\ {}_p\Psi_q\left[\begin{array}{c}(a_i,\alpha_i)_{1,p}\\ \\ (b_j,\beta_j)_{1,q}\end{array}\middle| at^{\mu}\right]\right)\right)(x)$$

$$= \left(\frac{d}{dx}\right)^n \left(x^{\gamma+n-\alpha-1}\ {}_{p+1}\Psi_{q+1}\left[\begin{array}{c}(a_i,\alpha_i)_{1,p},(\gamma,\mu)\\ \\ (b_j,\beta_j)_{1,q},(\gamma+n-\alpha,\mu)\end{array}\middle| ax^{\mu}\right]\right)$$

$$= \left(\frac{d}{dx}\right)^n \left[\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i k)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j k)}\frac{\Gamma(\gamma+\mu k)}{\Gamma(\gamma+n-\alpha+\mu k)}\frac{a^k}{k!}x^{\gamma+n-\alpha+\mu k-1}\right]. \tag{15}$$

According to [17, Lemma 15.1], a term-by-term differentiation of the series on the right-hand side of (15) is possible. Therefore

$$\left(D_{0+}^{\alpha}\left(t^{\gamma-1}\ {}_p\Psi_q\left[\begin{array}{c}(a_i,\alpha_i)_{1,p}\\ \\ (b_j,\beta_j)_{1,q}\end{array}\middle| at^{\mu}\right]\right)\right)(x)$$

$$= \sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i k)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j k)}\frac{\Gamma(\gamma+\mu k)}{\Gamma(\gamma-\alpha+\mu k)}\frac{a^k}{k!}x^{\gamma-\alpha+\mu k-1}.$$

Thus, in accordance with (1), (14) is proved. ■

The next result yields the right-hand sided fractional differentiation (8) of the generalized Wright function (1).

THEOREM 5. *Let $\alpha,\gamma\in\mathbf{C}$ be complex numbers such that* $\mathrm{Re}(\alpha)>0$ *and* $\mathrm{Re}(\gamma)>[\mathrm{Re}(\alpha)]+1-\mathrm{Re}(\alpha)$, *and let $a\in\mathbf{C}$, $\mu>0$. If condition* (4) *is satisfied, then the fractional differentiation D_{-}^{α} of the generalized Wright function* (1) *is given by*

$$\left(D_{-}^{\alpha}\left(t^{-\gamma}\ {}_p\Psi_q\left[\begin{array}{c}(a_i,\alpha_i)_{1,p}\\ \\ (b_j,\beta_j)_{1,q}\end{array}\middle| at^{-\mu}\right]\right)\right)(x)$$

$$= x^{-\alpha-\gamma}\ {}_{p+1}\Psi_{q+1}\left[\begin{array}{c}(a_i,\alpha_i)_{1,p},(\gamma+\alpha,\mu)\\ \\ (b_j,\beta_j)_{1,q},(\gamma,\mu)\end{array}\middle| ax^{-\mu}\right]. \tag{16}$$

P r o o f. By (4) and Corollary 1.1, the generalized Wright functions in both sides of (16) exist for $x > 0$. Let $n = [\mathrm{Re}(\alpha)] + 1$. Using (8) and (1) and taking into account (13) with α replaced by $n - \alpha$, similarly to the proof of Theorem 4, we obtain

$$\left(D_-^\alpha\left(t^{-\gamma}\ {}_p\Psi_q\left[\begin{array}{c|c}(a_i,\alpha_i)_{1,p} & \\ & at^{-\mu}\\ (b_j,\beta_j)_{1,q} & \end{array}\right]\right)\right)(x)$$

$$= \left(-\frac{d}{dx}\right)^n \left(I_-^{n-\alpha}\left(t^{-\gamma}\ {}_p\Psi_q\left[\begin{array}{c|c}(a_i,\alpha_i)_{1,p} & \\ & at^{-\mu}\\ (b_j,\beta_j)_{1,q} & \end{array}\right]\right)\right)(x)$$

$$= \left(-\frac{d}{dx}\right)^n \left(x^{n-\alpha-\gamma}\ {}_{p+1}\Psi_{q+1}\left[\begin{array}{c|c}(a_i,\alpha_i)_{1,p}, (\gamma-n+\alpha,\mu) & \\ & ax^{-\mu}\\ (b_j,\beta_j)_{1,q}, (\gamma,\mu) & \end{array}\right]\right)$$

$$= \left(-\frac{d}{dx}\right)^n \left[\sum_{k=0}^{\infty} \frac{\prod_{i=1}^p \Gamma(a_i+\alpha_i k)}{\prod_{j=1}^q \Gamma(b_j+\beta_j k)} \frac{\Gamma(\gamma-n+\alpha+\mu k)}{\Gamma(\gamma+\mu k)} \frac{a^k}{k!} x^{n-\alpha-\gamma-\mu k}\right]$$

$$= \sum_{k=0}^{\infty} \frac{\prod_{i=1}^p \Gamma(a_i+\alpha_i k)}{\prod_{j=1}^q \Gamma(b_j+\beta_j k)} (-1)^n \frac{\Gamma(\gamma-n+\alpha+\mu k)}{\Gamma(\gamma+\mu k)}$$

$$\times \frac{\Gamma(1+n-\alpha-\gamma-\mu k)}{\Gamma(1-\gamma-\alpha-\mu k)} \frac{a^k}{k!} x^{-\alpha-\gamma-\mu k}. \tag{17}$$

By the reflection formula for the gamma-function, see for example, [17, (1.60)],

$$\frac{1}{\Gamma(1-\gamma-\alpha-\mu k)} = \frac{\Gamma(\gamma+\alpha+\mu k)}{\Gamma(\gamma+\alpha+\mu k)\Gamma(1-\gamma-\alpha-\mu k)}$$

$$= \frac{\Gamma(\gamma+\alpha+\mu k)\sin[(\gamma+\alpha+\mu k)\pi]}{\pi}$$

and

$$\Gamma(\gamma-n+\alpha+\mu k)\Gamma(1+n-\alpha-\gamma-\mu k) = \frac{\pi}{\sin[(\gamma-n+\alpha+\mu k)\pi]}$$

$$= \frac{\pi}{\sin[(\gamma+\alpha+\mu k)\pi]\cos(n\pi)} = \frac{(-1)^n\pi}{\sin[(\gamma+\alpha+\mu k)\pi]}.$$

Substituting these relations into (17) we obtain

$$\left(D_{-}^{\alpha}\left(t^{-\gamma}\ {}_{p}\Psi_{q}\left[\begin{array}{c|c}(a_i,\alpha_i)_{1,p} & \\ & at^{-\mu}\\ (b_j,\beta_j)_{1,q} & \end{array}\right]\right)\right)(x)$$

$$= x^{-\alpha-\gamma}\sum_{k=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+\alpha_i k)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j k)}(-1)^{n}\frac{\Gamma(\gamma+\alpha+\mu k)}{\Gamma(\gamma+\mu k)}\frac{(ax^{-\mu})^{k}}{k!},$$

which, in accordance with (1), yields (16). ■

5. Fractional calculus of the Wright and the Bessel-Maitland functions

In this section we establish fractional integration and differentiation of the Wright function $\phi(\beta, b; z)$ and Bessel-Maitland function $J_{\nu}^{\delta}(z)$. Using (2), from Theorems 2-3 and Theorems 4-5 we deduce formulas for the fractional integration and differentiation of $\phi(\beta, b; z)$.

THEOREM 6. *Let $\alpha, \gamma, b, a \in \mathbf{C}$ and $\mu > 0$ and $\beta > -1$.*

(a) If $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\gamma) > 0$, then the fractional integration I_{0+}^{α} of the Wright function (2) is given for $x > 0$ by

$$\left(I_{0+}^{\alpha}\left[t^{\gamma-1}\phi\left(\beta,b;at^{\mu}\right)\right]\right)(x) = x^{\gamma+\alpha-1}\ {}_{1}\Psi_{2}\left[\begin{array}{c|c}(\gamma,\mu) & \\ & ax^{\mu}\\ (b,\beta),(\gamma+\alpha,\mu) & \end{array}\right]. \tag{18}$$

(b) If $\mathrm{Re}(\gamma) > \mathrm{Re}(\alpha) > 0$, then the fractional integration I_{-}^{α} of the Wright function (2) is given for $x > 0$ by

$$\left(I_{-}^{\alpha}\left[t^{-\gamma}\phi\left(\beta,b;at^{-\mu}\right)\right]\right)(x) = x^{\alpha-\gamma}\ {}_{1}\Psi_{2}\left[\begin{array}{c|c}(\gamma-\alpha,\mu) & \\ & ax^{-\mu}\\ (b,\beta),(\gamma,\mu) & \end{array}\right]. \tag{19}$$

COROLLARY 6.1. *Let $\alpha, \gamma, a \in \mathbf{C}$ and $\mu > 0$.*

(a) If $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(\gamma) > 0$, then

$$\left(I_{0+}^{\alpha}\left[t^{\gamma-1}\phi\left(\mu,\gamma;at^{\mu}\right)\right]\right)(x) = x^{\gamma+\alpha-1}\phi\left(\mu,\gamma+\alpha;ax^{\mu}\right). \tag{20}$$

(b) If $\mathrm{Re}(\gamma) > \mathrm{Re}(\alpha) > 0$, then

$$\left(I_{-}^{\alpha}\left[t^{-\gamma}\phi\left(\mu,\gamma-\alpha;at^{-\mu}\right)\right]\right)(x) = x^{\alpha-\gamma}\ \phi\left(\mu,\gamma;ax^{-\mu}\right). \tag{21}$$

THEOREM 7. *Let $\alpha, \gamma, b, a \in \mathbf{C}$ and $\mu > 0$ and $\beta > -1$.*

(a) If $\mathrm{Re}(\alpha) > 0$ *and* $\mathrm{Re}(\gamma) > 0$, *then the fractional differentiation* D_{0+}^{α} *of the Wright function* (2) *is given for* $x > 0$ *by*

$$\left(D_{0+}^{\alpha}\left[t^{\gamma-1}\phi\left(\beta, b; at^{\mu}\right)\right]\right)(x) = x^{\gamma-\alpha-1}\ {}_1\Psi_2\left[\begin{array}{c|c} (\gamma,\mu) & \\ & ax^{\mu} \\ (b,\beta),(\gamma-\alpha,\mu) & \end{array}\right]. \tag{22}$$

(b) If $\mathrm{Re}(\gamma) > [\mathrm{Re}(\alpha)] + 1 - \mathrm{Re}(\alpha)$, *then the fractional differentiation* D_{-}^{α} *of the Wright function* (2) *is given for* $x > 0$ *by*

$$\left(D_{-}^{\alpha}\left[t^{-\gamma}\phi\left(\beta, b; at^{-\mu}\right)\right]\right)(x) = x^{-\alpha-\gamma}\ {}_1\Psi_2\left[\begin{array}{c|c} (\gamma+\alpha,\mu) & \\ & ax^{-\mu} \\ (b,\beta),(\gamma,\mu) & \end{array}\right]. \tag{23}$$

COROLLARY 7.1. *Let $\alpha, \gamma, a \in \mathbf{C}$ and $\mu > 0$.*

(a) If $\mathrm{Re}(\alpha) > 0$ *and* $\mathrm{Re}(\gamma) > 0$, *then*

$$\left(D_{0+}^{\alpha}\left[t^{\gamma-1}\phi\left(\mu, \gamma; at^{\mu}\right)\right]\right)(x) = x^{\gamma-\alpha-1}\phi\left(\mu, \gamma-\alpha; ax^{\mu}\right). \tag{24}$$

(b) If $\mathrm{Re}(\gamma) > [\mathrm{Re}(\alpha)] + 1 - \mathrm{Re}(\alpha)$, *then*

$$\left(I_{-}^{\alpha}\left[t^{-\gamma}\phi\left(\mu, \gamma+\alpha; at^{-\mu}\right)\right]\right)(x) = x^{\alpha-\gamma}\ \phi\left(\mu, \gamma; ax^{-\mu}\right). \tag{25}$$

Similarly, in accordance with (3), from Theorems 2-3 and Theorems 4-5 we obtain the fractional integration and differentiation of $J_{\nu}^{\delta}(z)$.

THEOREM 8. *Let $\alpha, \gamma, \nu, a \in \mathbf{C}$ and $\mu > 0$ and $\delta > -1$.*

(a) If $\mathrm{Re}(\alpha) > 0$ *and* $\mathrm{Re}(\gamma) > 0$, *then the fractional integration* I_{0+}^{α} *of the Bessel-Maitland function* (3) *is given for* $x > 0$ *by*

$$\left(I_{0+}^{\alpha}\left[t^{\gamma-1}J_{\nu}^{\delta}\left(at^{\mu}\right)\right]\right)(x) = x^{\gamma+\alpha-1}\ {}_1\Psi_2\left[\begin{array}{c|c} (\gamma,\mu) & \\ & ax^{\mu} \\ (\nu+1,\delta),(\gamma+\alpha,\mu) & \end{array}\right]. \tag{26}$$

(b) If $\mathrm{Re}(\gamma) > \mathrm{Re}(\alpha) > 0$, *then the fractional integration* I_{-}^{α} *of the Bessel-Maitland function* (3) *is given for* $x > 0$ *by*

$$\left(I_{-}^{\alpha}\left[t^{-\gamma}J_{\nu}^{\delta}\left(at^{-\mu}\right)\right]\right)(x) = x^{\alpha-\gamma}\ {}_1\Psi_2\left[\begin{array}{c|c} (\gamma-\alpha,\mu) & \\ & ax^{-\mu} \\ (\nu+1,\delta),(\gamma,\mu) & \end{array}\right]. \tag{27}$$

Corollary 8.1. *Let $\alpha, \nu, a \in \mathbf{C}$ be complex numbers such that* $\mathrm{Re}(\alpha) > 0$ *and* $\mathrm{Re}(\nu) > -1$, *and let* $\mu > 0$. *Then there hold the relations*

$$\left(I_{0+}^{\alpha}\left[t^{\nu} J_{\nu}^{\mu}\left(a t^{\mu}\right)\right]\right)(x)=x^{\nu+\alpha} J_{\nu+1+\alpha}^{\mu}\left(a x^{\mu}\right). \tag{28}$$

and

$$\left(I_{-}^{\alpha}\left[t^{-\alpha-\nu-1} J_{\nu}^{\mu}\left(a t^{-\mu}\right)\right]\right)(x)=x^{-\nu-1} J_{\nu+1+\alpha}^{\mu}\left(a x^{-\mu}\right). \tag{29}$$

Theorem 9. *Let $\alpha, \gamma, b, \nu \in \mathbf{C}$ and $\mu > 0$ and $\delta > -1$.*

(a) If $\mathrm{Re}(\alpha) > 0$ *and* $\mathrm{Re}(\gamma) > 0$, *then the fractional differentiation* D_{0+}^{α} *of the Bessel-Maitland function* (3) *is given for* $x > 0$ *by*

$$\left(D_{0+}^{\alpha}\left[t^{\gamma-1} J_{\nu}^{\delta}\left(a t^{\mu}\right)\right]\right)(x)=x^{\gamma-\alpha-1}\,{}_{1}\Psi_{2}\left[\begin{array}{c|c} (\gamma, \mu) & \\ & a x^{\mu} \\ (\nu+1, \delta),(\gamma-\alpha, \mu) & \end{array}\right]. \tag{30}$$

(b) If $\mathrm{Re}(\gamma) > [\mathrm{Re}(\alpha)] + 1 - \mathrm{Re}(\alpha)$, *then the fractional differentiation* D_{-}^{α} *of the Bessel-Maitland function* (3) *is given for* $x > 0$ *by*

$$\left(D_{-}^{\alpha}\left[t^{-\gamma} J_{\nu}^{\delta}\left(a t^{-\mu}\right)\right]\right)(x)=x^{-\alpha-\gamma}\,{}_{1}\Psi_{2}\left[\begin{array}{c|c} (\gamma+\alpha, \mu) & \\ & a x^{-\mu} \\ (\nu+1, \delta),(\gamma, \mu) & \end{array}\right]. \tag{31}$$

Corollary 9.1. *Let $\alpha, \nu, a \in \mathbf{C}$ and $\mu > 0$.*

(a) If $\mathrm{Re}(\alpha) > 0$ *and* $\mathrm{Re}(\nu) > -1$, *then*

$$\left(D_{0+}^{\alpha}\left[t^{\nu} J_{\nu}^{\mu}\left(a t^{\mu}\right)\right]\right)(x)=x^{\nu-\alpha} J_{\nu+1-\alpha}^{\mu}\left(a x^{\mu}\right). \tag{32}$$

(b) If $\mathrm{Re}(\alpha) > 0$ *and* $\mathrm{Re}(\nu) > [\mathrm{Re}(\alpha)]$, *then*

$$\left(D_{-}^{\alpha}\left[t^{\alpha-\nu-1} J_{\nu}^{\mu}\left(a t^{-\mu}\right)\right]\right)(x)=x^{-\nu-1} J_{\nu+1-\alpha}^{\mu}\left(a x^{-\mu}\right). \tag{33}$$

References

[1] E. Buckwar E. and Yu. Luchko, Invariance of partial differential equations of fractonal order under the Lie group of scaling transformations. *J. Math. Anal. Appl.* **237**, No 1 (1998), 81-97.

[2] M.R. Dotsenko, On some applications of Wright's hypergeometric function. *C.R. Acad. Bulgare Sci.* **44**, No 6 (1991), 13-16.

[3] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions*, Vol. **I**. McGraw-Hill, New York (1953); Reprinted: Krieger, Melbourne-Florida (1981).

[4] A. Erdélyi, W. Magnus, F. Oberhettinger and F.G. Tricomi, *Higher Transcendental Functions*, Vol. **III**. McGraw-Hill, New York (1954); Reprinted: Krieger, Melbourne-Florida (1981).

[5] L. Gajic and B. Stankovic, Some properties of Wright's function. *Publ. l'Institut Math. Beograd, Nouvelle Ser.* **20**, No 34 (1976), 91-98.

[6] A.A. Kilbas, M. Saigo and J.J. Trujillo, On the generalized Wright function. *Fract. Calc. Appl. Anal.* bf 5, No 4 (2002), 437-460.

[7] V.S. Kiryakova, *Generalized Fractional Calculus and Applications.* Pitman Research Notes in Mathematics, **301**, John Wiley and Sons, New York (1994).

[8] Yu.F. Luchko, Asymptotics of zeros of the Wright function. *Z. Anal. Anwendungen* **19**, No 2 (2000), 583-595.

[9] Yu. Luchko, On the distribution of zeros of the Wright function. *Integral Transform. Spec. Funct.* **11**, No 2 (2001), 195-200.

[10] Yu. Luchko and R. Gorenflo, Scale-invariant solutions of a partial differential equation of fractional order. *Fract. Calc. Appl. Anal.* **1**, No 1 (1998), 63-78.

[11] F. Mainardi, On the initial value problem for the fractional diffusion-wave equation. *Waves and Stability in Continuous Media* (Bologna, 1993), *Ser. Adv. Math. Appl. Sci.* **23** (1994), 246-251.

[12] F. Mainardi, Fractional calculus: Some basic problems in continuum and statistical mechanics. In: *Fractals and Fractional Calculus in Continuum Mechanics* (A.Carpintery and F. Mainardi, Editors) (Udine, 1996), *CIAM Courses and Lectures* **378** (1997), 291-348.

[13] F. Mainardi and M. Timorotti, On a special function arising in the time fractional diffusion-wave equation. *Transform Methods and Special Functions, Sofia'94 (Proc. 1st Intern. Workshop)*, SCTP, Singapore (1995), 171-183.

[14] O.I. Marichev, *Handbook of Integral Transforms and Higher Transcendental Functions. Theory and Algorithmic Tables*. Ellis Horwood, Chichester [John Wiley and Sons], New York (1983).

[15] J. Mikusinski, On the function whose Laplace transform is $\exp(-s^{\alpha})$, $0 < \alpha < 1$. *Studia Math. J.* **18**, (1959), 191-198.

[16] I. Podlubny, *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solutions and Some of Their Applications*. Mathematics in Sciences and Engineering. **198**, Academic Press, San-Diego (1999).

[17] S.G. Samko, A.A. Kilbas, and O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*. Gordon and Breach, New York (1993).

[18] B. Stankovic, On the function of E.M. Wright. *Publ. de l'Institut Mathematique, Nouvelle Ser.* **10**, NO 24 (1970), 113-124.

[19] E.M. Wright, On the coefficients of power series having exponential singularities. *J. London Math. Soc.* **8** (1933), 71-79.

[20] E.M. Wright, The asymptotic expansion of the generalized Bessel function. *Proc. London Math. Soc. (2)* **38** (1934), 257-270.

[21] E.M. Wright, The asymptotic expansion of the generalized hypergeometric function. *J. London Math. Soc.* **10** (1935), 287-293.

[22] E.M. Wright, The asymptotic expansion of integral functions defined by Taylor series. *Philos. Trans. Roy. Soc. London, Ser. A* **238** (1940), 423-451.

[23] E.M. Wright, The asymptotic expansion of the generalized hypergeometric funciton. *Proc. London Math. Soc. (2)* **46**, (1940), 389-408.

[24] E.M. Wright, The generalized Bessel functions of order greater than one. *Quart. J. Math. Oxford Ser.* **11** (1940), 36-48.

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